

AN INVESTIGATION ON HYBRID PROJECTIVE COMBINATION DIFFERENCE SYNCHRONIZATION SCHEME BETWEEN CHAOTIC PREY-PREDATOR SYSTEMS VIA ACTIVE CONTROL METHOD

TAQSEER KHAN AND HARINDRI CHAUDHARY[†]

Date of Receiving : 05. 06. 2020
Date of Revision : 17. 08. 2020
Date of Acceptance : 07. 10. 2020

Abstract. This research addresses a systematic design for investigating hybrid projective combination difference synchronization (HPCDS) scheme between chaotic prey-predator systems via active control method. The presented work deals with generalized Lotka and Volterra (GLV) biological system. The considered system analyzes the interactions among three species prey (one) and predators (two) that comprises of a system of ordinary differential equations. An active control approach has been investigated which is primarily based on Lyapunov stability theory (LST). The discussed scheme derives the asymptotic stability globally using HPCDS technique. Numerical simulations are thereafter implemented to validate the efficiency and feasibility of the discussed strategy using MATLAB. Interestingly, both the computational and theoretical results agree remarkably. In addition, a comparison analysis has been done which shows the significance of considered approach over prior published researches. Furthermore, the considered HPCDS scheme is useful in secure communication and encrypting images.

1. Introduction

Currently, the preservation of ecological as well as biological systems is a prime concern to a vast spectrum of scientific domains. Subsequently, controlling and surveying the immensely complex and irregular dynamic behaviour found in the afore-said systems is primarily a huge challenging topic for scientists and researchers from varied fields. This extreme complexity found in these systems has been introduced fundamentally due to the oscillatory interactions occurring in the populations. In this

2010 *Mathematics Subject Classification.* 34K23, 34K35, 37B25, 37N35.

Key words and phrases. Chaotic system, prey-predator model, hybrid projective synchronization, difference synchronization, active control method, Lyapunov stability theory.

Communicated by. Sundarapandian Vaidyanathan

[†] *Corresponding author*

configuration, prey-predator model develops the well known models for interplaying populations. In fact, the interplaying in prey and predator along with extra conditions has been an indicative idea in mathematical biology since many years. The predation and co-operation/competition species are basically the most significant interactions in all the corresponding interactions. This described competitiveness succeeds either both the considering population are independent of one another or both interact and each one of the population employs descending pressure on other which in turn generates a hugely contrasting complex natural habitat. Such interaction are basically described by nonlinear polynomial models. Interestingly, Alfred J. Lotka [28] and Vito Volterra [44] have independently in 1920s developed similar equations for these interactions almost at the same time to study various key aspects of population dynamics like predation or parasitism between two species. As a result, these equations are famously known as Lotka-Volterra (LV) equations or Predator-Prey model, to credit the seminal work of these mathematicians. Nevertheless, there is a huge dissimilarity in their respective approaches.

Initially, LV model has been developed as a biological concept, yet it is utilised to numerous diverse areas of research. For instance, Gavin et al. [14] studied LV type model to understand the structure of marine phage populations. Further, Antoniou and Pitsillides [1] applied LV model for controlling congestion in wireless sensor networks to avoid packet loss and delay. Also, Gatabazi et al. [13, 12] studied the transaction counts and interactions between crypto currencies like Bitcoin, Litecoin and Ripple using LV models. Further, Perhar et al. [36] employed the LV model to discuss zooplankton growth and phytoplankton dynamics in the context of variations in resource allocation. Also, Tonnang et al. [51] applied LV model to study the impact of biological control on diamondback moth (*Plutellaxylostella*) population. In addition, Tsai et al. [50] used LV model to describe the feasibility of using low-carbon energy sources to reduce fossil fuel consumption accurately in the United States. Also, Reichenbach et al. [39] calculated extinction probabilities LV model. Furthermore, Silva-Dias and Lopez-Castillo [46] discussed spontaneous symmetry breaking of population without population excess. Additionally, Hening and Nguyen [16] observed the persistence and extinction of species in a simple food chain using LV model. Moreover, Xiong et al. [58] discussed the role of random perturbations for inter-specific competition rates and the coexistence equilibrium. Very recently, Sayan Nag [32] studied the effect of novel corona virus, popularly known as COVID-19, on population's dynamics using a modified version of LV model.

Nowadays, generalized Lotka-Volterra (GLV) biological model containing three species has been the most prominent among all existent population's oscillatory interactions. Arnedo, Tresser and Coulet [2] in 1980 have depicted that it may acquire chaotic, that is, extremely confused designing for a properly selected set of parameters. Also, these models basically comprising of one prey and two predators populations. Significantly, in 1988 Samardzija and Greller [43] carried out a comprehensive study in GLV biological model illustrating the chaoticity of GLV model. Furthermore, they deduced some very intriguing properties of GLV model. Theoretically, chaotic systems are nonlinear systems having at least one positive Lyapunov exponent. Chaos synchronization (CS)

of chaotic systems is prescribed as a methodology of adapting chaotic systems (identical or non-identical) in a typical manner that both exhibit the similar execution owing to pairing to gain stability.

Synchronization phenomenon in chaotic systems has been an intriguing and fascinating field for researchers and scientists from varied fields ever since its initiation in 1990 by Pecora and Carroll [35] using master-slave framework that was unprecedented for almost 30 years. Later on, a wide ranging of newly prescribed chaos synchronization and controlling techniques have been introduced and developed. Precisely, enormous synchronization approaches such as complete [47], hybrid [48], anti [24], hybrid projective [20, 19], function projective [61], phase [29], projective [8], combination synchronization [19], lag [23], combination-combination [20], modified projective [25], compound [18], triple compound [59], combination difference [21], etc. in chaotic systems are achieved by using several control techniques, for example, active [7], adaptive or parameter identification method [21], backstepping design [38], impulsive [26], feedback [5], sliding mode [52], etc. available in the current literature. Chaos synchronization in chaotic systems using active control technique was firstly introduced and studied by Er-Wei Bai and Karl E Lonngren [3] in 1997. Mainieri and Rehacek [30] in 1999 advocated the idea of projective synchronization in chaotic systems. Furthermore, combination synchronization was first described in 2011 by Runzi et al [40]. Some further research works [56, 41] have been reported in this direction. Moreover, Dongmo et al. [9] in 2018 introduced a new technique described as difference synchronization of chaotic systems. Recently, Yadav et al.[60] have studied difference synchronization in chaotic systems with exponential terms in the year 2019.

Interestingly, chaos theory has a broad spectrum of applications in science and engineering such as secure communication [22], image encryption [49], neural networks [4], ecological models [42], robotics [34] and so on. In recent times, several kinds of secure communication techniques have been presented [11, 57, 22, 15] such as chaos modulation [57, 33, 27, 31], chaos masking [31, 55] and chaos shift keying [6, 17]. In chaos communication techniques, the basic idea of transmitting a message using chaotic/hyperchaotic systems is that a message signal has been embedded in transmitter system that generates a chaotic signal. Thereafter, this chaotic signal is emitted to the receiver via a public channel. The message signal is recovered finally by the receiver. A chaotic system is specifically used both as the transmitter and receiver. As a consequence, this theory has sought significant consideration in varied research fields. Also, an optimal control technique and synchronization among LV model has been examined rigorously in [10]. Furthermore, in [54] and [53], adaptive control technique is discussed to synchronize GLV biological system.

Keeping the aforementioned discussions in mind, the immediate goal in this paper is to investigate a hybrid projective combination difference synchronization (HPCDS) in three identical integer order chaotic GLV systems by using active control approach. Traditionally, in combination difference synchronization strategy, three chaotic systems (identical or non-identical) are investigated by choosing two systems as master systems and one as a slave system. We have selected in this work the GLV model (master as

well as slave system) since it possesses numerous oscillatory properties resembling to population's, yet it is non-realistic biological model. In addition, based on a famously known Lyapunov stability theory (LST), we investigate in detail the active biological control laws and the convergence of synchronization error functions to achieve HPCDS synchronized state.

The manuscript is systematized as: Sect. 2 presents some preliminaries having few notations and basic terminology to be employed in the following sections. Sect. 3 prescribes some basic structural characteristics of GLV model on which HPCDS using active control approach has been investigated. Also here some methodological considerations of HPCDS scheme are presented in a systematic manner. Further, the active nonlinear controllers have been appropriately designed for achieving HPCDS strategy. Sect. 4 comprises of the discussions with regard to numerical simulations and demonstrations of the graphical results performed in MATLAB software. In addition, a comparative analysis with previously published researches has been carried out. In the end, some conclusions and future works have been presented in Sect. 5.

2. Problem Formulation

In this section, a methodology to describe combination synchronization [40] based on master-slave framework in three chaotic systems has been presented that is necessary for the following sections.

Let the first master system be

$$\dot{y}_{m1} = f_1(y_{m1}), \quad (2.1)$$

and the second master system be

$$\dot{y}_{m2} = f_2(y_{m2}). \quad (2.2)$$

Let the slave system be

$$\dot{y}_{s1} = f_3(y_{s1}) + U(y_{m1}, y_{m2}, y_{s1}), \quad (2.3)$$

where $y_{m1} = (y_{m11}, y_{m12}, \dots, y_{m1n})^T \in R^n$, $y_{m2} = (y_{m21}, y_{m22}, \dots, y_{m2n})^T \in R^n$, $y_{s1} = (y_{s11}, y_{s12}, \dots, y_{s1n})^T \in R^n$ are the state vectors of master and slave systems respectively, $f_1, f_2, f_3 : R^n \rightarrow R^n$ are three nonlinear continuous vector functions, $U = (U_1, U_2, \dots, U_n)^T : R^n \times R^n \times R^n \rightarrow R^n$ are the controllers to be properly determined. We define the combination difference synchronization error as

$$E = Ry_{s1} - (Qy_{m2} - Py_{m1}),$$

where $P = \text{diag}(p_1, p_2, \dots, p_n)$, $Q = \text{diag}(q_1, q_2, \dots, q_n)$, $R = \text{diag}(r_1, r_2, \dots, r_n)$ and $R \neq 0$.

Definition 2.1. The combination of two chaotic master systems (2.1)-(2.2) are said to be in combination difference synchronization (CDS) with the slave system (2.3) if

$$\lim_{t \rightarrow \infty} \|E(t)\| = \lim_{t \rightarrow \infty} \|Ry_{s1}(t) - (Qy_{m2}(t) - Py_{m1}(t))\| = 0.$$

Remark 2.2. The matrices P, Q and R are called the scaling matrices. Moreover, P, Q and R can be extended as matrices of functions of state variables y_{m1}, y_{m2} and y_{s1} .

Remark 2.3. The problem of combination synchronization would be converted into traditional chaos control issue for $P = Q = 0$.

Remark 2.4. Definition 1 exhibits that combination of two master systems and corresponding one slave system may be expanded to more such systems. Moreover, the chosen master systems as well as slave system of combination synchronization scheme may be identical or non-identical.

Remark 2.5. If $R = I$ and $P = Q = \eta I$, then for $\eta = 1$ it will be reduced to combination complete synchronization and for $\eta = -1$ it turns into combination anti-synchronization. Therefore, the combination of anti-synchronization and complete synchronization makes hybrid projective synchronization. Hence, the hybrid projective combination difference synchronization (HPCDS) error takes the form:

$$E = y_{s1} - \eta(y_{m2} - y_{m1}), \quad (2.4)$$

where $\eta = \text{diag}(\eta_1, \eta_2, \dots, \eta_n)$.

Definition 2.6. The combination of two chaotic master systems (2.1)-(2.2) are said to be in hybrid projective combination difference synchronization (HPCDS) with the slave system (2.3) if

$$\lim_{t \rightarrow \infty} \|E(t)\| = \lim_{t \rightarrow \infty} \|y_{s1}(t) - \eta(y_{m2}(t) - y_{m1}(t))\| = 0.$$

The following section presents the CDS scheme to control chaos generated by chaotic systems (2.1)-(2.3) using active control approach.

3. Stability Analysis via Active Control Approach

We now describe the synchronization theory to achieve CDS scheme in two chaotic master systems (2.1)-(2.2) and one chaotic slave system (2.3). We next design the controllers by

$$U_i = \frac{\theta_i}{r_i} - (f_3)_i - \frac{K_i E_i}{r_i}, \quad (3.1)$$

where $\theta_i = (q_i(f_2)_i - p_i(f_1)_i)$, $i = 1, 2, \dots, n$.

Theorem 3.1. To achieve the CDS scheme among the chaotic systems (2.1)-(2.3) globally and asymptotically, we select the active controllers as described in (3.5).

Proof. The errors are given by

$$E_i = r_i y_{s1i} - (q_i y_{m2i} - p_i y_{m1i}), \text{ for } i = 1, 2, \dots, n.$$

The error dynamical system turns into

$$\begin{aligned} \dot{E}_i &= r_i \dot{y}_{s1i} - (q_i \dot{y}_{m2i} - p_i \dot{y}_{m1i}) \\ &= r_i((f_3)_i + U_i) - (q_i(f_2)_i - p_i(f_1)_i) \\ &= r_i((f_3)_i + \frac{\theta_i}{r_i} - (f_3)_i - \frac{K_i E_i}{r_i}) - \theta_i \\ &= \theta_i - K_i E_i - \theta_i \\ &= -K_i E_i \end{aligned} \quad (3.2)$$

The classic Lyapunov function is defined as:

$$V(E(t)) = \frac{1}{2} E^T E = \frac{1}{2} \sum E_i^2 \quad (3.3)$$

On differentiating $V(E(t))$ as given in eq (3.7), we have

$$\dot{V}(E(t)) = \sum E_i \dot{E}_i$$

Using eq (3.6), one finds that

$$\begin{aligned} \dot{V}(E(t)) &= \sum E_i (-K_i E_i). \\ &= - \sum K_i E_i^2. \end{aligned} \quad (3.4)$$

We now select (K_1, K_1, \dots, K_n) so that $\dot{V}(E(t))$ given by eq (3.8) becomes negative definite. Thus, by Lyapunov stability theory [37, 45], we obtain

$$\lim_{t \rightarrow \infty} E_i(t) = 0 \text{ for } (i = 1, 2, 3).$$

Therefore, the master systems (2.1)-(2.2) and slave system (2.3) have achieved desired CDS scheme. \square

4. An Illustrative example

In this section, we firstly describe in short the chaotic system, famously known as Generalized Lotka-Volterra (GLV) three species biological system, to be picked up for HPCDS technique using active control approach. Samardzija and Greller [43] prominently in 1988 have depicted that GLV system acquires chaotic behavior.

We now represent GLV model as the first master system:

$$\begin{cases} \dot{y}_{m11} = & y_{m11} - y_{m11}y_{m12} + b_3y_{m11}^2 - b_1y_{m11}^2y_{m13} \\ \dot{y}_{m12} = & -y_{m12} + y_{m11}y_{m12} \\ \dot{y}_{m13} = & -b_2y_{m13} + b_1y_{m11}^2y_{m13}, \end{cases} \quad (4.1)$$

where $(y_{m11}, y_{m12}, y_{m13})^T \in R^3$ is the state vector of the system and b_1, b_2 and b_3 are positive parameters. Also, in (4.9), y_{m11} represents the prey population and y_{m12}, y_{m13} denotes the predator populations. For parameter values $b_1 = 2.9851, b_2 = 3, b_3 = 2$ and initial values (27.5, 23.1, 11.4), the first master GLV system exhibits chaotic behaviour as shown in Fig.1 (a) and Fig.2 (a).

The second identical master GLV chaotic system prescribed respectively as:

$$\begin{cases} \dot{y}_{m21} = & y_{m21} - y_{m21}y_{m22} + b_3y_{m21}^2 - b_1y_{m21}^2y_{m23} \\ \dot{y}_{m22} = & -y_{m22} + y_{m21}y_{m22} \\ \dot{y}_{m23} = & -b_2y_{m23} + b_1y_{m21}^2y_{m23}, \end{cases} \quad (4.2)$$

where $(y_{m21}, y_{m22}, y_{m23})^T \in R^3$ is the state vector of the system and b_1, b_2 and b_3 are positive parameters. Also, in (4.10), y_{m11} represents the prey population and y_{m12}, y_{m13} denotes the predator populations. For parameter values $b_1 = 2.9851, b_2 = 3, b_3 = 2$, this second master GLV system displays chaotic behaviour for chosen initial conditions (1.2, 1.2, 1.2) as depicted in Fig.1 (b) and Fig.2 (b).

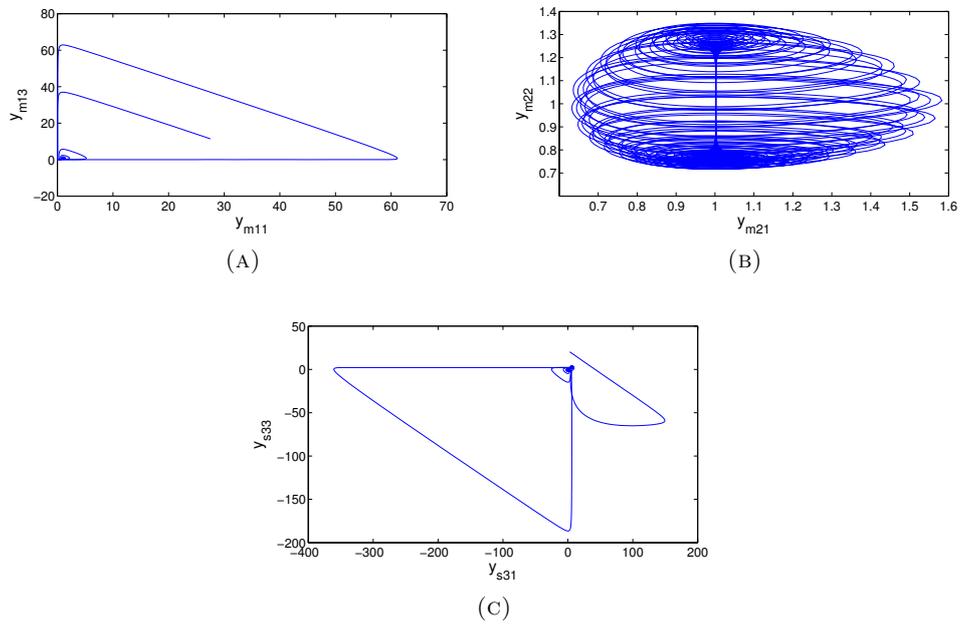


FIGURE 1. Phase graphs for chaotic GLV system in (A) $y_{m11} - y_{m13}$ plane, (B) $y_{m21} - y_{m22}$ plane, (C) $y_{s31} - y_{s33}$ plane

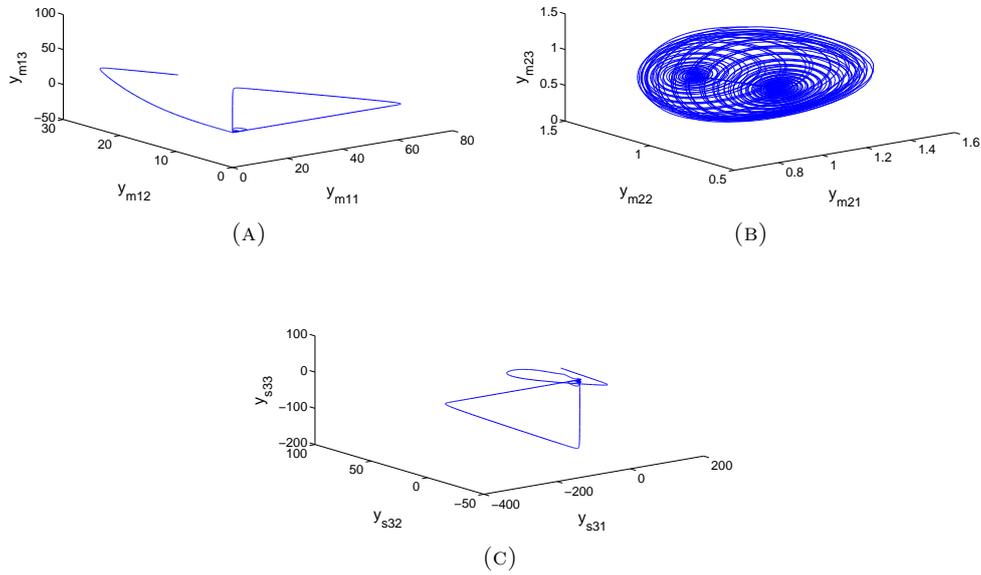


FIGURE 2. Phase graphs for chaotic GLV system in (A) $y_{m11} - y_{m12} - y_{m13}$ space, (B) $y_{m21} - y_{m22} - y_{m23}$ space, (C) $y_{s31} - y_{s32} - y_{s33}$ space

The slave system, prescribed by the identical chaotic GLV system, is described as:

$$\begin{cases} \dot{y}_{s31} = & y_{s31} - y_{s31}y_{s32} + b_3y_{s31}^2 - b_1y_{s31}^2y_{s33} + U_1 \\ \dot{y}_{s32} = & -y_{s32} + y_{s31}y_{s32} + U_2 \\ \dot{y}_{s33} = & -b_2y_{s33} + b_1y_{s31}^2y_{s33} + U_3, \end{cases} \quad (4.3)$$

where $(y_{s11}, y_{s12}, y_{s13})^T \in R^3$ is the state vector of the system and b_1, b_2 and b_3 are positive parameters. Also, in (4.11), y_{m11} represents the prey population and y_{m12}, y_{m13} denotes the predator populations. For parameter values $b_1 = 2.9851, b_2 = 3, b_3 = 2$ and initial conditions $(2, 9, 12.8, 20.3)$, the slave GLV system exhibits chaotic behaviour as shown in Fig.1 (c) and Fig.2 (c). Moreover, the detailed theoretical study and numerical simulation results for (4.9)-(4.11) can be found in [43]. Further, U_1, U_2 and U_3 are active controllers to be determined in such a manner that HPCDS between three identical GLV chaotic systems will be attained.

Next, the HPCDS scheme is proposed to synchronize the states of GLV model. Lyapunov stability theory (LST) based active control approach is employed and required stability criterion is derived.

Defining now the error functions (E_1, E_2, E_3) as

$$\begin{cases} E_1 = & y_{s31} - \eta_1(y_{m21} - y_{m11}) \\ E_2 = & y_{s32} - \eta_2(y_{m22} - y_{m12}) \\ E_3 = & y_{s33} - \eta_3(y_{m23} - y_{m13}). \end{cases} \quad (4.4)$$

The immediate objective in this paper is the designing of the controllers $U_i, (i = 1, 2, 3)$ which ensure that error functions described in (4.12) satisfy $\lim_{t \rightarrow \infty} E_i(t) = 0$ for $(i = 1, 2, 3)$. Then, resulting error dynamics becomes:

$$\begin{cases} \dot{E}_1 = & E_1 - y_{s31}y_{s32} + b_3y_{s31}^2 - b_1y_{s31}^2y_{s33} - \eta_1(-y_{m21}y_{m22} + b_3y_{m21}^2 \\ & - b_1y_{m21}^2y_{m23} + y_{m11}y_{m12} - b_3y_{m11}^2 + b_1y_{m11}^2y_{m13}) + U_1 \\ \dot{E}_2 = & -E_2 + y_{s31}y_{s32} - \eta_2(-y_{m11}y_{m12} + y_{m21}y_{m22}) + U_2 \\ \dot{E}_3 = & -b_2E_3 + b_1y_{s31}^2y_{s33} - \eta_3(b_1y_{m21}^2y_{m23} - b_1y_{m11}^2y_{m13}) + U_3 \end{cases} \quad (4.5)$$

Let us now define the active controllers as:

$$U_1 = \frac{\theta_1}{r_1} - (f_3)_1 - \frac{K_1 E_1}{r_1},$$

where $\theta_1 = (q_1(f_2)_1 - p_1(f_1)_1)$ as described in (3.5). On putting the values of $p_1, q_1, \theta_1, (f_3)_1$ in (4.14) and simplifying, we get

$$\begin{aligned} U_1 = & -E_1 + y_{s31}y_{s33} - b_3y_{s31}^2 + b_1y_{s31}^2y_{s33} + \eta_1(-y_{m21}y_{m22} + b_3y_{m21}^2 \\ & - b_1y_{m21}^2y_{m23} + y_{m11}y_{m12} - b_3y_{m11}^2 + b_1y_{m11}^2y_{m13}) - K_1 E_1. \end{aligned} \quad (4.6)$$

Considering (3.5), we have

$$U_2 = \frac{\theta_2}{r_2} - (f_3)_2 - \frac{K_2 E_2}{r_2}, \quad (4.7)$$

where $\theta_2 = (q_2(f_2)_2 - p_2(f_1)_2)$. By substituting the values of $p_2, q_2, \theta_2, (f_3)_2$ in (4.16) and solving, we find that

$$U_2 = E_2 - y_{s31}y_{s32} + \eta_2(-y_{m11}y_{m12} + y_{m21}y_{m22}) - K_2 E_2. \quad (4.8)$$

Again using (3.5), we obtain

$$U_3 = \frac{\theta_3}{r_3} - (f_3)_3 - \frac{K_3 E_3}{r_3}, \quad (4.9)$$

where $\theta_3 = (q_3(f_2)_3 - p_3(f_1)_3)$. By putting the values of $p_3, q_3, \theta_3, (f_3)_3$ in (4.18) and combining, we have

$$U_3 = b_2 E_3 - b_1 y_{s31}^2 y_{s33} + \eta_3 (b_1 y_{m21}^2 y_{m23} - b_1 y_{m11}^2 y_{m13}) - K_3 E_3, \quad (4.10)$$

where $K_1 > 0, K_2 > 0$ and $K_3 > 0$ are gain constants.

On substituting the active controllers (4.15), (4.17) and (4.19) into error dynamics (4.13), we get

$$\dot{E}_1 = -K_1 E_1, \quad \dot{E}_2 = -K_2 E_2, \quad \dot{E}_3 = -K_3 E_3. \quad (4.11)$$

Lyapunov function is now described as

$$V(E(t)) = \frac{1}{2}[E_1^2 + E_2^2 + E_3^2]. \quad (4.12)$$

It is obvious that Lyapunov function $V(E(t))$ is positive definite in R^3 .

Then, the derivative of Lyapunov function $V(E(t))$ may be expressed as:

$$\dot{V}(E(t)) = E_1 \dot{E}_1 + E_2 \dot{E}_2 + E_3 \dot{E}_3. \quad (4.13)$$

Using (4.20) in (4.22), we obtain

$$\dot{V}(E(t)) = -K_1 E_1^2 - K_2 E_2^2 - K_3 E_3^2 < 0,$$

which depicts clearly that $\dot{V}(E(t))$ is negative definite.

Therefore, by Lyapunov stability theory, we deduce that HPCDS error dynamics is asymptotic stable globally, i.e., the synchronization error $E(t) \rightarrow 0$ asymptotically as $t \rightarrow \infty$ for each initial values $E(0) \in R^3$.

5. Numerical Simulations and Discussions

In this section, we carry out few simulation experiments for illustrating the effectiveness of proposed HPCDS scheme using active control approach. For achieving this, we employ the typical 4th-order Runge-Kutta method in solving systems containing ordinary differential equations. Selected parameters of given GLV model are $b_1 = 2.9851$, $b_2 = 3$ and $b_3 = 2$ which depict that given GLV system behaves chaotically without the controllers. Initial conditions of master systems (4.9)-(4.10) and corresponding slave system (4.11) are $(y_{m11}(0) = 27.5, y_{m12}(0) = 23.1, y_{m13}(0) = 11.4)$, $(y_{m21}(0) = 1.2, y_{m22}(0) = 1.2, y_{m23}(0) = 1.2)$ and $(y_{s31}(0) = 2.9, y_{s32}(0) = 12.8, y_{s33}(0) = 20.3)$ respectively. We attain HPCDS scheme among two master (4.9)-(4.10) and corresponding one slave systems (4.11) by picking up the matrix η with $\eta_1 = 6, \eta_2 = -4, \eta_3 = 3$. Further, the control gains (K_1, K_2, K_3) have been taken as $K_i = 4$ for $i = 1, 2, 3$. Also, Fig. 3(a-c) display the HPCDS synchronized state trajectories of master (4.9)-(4.10) and slave system (4.11) respectively. Moreover, synchronization error functions $(E_1, E_2, E_3) = (160.7, -74.8, 50.9)$ approach zero for t tending to infinity as shown in Fig. 4(a-d). Therefore, the discussed HPCDS approach for master and slave systems has been illustrated computationally .

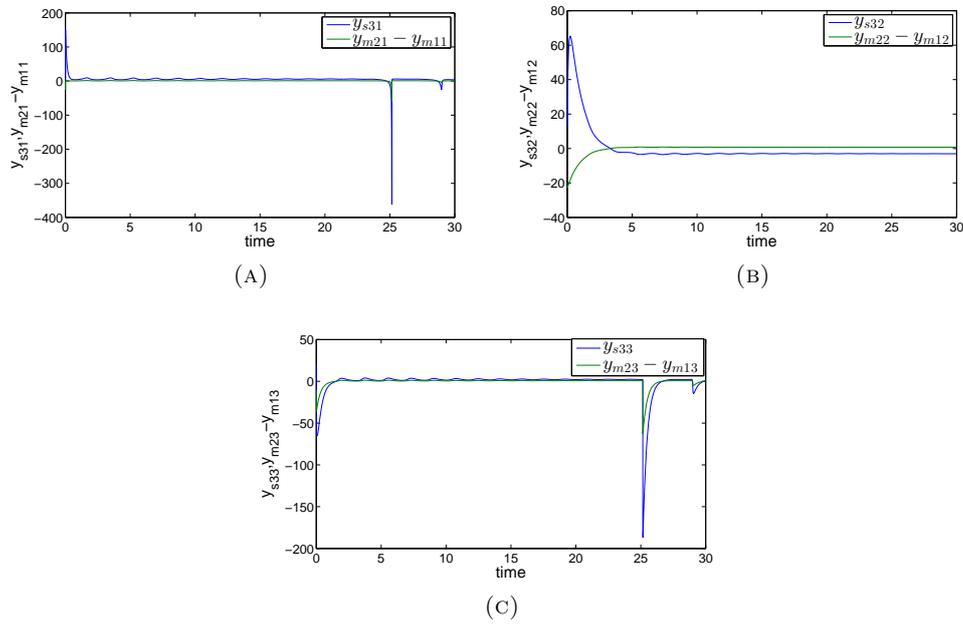


FIGURE 3. HPCDS synchronized trajectories for GLV system (A) between $y_{s31}(t)$ and $y_{m21}(t) - y_{m11}(t)$, (B) between $y_{s32}(t)$ and $y_{m22}(t) - y_{m12}(t)$, (C) between $y_{s33}(t)$ and $y_{m23}(t) - y_{m13}(t)$

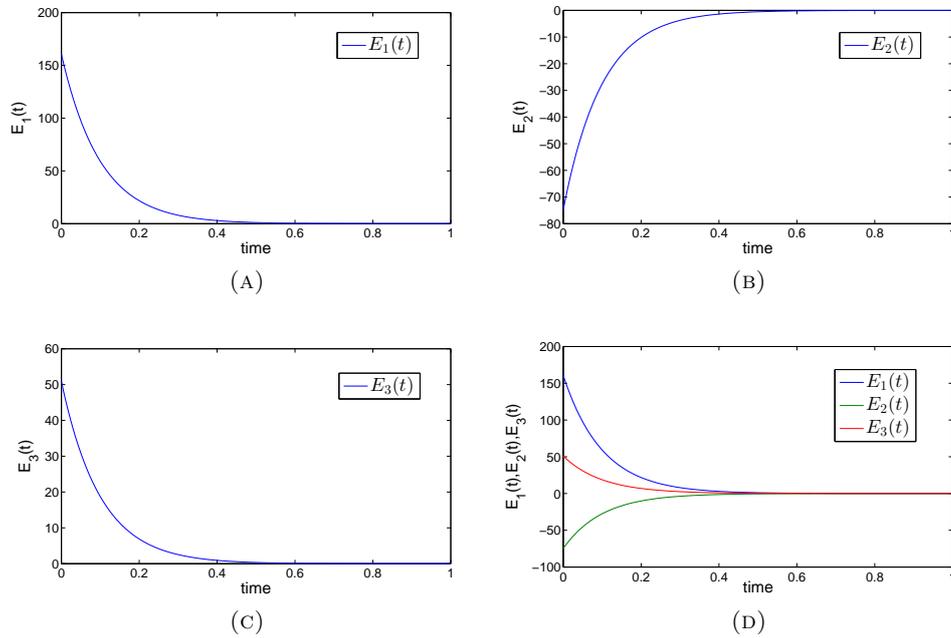


FIGURE 4. Time series for HPCDS error in GLV system (A) (t, E_1), (B) (t, E_2), (C) (t, E_3), (D) (t, E_1, E_2, E_3)

5.1. A Comparative Analysis. Researchers have achieved hybrid synchronization among two chaotic systems via adaptive control method in [54] when performed on the same GLV system with similar parameters. It is noticed that synchronized error converges to zero at $t = 0.8$ (*approx*), whereas in our study, the HPCDS scheme has been attained via active control approach, in which it is noted that the synchronization errors converge to zero at $t = 0.4$ (*approx*) as exhibited in Fig.5. This obviously illustrates that our proposed HPCDS scheme utilizing active control approach is more preferable over previous published work.

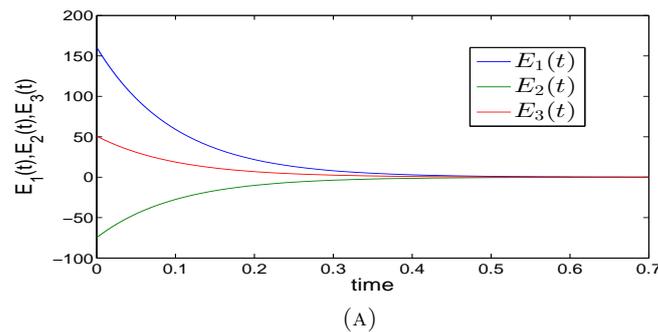


FIGURE 5. HPCDS synchronization error graph

6. Conclusion

In this paper, the suggested HPCDS scheme of identical chaotic GLV systems via active control approach has been explored. By constructing suitable active nonlinear controllers based on classic LST, the considered HPCDS strategy is attained. In addition, MATLAB performed numerical simulations indicate that the designed control functions are efficient in controlling the chaotic regime of GLV systems to desired set points which depicts the effectiveness of our proposed HPCDS technique. Remarkably, the analytical theory and the numerical outcomes both are in complete agreement. During comparison analysis, it is noticed that time taken by synchronization error functions for converging to zero with time tending to infinity is less in comparing to other previously published researches. Moreover, we understand that our considered HPCDS scheme among chaotic GLV system can be generalized by applying other control techniques.

References

- [1] P. Antoniou and A. Pitsillides, A bio-inspired approach for streaming applications in wireless sensor networks based on the lotka-volterra competition model, *Computer Communications*, 33(2010), no. 17, 2039-2047.
- [2] A. Arneodo, P. Couillet, and C. Tresser, Occurrence of strange attractors in three-dimensional volterra equations, *Physics Letters A*, 79(1980), no. 4, 259-263.
- [3] E.-W. Bai and K. E. Lonngren, Synchronization of two lorenz systems using active control, *Chaos, Solitons and Fractals*, 8(1997), no. 1, 51-58.

- [4] K. Bouallegue, A new class of neural networks and its applications. *Neurocomputing*, 249(2017), 28-47.
- [5] M. Chen and Z. Han, Controlling and synchronizing chaotic genesis system via nonlinear feedback control, *Chaos, Solitons and Fractals*, 17(2003), no. 4, 709-716.
- [6] H. Dedieu, M. P. Kennedy, and M. Hasler, Chaos shift keying: modulation and demodulation of a chaotic carrier using self-synchronizing chua's circuits, *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, 40(1993), no. 10, 634-642.
- [7] H. Delavari and M. Mohadeszadeh, Hybrid complex projective synchronization of complex chaotic systems using active control technique with nonlinearity in the control input, *Journal of Control Engineering and Applied Informatics*, 20(2018), no. 1, 67-74.
- [8] Z. Ding and Y. Shen, Projective synchronization of nonidentical fractional-order neural networks based on sliding mode controller. *Neural Networks*, 76 (2016), 97-105.
- [9] E. D. Dongmo, K. S. Ojo, P. Wofo, and A. N. Njah, Difference synchronization of identical and nonidentical chaotic and hyperchaotic systems of different orders using active backstepping design, *Journal of Computational and Nonlinear Dynamics*, 13(2018), no. 5, 051005, 9 pp.
- [10] A. El-Gohary and MT Yassen, Optimal control and synchronization of lotka-volterra model, *Chaos, Solitons and Fractals*, 12(2001), no. 11, 2087-2093.
- [11] K. Fallahi and H. Leung, A chaos secure communication scheme based on multiplication modulation, *Communications in Nonlinear Science and Numerical Simulation*, 15(2010), no. 2, 368-383.
- [12] P. Gatabazi, J. C. Mba, E Pindza, and C. Labuschagne, Grey lotka-volterra models with application to cryptocurrencies adoption, *Chaos, Solitons and Fractals*, 122(2010), 47-57.
- [13] P. Gatabazi, J. C. Mba, and E. Pindza, Modeling cryptocurrencies transaction counts using variable-order fractional grey Lotka-Volterra dynamical system, *Chaos, Solitons and Fractals*, 127(2019), 283-290.
- [14] C. Gavin, A. Pokrovskii, M. Prentice, and V. Sobolev, Dynamics of a lotka-volterra type model with applications to marine phage population dynamics, *J. Phys. Conf. Ser.*, 55(2006), 80-93.
- [15] J. He, J. Cai, and J. Lin, Synchronization of hyperchaotic systems with multiple unknown parameters and its application in secure communication, *Optik*, 127(2016), no. 5, 2502-2508.
- [16] A. Hening and D. H. Nguyen, Stochastic Lotka-Volterra food chains, *J. Math. Biol.*, 77(2018), no. 1, 135-163.
- [17] Y. Y. Hou, H. C. Chen, J. F. Chang, J. J. Yan, and T. L. Liao, Design and implementation of the Sprott chaotic secure digital communication systems, *Appl. Math. Comput.*, 218(2012), no. 24, 11799-11805.
- [18] L. S. Jahanzaib, P. Trikha, H. Chaudhary, Nasreen, and S. M. Haider, Compound synchronization using disturbance observer based adaptive sliding mode control technique, *J. Math. Comput. Sci.*, 10(2020), no.5, 1463-1480.
- [19] A. Khan and H. Chaudhary, Adaptive control and hybrid projective combination synchronization of chaos generated by generalized lotka-volterra biological systems, *BLOOMSBURY INDIA* (2019), 174-184.
- [20] A. Khan and H. Chaudhary, Hybrid projective combination-combination synchronization in non-identical hyperchaotic systems using adaptive control, *Arab. J. Math.* 9(2020), no. 3, 597-611
- [21] T. Khan and H. Chaudhary, Estimation and identifiability of parameters for generalized Lotka-Volterra biological systems using adaptive controlled combination difference anti-synchronization, *Differential Equations and Dynamical Systems*, 28(2020), 515-526.

- [22] W. Kinzel, A. Englert and I. Kanter, On chaos synchronization and secure communication, *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 368(2010), no. 1911, 379-389.
- [23] C. Li and X. Liao, Complete and lag synchronization of hyperchaotic systems using small impulses, *Chaos, Solitons and Fractals*, 22(2004), no. 4, 857-867.
- [24] G. H. Li and S. P. Zhou, Anti-synchronization in different chaotic systems, *Chaos, Solitons and Fractals*, 32(2007), no. 2, 516-520.
- [25] G. H. Li, Modified projective synchronization of chaotic system, *Chaos, Solitons and Fractals*, 32(2007), no. 5, 1786-1790.
- [26] D. Li and X. Zhang, Impulsive synchronization of fractional order chaotic systems with time-delay, *Neurocomputing*, 216(2016), 39-44.
- [27] J. S. Lin, C. F. Huang, T. L. Liao, and J. J. Yan, Design and implementation of digital secure communication based on synchronized chaotic systems, *Digital Signal Processing*, 20(2010), no. 1, 229-237.
- [28] A. J. Lotka, *Elements of physical biology*, *Science Progress in the Twentieth Century (1919-1933)*, 21(1926), no. 82, 341-343.
- [29] J. Ma, L. Mi, P. Zhou, Y. Xu, and T. Hayat, Phase synchronization between two neurons induced by coupling of electromagnetic field, *Applied Mathematics and Computation*, 307(2017), 321-328.
- [30] R. Mainieri and J. Rehacek, Projective synchronization in three-dimensional chaotic systems, *Phys. Rev. Lett.*, 82(1999), no. 15, 3042.
- [31] B. Naderi and H. Kheiri, Exponential synchronization of chaotic system and application in secure communication, *Optik*, 127(2016), no. 5, 2407-2412.
- [32] S. Nag, A mathematical model in the time of covid-19, (2020).
- [33] E. B. M. Ngouonkadi, H. B. Fotsin, and P. L. Fotso, Implementing a memristive van der pol oscillator coupled to a linear oscillator: synchronization and application to secure communication. *Physica Scripta*, 89(2014), no. 3, 035201.
- [34] B. K. Patle, D. R. K. Parhi, A. Jagadeesh, and S. K. Kashyap. Matrix-binary codes based genetic algorithm for path planning of mobile robot, *Computers and Electrical Engineering*, 67(2018), 708-728.
- [35] L. M. Pecora and T. L. Carroll, Synchronization in chaotic systems, *Phys. Rev. Lett.*, 64(1990), no. 8, 821.
- [36] G. Perhar, N. E. Kelly, F. J. Ni, M. J. Simpson, A. J. Simpson, and G. B. Arhonditsis, Using daphnia physiology to drive food web dynamics: A theoretical revisit of Lotka-Volterra models, *Ecological informatics*, 35(2016), 29-42.
- [37] L. Perko, *Differential equations and dynamical systems*, volume 7. Springer Science & Business Media, 2013.
- [38] S. Rasappan and S. Vaidyanathan, Synchronization of hyperchaotic liu system via backstepping control with recursive feedback, In *International Conference on Eco-friendly Computing and Communication Systems*, pages 212-221, Springer, 2012.
- [39] T. Reichenbach, M. Mobilia, and E. Frey, Coexistence versus extinction in the stochastic cyclic Lotka-Volterra model, *Physical Review E*, 74(2006), no. 5, 051907.
- [40] L. Runzi, W. Yinglan, and D. Shucheng, Combination synchronization of three classic chaotic systems using active backstepping design, *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 21(2011), no. 4, 043114.
- [41] L. Runzi and W. Yinglan, Finite-time stochastic combination synchronization of three different chaotic systems and its application in secure communication, *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(2012), no. 2, 023109.
- [42] B. Sahoo and S Poria. The chaos and control of a food chain model supplying additional food to top-predator. *Chaos, Solitons and Fractals*, 58 (2014), 52-64.

- [43] N. Samardzija and L. D. Greller. Explosive route to chaos through a fractal torus in a generalized lotka-volterra model. *Bulletin of Mathematical Biology*, 50(1988), no. 5, 465-491.
- [44] F. M. Scudo, Vito volterra and theoretical ecology, *Theoretical population biology*, 2(1971), no. 1, 1-23.
- [45] D. Shevitz and B. Paden, Lyapunov stability theory of nonsmooth systems, *IEEE Transactions on automatic control*, 39(1994), no. 9, 1910-1914.
- [46] L. S. Dias and A. L. Castillo, Spontaneous symmetry breaking of population: stochastic Lotka-Volterra model for competition among two similar preys and predators. *Mathematical biosciences*, 300(2018), 36-46.
- [47] A. K. Singh, V. K. Yadav, and S. Das, Synchronization between fractional order complex chaotic systems, *International Journal of Dynamics and Control*, 5(3) (2017), 756-770.
- [48] K. S. Sudheer and M. Sabir, Hybrid synchronization of hyperchaotic Lu system, *Pramana*, 73(2009), no. 4, 781.
- [49] X. J. Tong, M. Zhang, Z. Wang, Y. Liu, and J. Ma, An image encryption scheme based on a new hyperchaotic finance system, *Optik*, 126(2015), no. 20, 2445-2452.
- [50] B. H. Tsai, C. J. Chang, and C. H. Chang, Elucidating the consumption and CO₂ emissions of fossil fuels and low-carbon energy in the united states using Lotka-Volterra models. *Energy*, 100(2016), 416-424.
- [51] H. E. Z. Tonnang, L. V. Nedorezov, H. Ochanda, J. Owino, and B. Lohr, Assessing the impact of biological control of *Plutella xylostella* through the application of Lotka-Volterra model, *Ecological Modelling*, 220(2009), no. 1, 60-70.
- [52] S. Vaidyanathan and S. Sampath, Anti-synchronization of four-wing chaotic systems via sliding mode control, *International Journal of Automation and Computing*, 9(2012), no.3, 274-279.
- [53] S. Vaidyanathan, Adaptive biological control of generalized Lotka-Volterra three-species biological system, *International Journal of PharmTech Research*, 8(2015), no. 4 622-631.
- [54] S. Vaidyanathan, Hybrid synchronization of the generalized Lotka-Volterra three-species biological systems via adaptive control, *Int J PharmTech Res*, 9(2016), no. 1, 179-192.
- [55] X. Wu, H. Wang, and H. Lu, Modified generalized projective synchronization of a new fractional-order hyperchaotic system and its application to secure communication, *Nonlinear Analysis: Real World Applications*, 13(2012), no. 3, 1441-1450.
- [56] Z. Wu and X. Fu, Combination synchronization of three different order nonlinear systems using active backstepping design, *Nonlinear Dynamics*, 73(2013), no. 3, 1863-1872.
- [57] X. Wu, Z. Fu, and J. Kurths, A secure communication scheme based generalized function projective synchronization of a new 5D hyperchaotic system, *Physica Scripta*, 90(2015), no. 4, 045210.
- [58] J. Xiong, X. Li, and H. Wang, The survival analysis of a stochastic Lotka-Volterra competition model with a coexistence equilibrium, *Mathematical Biosciences and Engineering*, 16(2019), no. 4, 2717-2737.
- [59] V. K. Yadav, G. Prasad, M. Srivastava, and S. Das, Triple compound synchronization among eight chaotic systems with external disturbances via non-linear approach, *Differential Equations and Dynamical Systems*, (2019), 1-24.
- [60] V. K. Yadav, V. K. Shukla, and S. Das, Difference synchronization among three chaotic systems with exponential term and its chaos control, *Chaos, Solitons and Fractals*, 124(2019), 36-51.
- [61] P. Zhou and W. Zhu, Function projective synchronization for fractional-order chaotic systems, *Nonlinear Analysis: Real World Applications*, 12(2011), no. 2, 811-816.

(Taqseer Khan) DEPARTMENT OF MATHEMATICS JAMIA MILLIA ISLAMIA, NEW DELHI,
INDIA

E-mail address: tkhan4@jmi.ac.in

(Harindri Chaudhary) DEPARTMENT OF MATHEMATICS JAMIA MILLIA ISLAMIA, NEW
DELHI, INDIA

E-mail address: harindri20dbc@gmail.com