

## REFINEMENTS OF CARLEMAN'S INEQUALITY

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**Abstract.** In this paper, we prove that the inequalities

$$\sum_{n=1}^{\infty} \left( \prod_{k=1}^n a_k \right)^{\frac{1}{n}} \leq e \sum_{n=1}^{\infty} \left( 1 - \frac{a}{n+13a} \right) a_n$$

and

$$\sum_{n=1}^{\infty} \left[ \left( 1 + \frac{c}{n+b} \right) \left( \prod_{k=1}^n a_k \right)^{\frac{1}{n}} \right] \leq e \sum_{n=1}^{\infty} a_n$$

hold if  $a_n \geq 0$  ( $n = 1, 2, \dots$ ) with  $0 < \sum_{n=1}^{\infty} a_n < +\infty$ , where  $a = 2.739$ ,  $b = [(4\sqrt{2}-3)e-6]/[(3-2\sqrt{2})e] = 2.6203\dots$  and  $c = (e/2-1)(1+b) = 1.3002\dots$ .

### 1. Introduction

Let  $a_n \geq 0$  ( $n = 1, 2, \dots$ ) with  $0 < \sum_{n=1}^{\infty} a_n < +\infty$ , then the well-known Carleman's inequality [1] is given by

$$\sum_{n=1}^{\infty} \left( \prod_{k=1}^n a_k \right)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} a_n.$$

In the last ninety years, the refinement, improvement, generalization, extension and application for the Carleman's inequality have attracted the attention of many researchers [2-19].

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