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DYNAMIC HARDY-TYPE INTEGRAL INEQUALITIES ON TIME SCALES VIA NABLA CALCULUS

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Abstract. In this paper, we derive dynamic Hardy-type integral inequalities on time scales using nabla calculus. By utilizing the chain rule, Hölder's inequality, and certain properties of multiple integrals within nabla calculus, we have derived novel integral inequalities that unify and extend classical results to both continuous and discrete settings. The results include, as special cases, the time scales $\mathbb{T} = \mathbb{R}$ (continuous case), which yield continuous inequalities, and $\mathbb{T} = \mathbb{Z}$ (discrete case), which yield discrete inequalities. Here, $\mathbb{R}$ and $\mathbb{Z}$ represent the sets of real numbers and integers, respectively. Furthermore, the derived inequalities on time scales are validated through illustrative examples and various applications.

1. Introduction

Hardy-type inequalities find applications in areas such as functional analysis, differential equations and mathematical physics providing fundamental estimates for integrals and sums. The development of time scale calculus, introduced by Hilger in 1988, provided a unified framework for treating differential and difference equations simultaneously. Early extensions of Hardy-type inequalities on time scales were primarily established using delta calculus, focusing on forward dynamic equations. More recently, attention has shifted to the nabla calculus, which handles backward dynamic processes. Extending Hardy inequalities within the nabla framework enriches the theory and broadens their applicability to a wider class of dynamic systems on discrete, continuous, and hybrid domains.

In 1925, Hardy [11] proved the following theorem:

Theorem 1.1. *If $\beta > 1$ and $\omega(\xi) \geq 0$ is an integrable function over the finite interval $(0, \xi)$ for every positive ξ, then*

$$\int_0^\infty \left(\frac{\Omega(\xi)}{\xi}\right)^\beta d\xi \leq \left(\frac{\beta}{\beta-1}\right)^\beta \int_0^\infty \omega^\beta(\xi)\, d\xi, \tag{1.1}$$

where $\Omega(\xi) = \int_0^\xi \omega(\bar{\xi})\, d\bar{\xi}$, for all $\xi > 0$.

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