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UPPER BOUNDS IN LIMIT LAWS FOR MARTINGALES

SANTOSH GHIMIRE

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Abstract. Salem and Zygmund established a limit law for the tail sums of lacunary trigonometric series and this limit law is popularly known as tail law of the iterated logarithm. Here, we propose a similar limit law in the context of martingales and provide an upper bound estimate. Furthermore, we establish an upper bound for a comparable limit law involving stopping times in martingales.

1. Introduction

Law of the Iterated Logarithm, abbreviated LIL, is a famous limit law that focuses on the asymptotic behavior of various contexts in mathematics and statistics. In fact, the LIL can be taken as a refinement of two classical limit laws in probability theory: the central limit theorem and the law of large numbers. In particular, the LIL offers a more precise characterization in scenarios where the sample mean exhibits minor fluctuations, yet the possibility of large deviations persists. The concept of the LIL was first introduced in 1924 by Khintchine [9], in the context of analyzing the magnitude of deviation from the expected value for Bernoulli random variables. Precisely, the first LIL was introduced to obtain the exact rate of convergence in the Borel's theorem and is given by:

Theorem 1.1. *If $N_n(t)$ denote the number of occurrences of the digit 1 in the first $n-$places of the binary expansion of a number $t \in [0, 1)$ and $S_n = 2N_n(t) - n$, then for a.e. t, we have*

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2n \ln \ln n}} = 1.$$

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