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ON A GENERALIZED PRODUCT SUMMABILITY OF FOURIER SERIES

XHEVAT Z. KRASNIQI

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Abstract. In this paper a generalized product of summability is introduced in order to make an advanced study on the special topic of summability. In addition, employing that product we establish a new theorem regarding to summability of Fourier series.

1. Introduction and Known Results

Let $f(t)$ be a periodic function with period 2π and integrable over the interval $(-\pi, \pi)$ in the sense of Lebesgue. Let

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \tag{1.1}$$

be its Fourier series.

For two sequences of real or complex numbers $p = \{p_n\}$ and $q = \{q_n\}$, let

$$\begin{aligned} P_n &= p_0 + p_1 + p_2 + \cdots + p_n = \sum_{v=0}^{n} p_v, \quad \text{for all} \quad n, \\ Q_n &= q_0 + q_1 + q_2 + \cdots + q_n = \sum_{v=0}^{n} q_v, \quad \text{for all} \quad n, \end{aligned}$$

and let the convolution $(p * q)_n$ be defined by

$$R_n := (p * q)_n := \sum_{v=0}^{n} p_v q_{n-v}.$$

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Let $\sum_{n=0}^{\infty} c_n$ be a given infinite series with the sequence of its $n-$th partial sums $\{s_n\}$. We write

$$t_n^{p,q} = \frac{1}{R_n}\sum_{v=0}^{n} p_{n-v}q_v s_v.$$

If $R_n \neq 0$ for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$.

Throughout this paper we shall write $w_n = \mathcal{O}(z_n)$ if $z_n > 0$, $(n = 0, 1, 2, \dots)$, and if there exists a constant $K > 0$ such that $\frac{w_n}{z_n} \leq K$, and we write $w_n = o(z_n)$ if $\frac{w_n}{z_n} \to 0$ as $n \to \infty$.

If $t_n^{p,q} \to s$ as $n \to \infty$, then the series $\sum_{n=0}^{\infty} c_n$ or $\{s_n\}$ is summable to s by generalized Nörlund method (see Borwein [2]) and is denoted by

$$s_n \longmapsto s(N, p, q).$$

The necessary and sufficient conditions for (N, p, q) method to be regular are

$$\sum_{v=0}^{n} |p_{n-v}q_v| = \mathcal{O}(|R_n|)$$

and $p_{n-v} = o(|R_n|)$, as $n \to \infty$ for every fixed $k \geq 0$, for which $q_v \neq 0$.

If

$$E_n^r = \frac{1}{(1+r)^n}\sum_{v=0}^{n}\binom{n}{v} r^{n-v}s_v \to s, \quad (r > 0)$$

then the series $\sum_{n=0}^{\infty} c_n$ is said to be (E, r) summable to s (see Hardy [3]).

Let $A := (a_{n,v})_{n,v\in\mathbb{N}\cup\{0\}}$ be a positive lower triangular matrix, i.e. $a_{n,v} > 0$ for $0 \leq v \leq n$, and $a_{n,v} = 0$ for $v > n$, such that $\sum_{v=0}^{n} a_{n,v} = 1$. Further, let us consider the transformation

$$T_{n,A(E,r)}(S) = \sum_{v=0}^{n}\frac{a_{n,v}}{(1+r)^v}\sum_{j=0}^{v}\binom{v}{j} r^{v-j}s_j$$

of the sequence $S = \{s_j\}$.

If $T_{n,A(E,r)}(S) \to s$, as $n \to \infty$, then we say that the series $\sum_{n=0}^{\infty} c_n$ or the sequence $S = \{s_n\}$ is summable to s by $(A)(E, r)$ summability.

In the special case, if we take $a_{n,v} = \frac{p_{n-v}q_v}{R_n}$ and $r = 1$, then $(A)(E, r)$ summability exactly coincides to $(N, p, q)(E, 1)$ summability employed in [5].

Despite of a good amount of work that is known regarding to ordinary summability of Fourier series, recently the authors of the paper [5] established a new theorem on $(N, p, q)(E, 1)$ summability of Fourier series.

Following, before we recall their result, in which several typos has been occured, we shall write some notations:

$$\phi_x(t) = \frac{1}{2}[f(x+t) + f(x-t) - 2f(x)], \quad \Phi_x(t) = \int_0^t |\phi_x(u)|du,$$

$$\tau = \left[\frac{1}{t}\right], \quad R\left(\frac{1}{t}\right) = R_\tau, \quad R_n = R(n),$$

and

$$K_n(t) = \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{n-v} q_v \frac{\cos^k(\frac{t}{2}) \cos(k+1)(\frac{t}{2})}{\sin(\frac{t}{2})},$$

where $t \in (-\pi, \pi)$ and x is a fixed value of t.

Theorem 1.1 ([5]). *Let $\{p_n\}$ and $\{q_n\}$ be positive monotonic non-increasing sequences of real numbers such that*

$$R_n = \sum_{v=0}^{n} p_k q_{n-v} \to \infty, \quad \text{as} \quad n \to \infty.$$

Let $\beta(t)$ be a positive and non-decreasing function of t. If

$$\Phi_x(t) = \int_0^t |\phi(u)| du = o\left(\frac{t}{\beta(1/t)}\right), \quad \text{as} \quad t \to +0,$$

$$\beta(n) \to \infty, \quad \text{as} \quad n \to \infty,$$

then a sufficient condition that the Fourier series (1.1) to be summable $(N,p,q)(E,1)$ to $f(t)$ at the point $t = x$ is

$$\int_1^n \frac{R(u)}{u\beta(u)} du = \mathcal{O}(R_n), \quad \text{as} \quad n \to \infty.$$

The main aim of this paper is to find the sufficient conditions under which the Fourier series (1.1) of the function $f(t)$ is $(A)(E,r)$ summable at the point $t = x$; $t \in (-\pi, \pi)$.

As is showed in [4] the infinite series

$$1 - 4\sum_{n=1}^{\infty} (-3)^{n-1}$$

is not $(E,1)$ summable nor $(C,1)$ summable. However, it is proved that the above series is $(C,1)(E,1)$ summable. Therefore the product summability $(C,1)(E,1)$ is more powerful than the individual methods $(C,1)$ and $(E,1)$. Thus, $(C,1)(E,1)$ mean (as particular case of $(A)(E,r)$ mean) gives some more general results for a wider class of Fourier series than the individual means $(C,1)$ and $(E,1)$.

To achieve the aim of this paper first we need to prove some helpful statements given in next section.

2. Auxiliary Lemmas

At first, we denote

$$A(1/t) = \sum_{v=0}^{\tau} a_{n,v}, \quad A(n) = 1,$$

and

$$G_n(t) = \sum_{v=0}^{n} \frac{a_{n,v}}{(1+r)^v} \sum_{j=0}^{v} \binom{v}{j} r^{v-j} \frac{\sin(2j+1)\frac{t}{2}}{\sin\left(\frac{t}{2}\right)} = \Im \sum_{v=0}^{n} a_{n,v} \frac{e^{\frac{it}{2}}(r+e^{it})^v}{(1+r)^v \sin(t/2)},$$

where $\Im(z)$ denotes the imaginary part of z.

Lemma 2.1 (Abel's Lemma, [1], page 18)**.** *Let $\{a_k\}_{k=0}^{\infty}$, $\{b_k\}_{k=0}^{\infty}$ two real sequences and $V_k=\sum_{i=0}^{n} b_i$. If $a_k\geq 0$, $a_k\downarrow$, $|V_k|\leq M$ for $m\leq k\leq n$, and $M>0$, then*

$$\left|\sum_{k=m}^{n} a_k b_k\right|\leq 2a_m M.$$

Lemma 2.2. *Let $0\leq t\leq 1/n$. Then the following estimation*

$$|G_n(t)|=\mathcal{O}\left(n\right),$$

holds true.

Proof. Obviously, we have

$$\begin{aligned}
|G_n(t)| &= \left|\sum_{v=0}^{n}\frac{a_{n,v}}{(1+r)^v}\sum_{j=0}^{v}\binom{v}{j}r^{v-j}\frac{\sin(2j+1)\frac{t}{2}}{\sin\left(\frac{t}{2}\right)}\right|\\
&\leq \sum_{v=0}^{n}\frac{a_{n,v}}{(1+r)^v}\sum_{j=0}^{v}\binom{v}{j}r^{v-j}\frac{(2j+1)\left|\sin\frac{t}{2}\right|}{\left|\sin\frac{t}{2}\right|}\\
&\leq \sum_{v=0}^{n}\frac{a_{n,v}}{(1+r)^v}(2v+1)\sum_{j=0}^{v}\binom{v}{j}r^{v-j}\\
&\leq \sum_{v=0}^{n}2\left(v+1\right)a_{n,v}\\
&= \mathcal{O}\left(n\right).
\end{aligned}$$

The proof is completed. □

Lemma 2.3. *Let $\{a_{n,k}\}$ be non-negative and non-increasing sequence with respect to k. Then for $0\leq i\leq j<\infty$, $(1/n)\leq t\leq \pi$, and any n, the following estimation*

$$\left|G_n^{(i,j)}(t)\right|:=\left|\sum_{v=i}^{j}\frac{a_{n,v}}{(1+r)^v}\sum_{j=0}^{v}\binom{v}{j}r^{v-j}\frac{\sin(2j+1)\frac{t}{2}}{\sin\left(\frac{t}{2}\right)}\right|=\mathcal{O}_r\left(\frac{A_{n,\tau}}{t}\right),$$

holds true.

Proof. We have

$$\begin{aligned}
\left|G_n^{(i,j)}(t)\right| &\leq \left|\sum_{v=i}^{\tau-1}\frac{a_{n,v}}{(1+r)^v}\sum_{j=0}^{v}\binom{v}{j}r^{v-j}\frac{\sin(2j+1)\frac{t}{2}}{\sin\left(\frac{t}{2}\right)}\right|\\
&\quad+\left|\sum_{v=\tau}^{j}\frac{a_{n,v}}{(1+r)^v}\sum_{j=0}^{v}\binom{v}{j}r^{v-j}\frac{\sin(2j+1)\frac{t}{2}}{\sin\left(\frac{t}{2}\right)}\right|:=|G_{n,1}(t)|+|G_{n,2}(t)|.
\end{aligned}$$

For the first term, appearing in the right hand side of the above inequality, we have

$$\begin{aligned}
|G_{n,1}(t)| &\leq \left(\frac{\pi}{t}\right)\sum_{v=i}^{\tau-1}\frac{a_{n,v}}{(1+r)^v}\sum_{j=0}^{v}\binom{v}{j}r^{v-j}\\
&\leq \left(\frac{\pi}{t}\right)\sum_{v=i}^{\tau-1}a_{n,v} = \mathcal{O}\left(\frac{1}{t}\sum_{v=i}^{\tau-1}a_{\tau,v}\right) = \mathcal{O}\left(\frac{A_{n,\tau}}{t}\right),
\end{aligned}$$

where we have used the basic inequality $\sin\theta \geq (2/\pi)\theta$ for $0 \leq \theta \leq \pi/2$.

Now we estimate the second term. Using Abel's lemma, we get

$$\begin{aligned}
G_{n,2}(t) &\leq \left|\sum_{v=\tau}^{j}\frac{a_{n,v}}{(1+r)^v}\sum_{j=0}^{v}\binom{v}{j}r^{v-j}\frac{\sin(2j+1)\frac{t}{2}}{\sin\left(\frac{t}{2}\right)}\right|\\
&= \left|\Im\sum_{v=\tau}^{j}\frac{a_{n,v}}{(1+r)^v}\sum_{j=0}^{v}\binom{v}{j}r^{v-j}\frac{e^{i(2j+1)\frac{t}{2}}}{\sin\left(\frac{t}{2}\right)}\right|\\
&\leq \frac{\pi}{t}\left|\sum_{v=\tau}^{j}\frac{a_{n,v}}{(1+r)^v}\sum_{j=0}^{v}\binom{v}{j}r^{v-j}\left(e^{it}\right)^j\right|\\
&\leq \frac{\pi}{t}\left|\sum_{v=\tau}^{j}a_{n,v}\left(\frac{r+e^{it}}{1+r}\right)^v\right|\\
&\leq \frac{2\pi a_{n,\tau}}{t}\max_{\tau\leq v\leq j}\left|\sum_{m=0}^{v}\left(\frac{r+e^{it}}{1+r}\right)^m\right|\\
&= \frac{2\pi a_{n,\tau}}{t}\max_{\tau\leq v\leq j}\left|\frac{1-\left(\frac{r+e^{it}}{1+r}\right)^{v+1}}{\frac{1-e^{it}}{1+r}}\right|\\
&\leq \frac{4\pi a_{n,\tau}}{t}\frac{r+1}{\sin\frac{t}{2}} \leq \frac{4(r+1)\pi^2}{t}\frac{a_{n,\tau}}{t}\\
&\leq \frac{4(r+1)\pi^2}{t}\left(\frac{a_{n,\tau}}{A_{n,\tau}}\right)A_{n,\tau}\frac{1}{t}\\
&\leq \frac{4(r+1)\pi^2}{t}\frac{A_{n,\tau}}{(\tau+1)\,t} = \mathcal{O}_r\left(\frac{A_{n,\tau}}{t}\right),
\end{aligned}$$

since $(a_{n,\tau})/A_{n,\tau} \leq \frac{1}{1+\tau}$ and $\frac{1}{1+\tau} < t$. With this the proof is completed. □

3. New Result

We establish the following theorem.

Theorem 3.1. *Let $A := (a_{n,v})$ be a positive lower triangular matrix such that*

$$\sum_{v=0}^{n}a_{n,v} = 1, \quad (n = 0, 1, 2, \dots)$$

and its entries form a non-increasing sequence.

Let $\beta(t)$ be a positive and non-decreasing function of t. If

$$\Phi_x(t)=\int_0^t|\phi_x(u)|du=o\left(\frac{t}{\beta(1/t)}\right),\quad as\quad t\to+0,$$

$$\beta(n)\to\infty,\quad as\quad n\to\infty,$$

then a sufficient condition that the Fourier series (1.1) to be summable $(A)(E,r)$ to $f(t)$ at the point $t=x$ is

$$\int_1^n\frac{R(u)}{u\beta(u)}du=\mathcal{O}\left(1\right),\quad as\quad n\to\infty.$$

Proof. Following A. Zygmund (see [6], page 88, equality (5.6)) the j-th partial sums of the series (1.1) at the point $t=x$ is

$$s_j(x)=f(x)+\frac{1}{\pi}\int_0^\pi\phi_x(t)\frac{\sin(j+1/2)t}{\sin(t/2)}dt.$$

Therefore, we can write for (E,r) means of the series (1.1) at the point $t=x$, as follows

$$\begin{aligned}
E_v^r(x) &= \frac{1}{(1+r)^v}\sum_{j=0}^{v}\binom{v}{j}r^{v-j}s_j(x)\\
&= f(x)+\frac{1}{\pi(1+r)^v}\int_0^\pi\frac{\phi_x(t)}{\sin(t/2)}\left[\sum_{j=0}^{v}\binom{v}{j}r^{v-j}\sin\left(j+\frac{1}{2}\right)t\right]dt\\
&= f(x)+\frac{1}{\pi(1+r)^v}\int_0^\pi\frac{\phi_x(t)}{\sin(t/2)}\Im\left[\sum_{j=0}^{v}\binom{v}{j}r^{v-j}e^{i\left(j+\frac{1}{2}\right)t}\right]dt\\
&= f(x)+\frac{1}{\pi(1+r)^v}\int_0^\pi\frac{\phi_x(t)}{\sin(t/2)}\Im\left[e^{\frac{it}{2}}(r+e^{it})^v\right]dt\\
&= f(x)+\frac{1}{\pi}\int_0^\pi\phi_x(t)\frac{\Im\left[e^{\frac{it}{2}}(r+e^{it})^v\right]}{(1+r)^v\sin(t/2)}dt. \qquad (3.2)
\end{aligned}$$

Thus, using (3.2) transformation $T_{n,A(E,r)}(x)$ of the partial sums $s_j(x)$ takes the following form

$$T_{n,A(E,r)}(x)=\sum_{v=0}^{n}\frac{a_{n,v}}{(1+r)^v}\sum_{j=0}^{v}\binom{v}{j}r^{v-j}s_j(x)$$

$$\begin{aligned}
&= f(x)+\frac{1}{\pi}\int_0^\pi\sum_{v=0}^{n}a_{n,v}\phi_x(t)\frac{\Im\left[e^{\frac{it}{2}}(r+e^{it})^v\right]}{(1+r)^v\sin(t/2)}dt\\
&= f(x)+\frac{1}{\pi}\int_0^\pi\phi_x(t)G_n(t)dt
\end{aligned}$$

and for $0 < \delta < 1$ we can write

$$T_{n,A(E,r)}(x) - f(x) = \frac{1}{\pi}\left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi}\right) G_n(t)\phi_x(t)dt := \sum_{m=1}^{3} r_m(x). \tag{3.3}$$

Using Lemma 2.2 and conditions of the theorem, we have

$$\begin{aligned} |r_1(x)| &\leq \frac{1}{\pi}\int_0^{\frac{1}{n}} |G_n(t)||\phi_x(t)|dt \\ &= \mathcal{O}(n)\int_0^{\frac{1}{n}} |\phi_x(t)|dt = \mathcal{O}(n)\, o\left(\frac{1}{n\beta(n)}\right) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.4}$$

Now, using Lemma 2.3, we obtain

$$\begin{aligned} |r_2(x)| &\leq \frac{1}{\pi}\int_{\frac{1}{n}}^{\delta} |G_n(t)||\phi_x(t)|dt \\ &= \mathcal{O}\left(\int_{\frac{1}{n}}^{\delta} \frac{A(1/t)}{t}|\phi_x(t)|dt\right) \\ &= \mathcal{O}\left(\int_{\frac{1}{n}}^{\delta} \frac{A(1/t)}{t}\Phi_x'(t)dt\right) \\ &= \mathcal{O}\left(\left(\frac{A(1/t)}{t}\Phi_x(t)\right)\Bigg|_{\frac{1}{n}}^{\delta} - \int_{\frac{1}{n}}^{\delta} \Phi_x(t)d\left(\frac{A(1/t)}{t}\right)\right) \\ &= \mathcal{O}\left(o\left(\frac{A(1/t)}{\alpha(1/t)}\right)\Bigg|_{\frac{1}{n}}^{\delta} - \int_{\frac{1}{n}}^{\delta} \Phi_x(t)d\left(\frac{A(1/t)}{t}\right)\right) \\ &= o(1) + o\left(\frac{A(n)}{\alpha(n)}\right) + \mathcal{O}\left(\int_{\frac{1}{n}}^{\delta} o\left(\frac{t}{\alpha(1/t)}\right) d\left(\frac{A(1/t)\alpha(1/t)}{t\alpha(1/t)}\right)\right) \\ &= o(1) + \mathcal{O}\left(\int_{\frac{1}{n}}^{\delta} o\left(\frac{t}{\alpha(1/t)}\right)\left(\frac{A(1/t)}{t\alpha(1/t)}d\alpha(1/t) + \alpha(1/t)d\left(\frac{A(1/t)}{t\alpha(1/t)}\right)\right)\right) \\ &= o(1) + o(1)\left(\int_{\frac{1}{n}}^{\delta} \frac{d\alpha(1/t)}{[\alpha(1/t)]^2} + \int_{\frac{1}{n}}^{\delta} td\left(\frac{A(1/t)}{t\alpha(1/t)}\right)\right) \\ &= o(1) + o(1)\left(-\frac{1}{\alpha(1/t)}\Bigg|_{\frac{1}{n}}^{\delta} + \frac{tA(1/t)}{t\alpha(1/t)}\Bigg|_{\frac{1}{n}}^{\delta} - \int_{\frac{1}{n}}^{1} \frac{A(1/t)}{t\alpha(1/t)}dt\right) \\ &= o(1) + o(1)\left(\int_1^n \frac{A(u)}{u\alpha(u)}du\right) \\ &= o(1) + o(1)\mathcal{O}(1) = o(1). \end{aligned} \tag{3.5}$$

Finally, using the well-known Riemann-Lebesgue theorem we immediately obtain

$$|r_3(x)| \leq \frac{1}{\pi}\int_{\delta}^{\pi} |G_n(t)||\phi_x(t)|dt = o(1) \quad \text{as} \quad n \to \infty. \tag{3.6}$$

Now, (3.3) with (3.4), (3.5), and (3.6) complete the proof of theorem. □

In the following cases can be specialized the product mean $(A)(E,r)$:

(i) If $a_{n,v} = \frac{p_{n-v}q_v}{R_n}$ then $(A)(E,r)$ mean reduces to $(N,p,q)(E,r)$ mean.
(ii) If $a_{n,v} = \frac{p_{n-v}}{P_n}$ then $(A)(E,r)$ mean reduces to $(N,p)(E,r)$ mean.
(iii) If $a_{n,v} = \frac{q_v}{Q_n}$ then $(A)(E,r)$ mean reduces to $(\overline{N},q)(E,r)$ mean.
(iv) If $a_{n,v} = \frac{1}{n+1}$ and $r = 1$ then $(A)(E,r)$ mean reduces to $(N,1/(n+1))(E,1)$ mean.
(v) If $a_{n,v} = \binom{n+\alpha-1}{\alpha-1}$, $\alpha > 0$ and $r = 1$, then $(A)(E,r)$ mean reduces to $(C,\alpha)(E,1)$ mean.

Now we present here only two of corollaries derived from the main result.

Corollary 3.2. Let all the conditions of Theorem 3.1 be satisfied. Then a sufficient condition that the Fourier series (1.1) to be summable $(N,p,q)(E,r)$ to $f(t)$ at the point $t = x$ is

$$\int_1^n \frac{R(u)}{u\beta(u)}du = O\left(1\right), \quad \text{as} \quad n \to \infty.$$

Corollary 3.3. Let all the conditions of Theorem 3.1 be satisfied. Then a sufficient condition that the Fourier series (1.1) to be summable $(C,\alpha)(E,1)$ to $f(t)$ at the point $t = x$ is

$$\int_1^n \frac{R(u)}{u\beta(u)}du = O\left(1\right), \quad \text{as} \quad n \to \infty.$$

Remark 3.4. Note that if we take $r = 1$ in Corollary 4.1. we immediately obtain Theorem 1.1. Therefore Theorem 3.1 is a proper generalization of the result presented in [5].

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State University of Prishtina "Hasan Prishtina", Faculty of Education, Department of Mathematics and Informatics, Avenue "Mother Theresa", 10000 Prishtina, Kosova

E-mail address: xhevat.krasniqi@uni-pr.edu