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HILBERT TRANSFORM OF IRREGULAR WAVE PACKET SYSTEM FOR $L^2(\mathbb{R})$

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Abstract. Let $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ be an irregular wave packet system and let H be the Hilbert transform on $L^2(\mathbb{R})$. In this paper we give necessary and sufficient conditions for the system $\{D_{a_j}T_{bk}E_{c_m}H\psi\}_{j,k,m\in\mathbb{Z}}$ to be a frame for $L^2(\mathbb{R})$.

1. Introduction and Preliminaries

A sequence $\{f_k\}$ in a separable Hilbert space $\mathcal{H}$ with inner product $\langle .,.\rangle$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$, if there exists finite positive constants A and B such that

$$A\|f\|^2 \leq \|\{\langle f, f_k\rangle\}\|^2_{\ell^2} \leq B\|f\|^2, \text{ for all } f \in \mathcal{H}. \tag{1.1}$$

The positive constants A and B are called *lower* and *upper* bounds of the frame, respectively. The inequality (1.1) is called the *frame inequality* of the frame. If upper inequality in (1.1) holds, then $\{f_k\}$ is called a *Bessel sequence*. The operator $T : \ell^2 \to \mathcal{H}$ given by

$$T(\{c_k\}) = \sum_{k=1}^{\infty} c_k f_k,\ \{c_k\} \in \ell^2,$$

is called the *synthesis operator* or the *pre-frame operator* of the frame. The adjoint operator $T^* : \mathcal{H} \to \ell^2$ of T is called the *analysis operator* and is given by

$$T^* : f \to \{\langle f, f_k\rangle\},\ f \in \mathcal{H}.$$

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Composing T and T^* we obtain the *frame operator* $S = TT^* : \mathcal{H} \to \mathcal{H}$ which is given by

$$S(f) = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \ f \in \mathcal{H}.$$

The frame operator S is a positive continuous invertible linear operator from $\mathcal{H}$ onto $\mathcal{H}$. Note that if $\{f_k\}$ is a Bessel sequence, then the frame operator S is well defined operator on $\mathcal{H}$.

Duffin and Schaeffer [11] introduced frames for Hilbert spaces, while addressing some deep problems in nonharmonic Fourier series. Later, in 1986, Daubechies, Grossmann and Meyer [10] found new applications to wavelet and Gabor transforms in which frames played an important role. The basic theory of frames and wavelets can be found in [6, 12, 13, 14, 20].

Cordoba and Fefferman [9], introduced and studied the wave packet systems by applying certain collections of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. Later, Labate et al. [16] adopted the same expression to describe any collection of functions which are obtained by applying the same operations to a finite family of functions. More precisely, Gabor systems, wavelet systems and the Fourier transform of wavelet systems are special cases of wave packet systems. Wave packet systems have recently been successfully applied to some problems in harmonic analysis and operator theory. Recent development on wave packet system can be found in [7, 8, 18].

Recently, the Hilbert transform of Gabor and Wilson system studied by Jarrah and Panwar in [15]. In this paper we consider an irregular wave packet system of the form $\{D_{a_j} T_{bk} E_{c_m} \psi\}_{j,k,m \in \mathbb{Z}}$, where $\psi \in L^2(\mathbb{R})$, $\{a_j\}_{j \in \mathbb{Z}}, \subset \mathbb{R}^+$, $\{c_m\}_{m \in \mathbb{Z}} \subset \mathbb{R}$ and $b > 0$. In this paper we give necessary and sufficient conditions for the system $\{D_{a_j} T_{bk} E_{c_m} H\psi\}_{j,k,m \in \mathbb{Z}}$ to be a frame for $L^2(\mathbb{R})$, where H is the Hilbert transform on $L^2(\mathbb{R})$.

In the rest part of this section, we recall basic notations and definitions which will be used throughout the paper . For $1 \leq p < \infty$, let $L^p(\mathbb{R})$ denote the Banach space of complex-valued Lebesgue integrable functions f on $\mathbb{R}$ satisfying

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(t)|^p dt \right)^{\frac{1}{p}} < \infty.$$

For $p = 2$, an inner product on $L^p(\mathbb{R})$ is given by

$$\langle f, g \rangle = \int_{\mathbb{R}} f \ \overline{g} \ dt,$$

where $\overline{g}$ denote the complex conjugate of g.

The Gabor system is based on the following bounded linear unitary operators on $L^2(\mathbb{R})$ which are given by :
Translation $\equiv T_a : f(t) \to f(t-a)$, $a \in \mathbb{R}, \ f \in L^2(\mathbb{R})$.
Modulation $\equiv E_b : f(t) \to e^{2\pi ibt} f(t)$, $b \in \mathbb{R}, \ f \in L^2(\mathbb{R})$.
Dilation $\equiv D_a : f(t) \to \frac{1}{\sqrt{|a|}} f(\frac{t}{a}), a \in \mathbb{R} \ a \neq 0, \ f \in L^2(\mathbb{R})$.

One can easily verify that for $g \in L^2(\mathbb{R})$, $E_{mb} T_{na} g(t) = e^{2\pi imbt} g(t - na)$. If $a, b > 0$ and $(g, a, b) = \{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, then we call (g, a, b) a *Gabor* *frame* or a *Weyl-Heisenberg frame* for $L^2(\mathbb{R})$. Casazza in [3] introduced and studied the irregular Weyl-Heisenberg frames for $L^2(\mathbb{R})$. A survey of the irregular Weyl-Heisenberg frames can be found in [17]. Let $(x_m, y_n) \in \mathbb{R}^2$ and let $g \in L^2(\mathbb{R})$. A system of the form $\{E_{x_m} T_{y_n} g(\bullet)\}_{m,n\in\mathbb{Z}}$ is called an *irregular Weyl-Heisenberg frame* (in short *IWH frame*) for $L^2(\mathbb{R})$, if $\{E_{x_m} T_{y_n} g(\bullet)\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. Let $\psi \in L^2(\mathbb{R})$,$\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+$, $b \neq 0$ and $\{c_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}$. A system of the form $\{D_{a_j} T_{bk} E_{c_m} \psi\}_{j,k,m\in\mathbb{Z}}$ is called an irregular Weyl-Heisenberg wave packet system (in short *IWHW packet system*).

Let $\psi, \widetilde{\psi} \in L^2(\mathbb{R})$. Then, $\{D_{a_j} T_{bk} E_{c_m} \psi\}_{j,k,m\in\mathbb{Z}}$ and $\{D_{a_j} T_{bk} E_{c_m} \widetilde{\psi}\}_{j,k,m\in\mathbb{Z}}$ form dual pair if the series

$$P(f,g) = \sum_{j,k,m\in\mathbb{Z}} \langle f, D_{a_j} T_{bk} E_{c_m} \psi\rangle \langle D_{a_j} T_{bk} E_{c_m} \widetilde{\psi}, g\rangle \tag{1.2}$$

converges in the sense of the convergence of the sequence

$$\lim_{m_1,m_2\to\infty} \lim_{j_1,j_2\to\infty} \lim_{k_1,k_2\to\infty} \sum_{m=-m_1}^{m_2} \sum_{j=-j_1}^{j_2} \sum_{k=-k_1}^{k_2} \langle f, D_{a_j} T_{bk} E_{c_m} \psi\rangle \langle D_{a_j} T_{bk} E_{c_m} \widetilde{\psi}, g\rangle.$$

and satisfies

$$P(f,g) = \langle f, g\rangle,\ f, g \in L^2(\mathbb{R}). \tag{1.3}$$

Lemma 1.1. [8] *Let $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+, b > 0, \{c_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}$ and $\psi, \widetilde{\psi} \in L^2(\mathbb{R})$. Assume that the wave packet systems $\{D_{a_j} T_{bk} E_{c_m} \psi\}_{j,k,m\in\mathbb{Z}}$ and $\{D_{a_j} T_{bk} E_{c_m} \widetilde{\psi}\}_{j,k,m\in\mathbb{Z}}$ are Bessel sequences and*

$$L(f) = \frac{1}{b} \sum_{j,m,n\in\mathbb{Z}} \int_{supp\widehat{f}} |\widehat{f}(\xi + \frac{a_j}{b} n)|^2 |\widehat{\psi}(a_j^{-1}\xi - c_m)|^2 d\xi < \infty,$$

and

$$L'(f) = \frac{1}{b} \sum_{j,m,n\in\mathbb{Z}} \int_{supp\widehat{f}} |\widehat{f}(\xi + \frac{a_j}{b} n)|^2 |\widehat{\widetilde{\psi}}(a_j^{-1}\xi - c_m)|^2 d\xi < \infty,$$

for all $f \in \mathcal{D}$, where $\mathcal{D} = \{f \in L^2(\mathbb{R}) : \widehat{f} \in L^\infty(\mathbb{R})$ and $supp\widehat{f}$ is compact in $\mathbb{R}\}$. Then, $\{D_{a_j} T_{bk} E_{c_m} \psi\}_{j,k,m\in\mathbb{Z}}$ and $\{D_{a_j} T_{bk} E_{c_m} \widetilde{\psi}\}_{j,k,m\in\mathbb{Z}}$ form dual pair if and only if

$$\frac{1}{b} \sum_{m\in\mathbb{Z}} \sum_{j\in\tau_\alpha} \overline{\widehat{\psi}(a_j^{-1}\xi - c_m)} \widehat{\widetilde{\psi}}(a_j^{-1}(\xi + \alpha) - c_m) = \delta_{\alpha,0} \tag{1.4}$$

for a.e. $\xi \in \mathbb{R}$, $\alpha \in \Lambda$, where $\Lambda = \bigcup_{j\in\mathbb{Z}} a_j b^{-1}\mathbb{Z}$, $\tau_\alpha = \{j \in \mathbb{Z} : a_j^{-1} b\alpha \in \mathbb{Z}\}$ and δ is the kronecker delta.

Definition 1.2. [4] Let $1 < p < \infty$. The Hilbert transform H for a complex-valued function $f \in L^p(\mathbb{R})$ is defined as

$$H(f(t)) = \frac{1}{\pi} \lim_{\epsilon\to 0} \int_{|s|>\epsilon} \frac{f(t-s)}{s} ds.$$

Definition 1.3. [2] A function $f \in L^2(\mathbb{R})$ is *analytic* if its imaginary part is the Hilbert transform of its real part. Its real and imaginary parts are said to form a Hilbert pair.

Recall that the Fourier transform of a function $f \in L^2(\mathbb{R})$ is defined as

$$\widehat{f}(u) = \int_R f(x)e^{-2\pi iux}dx.$$

Theorem 1.4. [1] *Let f,g be a complex functions in $L^2(\mathbb{R})$ of real variable x. Then, the following holds.*

(*i*) *If the Fourier transform $\hat{f}(u)$, of f(x) vanishes for $|u| > a$ and the Fourier transform, $\hat{g}(u)$ of g(x) vanishes for $|u| < a$, where a is an arbitrary positive constant, or*

(*ii*) *If f(x) and g(x) are analytic (i.e. their real and imaginary parts are Hilbert pairs), then the Hilbert transform of the product of f(x) and g(x) is given by*

$$H(f(x)g(x)) = f(x)H(g(x)). \tag{1.5}$$

For basic properties of the Hilbert transform an interested reader may refer to [1, 2, 4, 5, 19].

2. Irregular Wave Packet System for $L^2(\mathbb{R})$

Definition 2.1. The *IWHW packet system* $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ if there exists finite positive constants A and B such that

$$A\|f\|^2 \leq \sum_{j,k,m\in\mathbb{Z}} |\langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle|^2 \leq B\|f\|^2, \text{ for all } f \in L^2(\mathbb{R}). \tag{2.1}$$

In this case we say that $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is an *IWHW packet frame* for $L^2(\mathbb{R})$. If upper inequality in (2.1) holds, then we say that $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is an *IWHW packet Bessel system* for $L^2(\mathbb{R})$ with *Bessel bound B.*

Definition 2.2. If $\{D_{a_j}T_{bk}E_{c_m}H\psi\}_{j,k,m\in\mathbb{Z}}$ constitutes a frame for $L^2(\mathbb{R})$, then we say that $\{D_{a_j}T_{bk}E_{c_m}H\psi\}_{j,k,m\in\mathbb{Z}}$ is a *Hilbert irregular Weyl-Heisenberg wave packet frame* (in short *H-IWHW packet frame)* for $L^2(\mathbb{R})$.

Remark 2.3. Let $\psi \in L^2(\mathbb{R})$ be an analytic function. If $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is a *IWHW* packet Bessel system with frame operator S. Then, *H-IWHW* packet Bessel system $\{D_{a_j}T_{bk}E_{c_m}H\psi\}_{j,k,m\in\mathbb{Z}}$ also has the frame operator S. Indeed, let T denotes the frame operator for the *IWHW* packet Bessel sequence $\{D_{a_j}T_{bk}E_{c_m}H\psi\}_{j,k,m\in\mathbb{Z}}$. Then, for $\psi \in L^2(\mathbb{R})$ we have

$$Tf = \sum_{j,k,m\in\mathbb{Z}} \langle f, D_{a_j}T_{bk}E_{c_m}H\psi\rangle D_{a_j}T_{bk}E_{c_m}H\psi.$$

Therefore

$$Tf = \sum_{j,k,m\in\mathbb{Z}} \langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle D_{a_j}T_{bk}E_{c_m}\psi. \text{ (see Lemma 2.7)}$$

Therefore $Tf = Sf$, for all $f \in L^2(\mathbb{R})$.

Example 2.4. Consider the function $\psi = \cos\frac{2\pi x}{3}\chi_{[0,\frac{3}{4}]} \subset ?L^2(\mathbb{R})$. Choose $a_j = 1$ for all $j \in \mathbb{N}$, i.e., $D_{a_j} = I$ (the identity operator on $L^2(\mathbb{R})$ for all $j \in \mathbb{Z}$ and $c_m = am$, for all $m \in \mathbb{Z}$. Then, it is easy to observe that $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ and

$\{D_{a_j}T_{bk}E_{c_m}H\psi\}_{j,k,m\in\mathbb{Z}}$ are both frame for $0 < a \leq \frac{4}{3}$ and $0 < b \leq \frac{3}{4}$ (see Theorem 9.1.12 in [6] at page 206).

Remark 2.5. Let $\psi \in L^2(\mathbb{R})$ be such that $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is an *IWHW* packet frame. Then, in general $\{D_{a_j}T_{bk}E_{c_m}H\psi\}_{j,k,m\in\mathbb{Z}}$ is not a *H-IWHW* packet frame. This is given in the following example.

Example 2.6. If $\psi = \chi_{[0,1]},\ b = 1,\ c_m = m$, then $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is an IWHW packet frame but $\{D_{a_j}T_{bk}E_{c_m}H\psi\}_{j,k,m\in\mathbb{Z}}$ is not a *H-IWHW* packet frame for $L^2(\mathbb{R})$.

Now we give a result in the form of a lemma, which says that the image of the Hilbert transform on IWHW packet system is the scalar (pure complex) multiple of the given *IWHW* packet system. For the Gabor system, this result can be found in [15].

Lemma 2.7. *Let $\psi \in L^2(\mathbb{R})$ be an analytic function. Then,*

$$H(D_{a_j}T_{bk}E_{c_m}\psi(t)) = D_{a_j}T_{bk}E_{c_m}H\psi(t)) = -iD_{a_j}T_{bk}E_{c_m}\psi(t).$$

Proof. Since

$$\begin{aligned}&H(D_{a_j}T_{bk}E_{c_m}\psi(t))\\&= H\left(\frac{1}{\sqrt{a_j}}(\cos(2\pi(a_j^{-1}t - kb)c_m) + i\,\sin(2\pi(a_j^{-1}t - kb)c_m))\psi(a_j^{-1}t - kb)\right).\end{aligned}$$

Therefore, by using (1.5) in Theorem 1.4, we have

$$\begin{aligned}&H(D_{a_j}T_{bk}E_{c_m}\psi(t))\\&= \frac{1}{\sqrt{a_j}}(\sin(2\pi(a_j^{-1}t - kb)c_m) - i\cos(2\pi(a_j^{-1}t - kb)c_m))\psi(a_j^{-1}t - kb)\\&= -iD_{a_j}T_{bk}E_{c_m}\psi(t).\end{aligned} \tag{2.2}$$

Now

$$\begin{aligned}D_{a_j}T_{bk}E_{c_m}H\psi(t) &= \frac{1}{\sqrt{a_j}}\exp\left(2\pi(a_j^{-1}t - kb)c_m\right)H\psi(a_j^{-1}t - kb)\\&= -iD_{a_j}T_{bk}E_{c_m}\psi(t).\end{aligned} \tag{2.3}$$

Thus, by using (2.2) and (2.3), we have

$$H(D_{a_j}T_{bk}E_{c_m}\psi(t)) = -iD_{a_j}T_{bk}E_{c_m}\psi(t).$$

□

The following proposition provides a sufficient condition on a window function ψ associated with a given *IWHW* packet frame $\{(\bullet)\psi\}_{j,k,m\in\mathbb{Z}}$ such that $\{(\bullet)H\psi\}_{j,k,m\in\mathbb{Z}}$ constitutes a *H-IWHW* packet frame for $L^2(\mathbb{R})$.

Proposition 2.8. *Let $\psi \in L^2(\mathbb{R})$ be an analytic function. Then, $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is an IWHW packet frame for $L^2(\mathbb{R})$ if and only if $\{D_{a_j}T_{bk}E_{c_m}H\psi\}_{j,k,m\in\mathbb{Z}}$ is a H-IWHW packet frame for $L^2(\mathbb{R})$.*

Proof. By Lemma 2.7, we have

$$\sum_{j,k,m\in\mathbb{Z}} |\langle f, D_{a_j}T_{bk}E_{c_m}H\psi\rangle|^2 = \sum_{j,k,m\in\mathbb{Z}} |\langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle|^2, \text{ for all } f \in L^2(\mathbb{R}).$$

Therefore, $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is an $IWHW$ packet frame for $L^2(\mathbb{R})$ if and only if $\{D_{a_j}T_{bk}E_{c_m}H\psi\}_{j,k,m\in\mathbb{Z}}$ is a H-$IWHW$ packet frame for $L^2(\mathbb{R})$. □

Remark 2.9. The Hilbert transform is an isometry on $L^2(\mathbb{R})$. Therefore, the system $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ in Proposition 2.8 is an orthonormal basis for $L^2(\mathbb{R})$ if and only if $\{D_{a_j}T_{bk}E_{c_m}H\psi\}_{j,k,m\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.

Now we give give necessary and sufficient conditions for the Hilbert transform of $IWHW$ packet system to be a H-$IWHW$ packet frame for $L^2(\mathbb{R})$. This is an extension of Theroem 3.2 in [16] to H-$IWHW$ packet frame for $L^2(\mathbb{R})$.

Theorem 2.10. *Let* $\{a_j\}_{j\in\mathbb{Z}}\subset\mathbb{R}^+, b>0, \{c_m\}_{m\in\mathbb{Z}}\subset\mathbb{R}$ *and let* $\psi\in L^2(\mathbb{R})$ *be an analytic function. Assume that*

$$L(f)=\frac{1}{b}\sum_{j,m,n\in\mathbb{Z}}\int_{supp\widehat{f}}|\widehat{f}(\xi+\frac{a_j}{b}n)|^2|\widehat{\psi}(a_j^{-1}\xi-c_m)|^2d\xi<\infty, \textit{for all } f\in\mathcal{D},$$
$$\textit{where } \mathcal{D}=\{f\in L^2(\mathbb{R}):\widehat{f}\in L^\infty(\mathbb{R}) \textit{ and } supp\widehat{f} \textit{ is compact in } \mathbb{R}\}.$$

Then, the system $\{D_{a_j}T_{bk}E_{c_m}H\psi\}_{j,k,m\in\mathbb{Z}}$ *is a tight H-IWHW packet frame for* $L^2(\mathbb{R})$ *if and only if* ψ *satisfies*

$$\frac{1}{b}\sum_{m\in\mathbb{Z}}\sum_{j\in\tau_\alpha}\overline{\widehat{\psi}(a_j^{-1}\xi-c_m)}\widehat{\psi}(a_j^{-1}(\xi+\alpha)-c_m)=\delta_{\alpha,0} \tag{2.4}$$

for a.e. $\xi\in\mathbb{R}$, $\alpha\in\Lambda$, *where* $\Lambda=\bigcup_{j\in\mathbb{Z}}a_jb^{-1}\mathbb{Z}$, $\tau_\alpha=\{j\in\mathbb{Z}:a_j^{-1}b\alpha\in\mathbb{Z}\}$.

Proof. Assume first that $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is a tight H-$IWHW$ packet frame for $L^2(\mathbb{R})$. Choose $\psi=\widetilde{\psi}$. Then, by using (1.2), we have

$$\begin{aligned}P(f,f)&=\sum_{j,k,m\in\mathbb{Z}}\langle f,D_{a_j}T_{bk}E_{c_m}\psi\rangle\langle D_{a_j}T_{bk}E_{c_m}\widetilde{\psi},f\rangle\\&=\sum_{j,k,m\in\mathbb{Z}}\langle f,D_{a_j}T_{bk}E_{c_m}\psi\rangle\langle D_{a_j}T_{bk}E_{c_m}\psi,f\rangle\\&=\sum_{j,k,m\in\mathbb{Z}}|\langle f,D_{a_j}T_{bk}E_{c_m}\psi\rangle|^2\\&=\|f\|^2, \text{ for all } f\in L^2(\mathbb{R}).\end{aligned}$$

Let us write $L_\psi(\xi)=\sum_{m\in\mathbb{Z},j\in\mathbb{Z}}|\widehat{\psi}(a_j^{-1}\xi-c_m)|^2$. Then, $L_\psi\in L^\infty(\mathbb{R})$ and $P(f,g)=\langle f,g\rangle$ for all $f,g\in L^2(\mathbb{R})$. Thus, by Lemma 1.1, (1.4) holds for $\widetilde{\psi}=\psi$, i.e., equation (2.4) holds.

For the reverse part, we first observe that

$$P(f,f)=\frac{1}{b}\sum_{j,k,m\in\mathbb{Z}}\int_{-\infty}^{\infty}\overline{\widehat{f}(\xi)}\widehat{\psi}(a_j^{-1}\xi-c_m)\widehat{f}(\xi+\frac{a_j}{b}s)\overline{\widehat{\psi}(a_j^{-1}\xi-c_m+\frac{s}{b})}d\xi,$$

where the series certainly converges for $f \in \mathcal{D}$, provided that $L_\psi \in L^\infty(\mathbb{R})$. Indeed, for $\operatorname{supp}\widehat{f} \subset [-h, h]$, there exist an $j \in \mathbb{Z}$ such that

$$\begin{aligned}
P(f,f) &= \frac{1}{b}\left|\sum_{m\in\mathbb{Z}}\sum_{j\le J}\sum_{|s|\le S_j}\int_{-h}^{h}\overline{\widehat{f}(\xi)}\widehat{\psi}(a_j^{-1}\xi - c_m)\widehat{f}(\xi + \frac{a_j}{b}s)\times\overline{\widehat{\psi}(a_j^{-1}\xi - c_m + \frac{s}{b})}d\xi\right| \\
&\le \frac{1}{b}\sum_{m\in\mathbb{Z}}\sum_{j\le J}\sum_{|s|\le S_j}\left(\int_{-h}^{h}|\overline{\widehat{f}(\xi)}\widehat{f}(\xi + \frac{a_j}{b}s)\overline{\widehat{\psi}(a_j^{-1}\xi - c_m)}|^2 d\xi\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_{-h(1+b)}^{h(1+b)}|\widehat{\psi}(a_j^{-1}\xi - c_m)|^2 d\xi\right)^{\frac{1}{2}} \\
&\le \kappa\sum_{m\in\mathbb{Z}}\sum_{j\le J}a_j^{-1}\|\widehat{f}\|_\infty\left(\int_{-h(1+b)}^{h(1+b)}|\widehat{\psi}(a_j^{-1}\xi - c_m)|^2 d\xi\right)^{\frac{1}{2}} \\
&\le \kappa\|\widehat{f}\|_\infty\|\widehat{\psi}\|, \text{ where } \kappa \text{ is some constant.}
\end{aligned}$$

Choose $\Lambda = \cup_{j\in\mathbb{Z}}a_j b^{-1}\mathbb{Z},\ \tau_\alpha = \{j \in \mathbb{Z} : a_j^{-1}b\alpha \in \mathbb{Z}\}$. Then, $P(f, f)$ can be written as

$$P(f,f) = \frac{1}{b}\sum_{m\in\mathbb{Z}}\sum_{\alpha\in\Lambda}\int_{-\infty}^{\infty}\overline{\widehat{f}(\xi)}\widehat{f}(\xi+\alpha)\times\left[\sum_{j\in\tau_\alpha}\widehat{\psi}(a_j^{-1}\xi - c_m)\widehat{\psi}(a_j^{-1}(\xi+\alpha) - c_m)\right]d\xi. \tag{2.5}$$

Choose $\alpha = 0$ in (2.4), we have

$$\frac{1}{b}\sum_{j\in\tau_\alpha}\overline{\widehat{\psi}(a_j^{-1}\xi - c_m)}\widehat{\psi}(a_j^{-1}(\xi - c_m) = \frac{1}{b}\sum_{j\in\tau_\alpha}|\widehat{\psi}(a_j^{-1}\xi - c_m)|^2 = 1, \text{ for a.e. } \xi \in \mathbb{R}.$$

Then, we have $L_\psi(\xi) = b$ a.e., $\xi \in \mathbb{R}$ so that $L_\psi \in L^\infty$; and on the other hand, equation (2.5) can be simplify to

$$P(f,f) = \int_{-\infty}^{\infty}\overline{\widehat{f}(\xi)}\widehat{f}(\xi)d\xi = \|f\|^2, \text{ for all } f \in \mathcal{D}.$$

This gives (choose $f = g$ in (1.2))

$$\sum_{j,k,m\in\mathbb{Z}}\langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle\langle D_{a_j}T_{bk}E_{c_m}\widetilde{\psi}, f\rangle = \|f\|^2, \text{ for all } f \in \mathcal{D}.$$

In particular, for $\psi = \widetilde{\psi}$, we have

$$\sum_{j,k,m\in\mathbb{Z}}\langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle\langle D_{a_j}T_{bk}E_{c_m}\psi, f\rangle = \|f\|^2, \text{ for all } f \in \mathcal{D}.$$

That is

$$\sum_{j,k,m\in\mathbb{Z}}|\langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle|^2 = \|f\|^2, \text{ for all } f \in \mathcal{D}. \tag{2.6}$$

By using the density of $\mathcal{D}$ in $L^2(\mathbb{R})$, we conclude that (2.6) hold for all $f \in L^2(\mathbb{R})$. Therefore, $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is a tight *IWHW* packet frame for $L^2(\mathbb{R})$. Hence by Lemma 2.7, $\{D_{a_j}T_{bk}E_{c_m}H\psi\}_{j,k,m\in\mathbb{Z}}$ is a tight *H-IWHW* packet frame for $L^2(\mathbb{R})$. $\square$

To conclude the paper we discuss the commutativity of the frame operator associated with $IWHW$ packet Bessel system and Hilbert transform. More precisely, let $\{(\bullet)\psi\}_{j,k,m\in\mathbb{Z}}$ IWHW packet Bessel system with frame operator S. Then, in general, S does not commutes with the Hilbert transform H. The following proposition provides sufficient conditions on the system $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ such that the associated frame operator S commutes with the Hilbert transform H.

Proposition 2.11. *Let $\psi \in L^2(\mathbb{R})$ be an analytic function. Let $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ be an IWHW packet Bessel sequence with frame operator S and let H be the Hilbert transform on $L^2(\mathbb{R})$. Then, $SH = HS$.*

Proof. Since

$$Sf = \sum_{j,k,m\in\mathbb{Z}} \langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle D_{a_j}T_{bk}E_{c_m}\psi, \text{ for all } f \in L^2(\mathbb{R}). \tag{2.7}$$

Taking Hilbert transform on both sides in (2.7), we have

$$HSf = \sum_{j,k,m\in\mathbb{Z}} \langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle H(D_{a_j}T_{bk}E_{c_m}\psi), \text{ for all } f \in L^2(\mathbb{R}). \tag{2.8}$$

Since ψ is analytic, by (2.8), we have

$$HSf = -i\sum_{j,k,m\in\mathbb{Z}} \langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle D_{a_j}T_{bk}E_{c_m}\psi, \text{ for all } f \in L^2(\mathbb{R}). \tag{2.9}$$

By using the fact that $H^* = -H$, we compute

$$\begin{aligned}
SHf &= \sum_{j,k,m\in\mathbb{Z}} \langle Hf, D_{a_j}T_{bk}E_{c_m}\psi\rangle (D_{a_j}T_{bk}E_{c_m}\psi)\\
&= \sum_{j,k,m\in\mathbb{Z}} \langle f, H^*D_{a_j}T_{bk}E_{c_m}\psi\rangle (D_{a_j}T_{bk}E_{c_m}\psi)\\
&= \sum_{j,k,m\in\mathbb{Z}} \langle f, -HD_{a_j}T_{bk}E_{c_m}\psi\rangle (D_{a_j}T_{bk}E_{c_m}\psi)\\
&= -i\sum_{j,k,m\in\mathbb{Z}} \langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle D_{a_j}T_{bk}E_{c_m}\psi, \text{ for all } f \in L^2(\mathbb{R}).
\end{aligned} \tag{2.10}$$

By (2.9) and (2.10), S and H commutes. The claim is proved. □

Remark 2.12. The result given in Proposition 2.11 for the Gabor system can be found in [15].

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