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ON THE FINE SPECTRUM OF THE OPERATOR $B(r, s, t)$ OVER THE CLASS OF CONVERGENT SERIES

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Abstract. The main aim of this paper is to determine the fine spectrum of the operator $B(r, s, t)$ on γ of all convergent series. Also, we study the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $B(r, s, t)$ on γ.

1. Introduction

In functional analysis, the spectrum of an operator generalizes the notion of eigenvalues for matrices. The spectrum of an operator over a Banach space is partitioned into three parts, which are point spectrum, the continuous spectrum and residual spectrum. The calculation of these three parts of the spectrum of an operator is called calculating the fine spectrum of the operator. By w we denote the space of all complex-valued sequences. Any vector subspace of w is called a sequence space. We write l_∞, c, c_0, and l_p for the spaces of all bounded, convergent, null and p-absolutely summable sequences, where $1 \le p < \infty$. The space γ of all convergent series is defined by

$$\gamma = \left\{ (x_k) \in w : \left(\sum_{k=0}^{n} x_k \right) \in c \right\}.$$

If $T : \gamma \to \gamma$ is a bounded linear operator with the matrix A, then its adjoint operator $T^* : \gamma^* \to \gamma^*$ is defined by transpose of the matrix A and γ^* is isomorphic to l_1 with the norm $\|x\| = \sum_{k=0}^{\infty} |x_k|$.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Then, we say that A defines a

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matrix mapping from X into Y, and we denote it by writing $A : X \to Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}_{n\in\mathbb{N}}$, the A-transform of x, is in Y, where

$$(Ax)_n = \sum_k a_{nk}x_k \ \ (n \in \mathbb{N}) \tag{1.1}$$

By $(X : Y)$, we denote the class of all matrices A such that $A : X \to Y$. Thus, $A \in (X : Y)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in Y$ for all $x \in X$.
The spectrum and fine spectrum of linear operators which are defined by some triangle matrices over some sequence spaces are studied. One of them is the spectrum of the Cesaro operator on the sequence space c_0 which was studied by Reade [13]. Okutoyi studied the spectrum of the Cesaro operator C_1 on bv_0 [12]. The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c has been studied by Altay and Başar [1]. Same authors have studied the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces c_0 and c [2]. Then, Furkan, Bilgiç and Altay [7] have studied the fine spectrum of the operator $B(r, s, t)$ over c_0 and c. Bilgiç and Furkan [5] have studied the fine spectrum of operator $B(r, s, t)$ over l_1 and bv. The fine spectrum of operator $B(r, s, t)$ over the sequence spaces l_p and bv_p $(1 < p < \infty)$ has been studied by Furkan, Bilgiç and Başar [8]. Recently, Başar, Durna and Yıldırım [4] studied subdivisions of spectra for the triple band matrix over certain sequence spaces, and fine spectrum of the generalized difference operator $B(r, s)$ over the class of convergent series has been studied by Dutta and Tripathy [6]

2. Preliminaries and Notations

In this section, basic definitions which are benefit from [4], are given.
Let X and Y be Banach spaces and $T : X \to Y$ a bounded linear operator. We denote the range of T by $R(T)$, and $B(X)$, the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\varphi)(x) = \varphi(Tx)$ for all $\varphi \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$.
Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subset X$. By T, associate the operator $T_\alpha = T - \alpha I$, where α is a complex number and I is the identity operator on $D(T)$. If T_α has an inverse, which is linear, we denote it by $T_\alpha^{-1} = (T - \alpha I)^{-1}$, and it is called the resolvent operator of T. The name resolvent is appropriate, since T_α^{-1} helps to solve the equation $T_\alpha x = y$. Thus, $x = T_\alpha^{-1}y$ provided T_α^{-1} exist. More important, the investigation of properties of T_α^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_α and T_α^{-1} depend on α, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all α in the complex plane such that T_α^{-1} exist. Boundedness of T_α^{-1} is another property that will be essential. We shall also ask for what α's the domain of T_α^{-1} is dense in X, to name just a few aspects. For our investigation of T, T_α and T_α^{-1}, we need some basic concepts in spectral theory which are given as follows (see[[11] pp. 370–371]).
Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ also be a linear operator with domain $D(T) \subset X$. A regular value α of T is a complex number such that

(R1): T_α^{-1} exist
(R2): T_α^{-1} is bounded
(R3): T_α^{-1} is defined on a set which is dense in X.

The resolvent set $\rho(T,X)$ of T is the set of all regular values α of T. Its complement, in the complex plane $\mathbb{C}$, $\sigma(T,X) = \mathbb{C} \setminus \rho(T,X)$ is called the spectrum of T. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows:
The point spectrum $\sigma_p(T,X)$ is the set such that T_α^{-1} does not exist. An $\alpha \in \sigma_p(T,X)$ is called an eigenvalue of T.
The continuous spectrum $\sigma_c(T,X)$ is the set such that T_α^{-1} exist and satisfies (R3) but not (R2).
The residual spectrum $\sigma_r(T,X)$ is the set such that T_α^{-1} exist (and may be bounded or not) but not satisfy (R3).
Therefore, these three subspectras form disjoint subdivisions

$$\sigma(T,X) = \sigma_p(T,X) \cup \sigma_c(T,X) \cup \sigma_r(T,X). \tag{2.1}$$

In this section, following Appell et al. [3], we call the three more subdivisions of the spectrum called the approximate point spectrum, defect spectrum, and compression spectrum.
Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X as a Wely sequence for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$.
In what follows, we call the set

$$\sigma_{ap}(T,X) := \{\alpha \in \mathbb{C} : \text{there exists a Wely sequence for } T - \alpha I\} \tag{2.2}$$

the approximate point spectrum of T. Moreover, the subspectrum

$$\sigma_\delta(T,X) := \{\alpha \in \mathbb{C} : T - \alpha I \text{ is not surjective}\} \tag{2.3}$$

is called defect spectrum of T.
The two subspectra given by (2.2) and (2.3) form a (not necessarily disjoint) subdivision

$$\sigma(T,X) = \sigma_{\mathrm{ap}}(T,X) \cup \sigma_\delta(T,X)$$

of the spectrum. There is another subspectrum,

$$\sigma_{\mathrm{co}}(T,X) := \left\{\alpha \in : \overline{R(\alpha I - T)} \neq X\right\}$$

which is often called compression spectrum in the literature. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T,X) = \sigma_{\mathrm{ap}}(T,X) \cup \sigma_{\mathrm{co}}(T,X)$$

of spectrum. Clearly, $\sigma_{\mathrm{p}}(T,X) \subseteq \sigma_{\mathrm{ap}}(T,X)$ and $\sigma_{\mathrm{co}}(T,X) \subseteq \sigma_\delta(T,X)$. Moreover, comparing these subspectra with those in (2.1) we note that

$$\sigma_{\mathrm{r}}(T,X) = \sigma_{\mathrm{co}}(T,X) \setminus \sigma_{\mathrm{p}}(T,X), \text{ and } \sigma_c(T,X) = \sigma(T,X) \setminus [\sigma_{\mathrm{p}}(T,X) \cup \sigma_{\mathrm{co}}(T,X)].$$

Proposition 2.1. [3] *Spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by following relations:*

(a): $\sigma(T^*,X^*) = \sigma(T,X)$
(b): $\sigma_{\mathrm{c}}(T^*,X^*) \subseteq \sigma_{\mathrm{ap}}(T,X)$
(c): $\sigma_{\mathrm{ap}}(T^*,X^*) = \sigma_\delta(T,X)$
(d): $\sigma_\delta(T^*,X^*) = \sigma_{\mathrm{ap}}(T,X)$
(e): $\sigma_p(T^*,X^*) = \sigma_{\mathrm{co}}(T,X)$

(f): $\sigma_{\rm co}(T^*, X^*) \supseteq \sigma_{\rm p}(T, X)$
(g): $\sigma(T, X) = \sigma_{\rm ap}(T, X) \cup \sigma_{\rm p}(T^*, X^*) = \sigma_p(T, X) \cup \sigma_{\rm ap}(T^*, X^*)$.

Relations (c)-(f) show that the approximate point spectrum is in a certain sense dual to the defect spectrum, and the point spectrum dual to compression spectrum. The equality (g) implies, in particular, that $\sigma(T, X) = \sigma_{\rm ap}(T, X)$ if T is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operator on Hilbert space are most similar to matrices in finite dimensional spaces (see [3]).

From Goldberg [9], If $T \in B(X)$, X a Banach space, then there are three possibilities for $R(T)$,

(A): $R(T) = X$,
(B): $R(T) \neq \overline{R(T)} = X$,
(C): $\overline{R(T)} \neq X$,

and

(1): T^{-1} exists and is continuous,
(2): T^{-1} exists but is discontinuous,
(3): T^{-1} does not exist.

If these possibilities are combined in all ways, nine different states are created. These are labelled by: A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, and C_3. If an operator is in state C_2, for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous (see[9]).

Lemma 2.2. [9] *The adjoint operator T^* of T is onto if and only if T has a bounded inverse.*

Lemma 2.3. [9] *T has a dense range if and only if T^* is one to one.*

Lemma 2.4. [14] *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(l_1)$ from l_1 to itself if and only if the supremum of l_1 norms of the columns of A is bounded.*

Corollary 2.5. [9] $\sigma_r(T, X) \subseteq \sigma_p(T^*, X^*) \subseteq \sigma_r(T, X) \cup \sigma_p(T, X)$.

3. Main results

In this section, we study the fine spectrum of lower triangular triple-band matrix $B(r, s, t)$ over the sequence space γ. Now, $B(r, s, t)$ is determined by

$$B(r,s,t) = \begin{bmatrix} r & 0 & 0 & 0 & 0 & \cdots \\ s & r & 0 & 0 & 0 & \cdots \\ t & s & r & 0 & 0 & \cdots \\ 0 & t & s & r & 0 & \cdots \\ 0 & 0 & t & s & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where s and t are complex parameters which do not simultaneously vanish.

Theorem 3.1. *$B(r, s, t) : \gamma \to \gamma$ is a bounded linear operator with*

$$\|B(r,s,t)\|_{(\gamma,\gamma)} = |r| + |s| + |t|.$$

Proof. The proof is obvious. $\square$

Theorem 3.2. *Let* $\sqrt{s^2} = -s$ *and* $S_1 = \left\{\alpha \in \mathbb{C} : \left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| \leq 1\right\}$. *Then,* $\sigma(B(r,s,t),\gamma) = S_1$.

Proof. Now, we show that $(B(r,s,t)-\alpha I)^{-1}$ exists and $(B(r,s,t)-\alpha I)^{-1}$ is in $B(\gamma)$ for $\alpha \in \mathbb{C}$ with

$$\left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| > 1.$$

Then we indicate that $(B(r,s,t)-\alpha I)^{-1}$ does not exist for $\alpha \in S_1$.
Let $\alpha \notin S_1$. So, we have $\alpha \neq r$. Since $\alpha \neq r$, $B(r,s,t) - \alpha I$ is triangle and hence $(B(r,s,t)-\alpha I)^{-1}$ exists.
By solving the equation $[B(r,s,t)-\alpha I]\,x = y$. We obtain
$(r-\alpha)x_0 = y_0$
$sx_0 + (r-\alpha)x_1 = y_1$
$tx_o + sx_1 + (r-\alpha)x_2 = y_2$
$\vdots$
and in this way we can get
$x_0 = \frac{y_0}{r-\alpha}$,
$x_1 = \frac{1}{r-\alpha}y_1 + \frac{-s}{(r-\alpha)^2}y_o$,
$x_2 = \frac{1}{r-\alpha}y_2 + \frac{-s}{(r-\alpha)^2}y_1 + \frac{s^2-(r-\alpha)t}{(r-\alpha)^3}y_0$,
$\vdots$

Let $a_1 = \frac{1}{r-\alpha}$, $a_2 = \frac{-s}{(r-\alpha)^2}$, $\ldots$, $a_n = \frac{-(sa_{n-1}+ta_{n-2})}{r-\alpha}$, $(n \geq 3)$. So we can observe

$$(B(r,s,t)-\alpha I)^{-1} = \begin{bmatrix} a_1 & 0 & 0 & \cdots \\ a_2 & a_1 & 0 & \cdots \\ a_3 & a_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Clearly, when $s^2 \neq 4t(r-\alpha)$, $a_n = \frac{(u_1)^n-(u_2)^n}{\sqrt{s^2-4t(r-\alpha)}}$ for $n \geq 1$ where
$u_1 = \frac{-s+\sqrt{s^2-4t(r-\alpha)}}{2(r-\alpha)}$ and $u_2 = \frac{-s-\sqrt{s^2-4t(r-\alpha)}}{2(r-\alpha)}$.
Since $\alpha \notin S_1$, then $|u_1| < 1$. So, we have $|u_2| < |u_1|$ and hence $|u_2| < 1$.
Since $|u_1| < 1$ and $|u_2| < 1$, we can see that

$$\begin{aligned}\left\|(B(r,s,t)-\alpha I)^{-1}\right\|_{(\gamma,\gamma)} &= \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} |a_k| = \sum_{k=1}^{\infty} |a_k| \\ &\leq \frac{1}{\left|\sqrt{s^2-4t(r-\alpha)}\right|}\left(\sum_{k=1}^{\infty}\left(|u_1|^k + |u_2|^k\right)\right) < \infty.\end{aligned}$$

This shows that $\sigma(B(r,s,t),\gamma) \subseteq S_1$.
Let $\alpha \in S_1$ and $\alpha = r$. So,

$$(B(r,s,t)-\alpha I)=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ s & 0 & 0 & 0 & \cdots \\ t & s & 0 & 0 & \cdots \\ 0 & t & s & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$(B(r,s,t)-\alpha I)$ does not have a dense range, so it is not invertible.
If $s^2=4t(r-\alpha)$, then $a_n=\left(\frac{2n}{-s}\right)\left(\frac{-s}{2(r-\alpha)}\right)^n$ for all $n\geq 1$ where
$(r-\alpha)\,a_n+sa_{n-1}+ta_{n-2}=0$. Since $\left|\frac{-s}{2(r-\alpha)}\right|\geq 1$, then $(B\,(r,s,t)-\alpha I)^{-1}\notin B(\gamma)$.
We may assume that $\alpha\neq r$ and $s^2\neq 4t(r-\alpha)$. Since $\alpha\neq r$, then $(B\,(r,s,t)-\alpha I)^{-1}$ exists, but since $s^2\neq 4t(r-\alpha)$ and $|u_1|>|u_2|>1$, then $(B\,(r,s,t)-\alpha I)^{-1}$ is not element $B\,(\gamma)$ as in [5].
This shows that $S_1\subseteq\sigma(B(r,s,t),\gamma)$. □

Theorem 3.3. $\sigma_p(B(r,s,t),\gamma)=\emptyset$.

Proof. Let $B(r,s,t)x=\alpha x$ for $x\neq\theta$ in γ. This gives
$rx_0=\alpha x_0$
$sx_0+rx_1=\alpha x_1$
$tx_0+sx_1+rx_2=\alpha x_2$
$\vdots$
If x_{n_0} is the first nonzero entry of the sequence $x=(x_n)$, then $\alpha=r$ and

$$tx_{n_0-1}+sx_{n_0}+rx_{n_0+1}=\alpha x_{n_0+1}$$

which implies that $x_{n_0}=0$. This contradicts the fact that $x_{n_0}\neq 0$ which means that $\sigma_p(B(r,s,t),\gamma)=\emptyset$ and this completes the proof. □

Theorem 3.4. *Let* $S_2=\left\{\alpha\in\mathbb{C}:\left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right|<1\right\}$. *Then* $\sigma_p\left(B(r,s,t)^*,\gamma^*\right)=S_2$.

Proof. Let $B(r,s,t)^*x=\alpha x$ for $x\neq\theta$ in $\gamma^*\cong l_1$. Then, by solving the system of linear equations as in [2],
$rx_0+sx_1+tx_2=\alpha x_0$
$rx_1+sx_2+tx_3=\alpha x_1$
$rx_2+sx_3+tx_4=\alpha x_2$
$\vdots$
If $\alpha=r$, then $x=(x_0,0,0,0,\ldots)$ for $x_0\neq 0$ is an eigenvector corresponding to $\alpha=r$.
Suppose that $\alpha\neq r$, we get
$x_n=\frac{P_{n-1}}{t^{n-1}}(\alpha-r)x_0+\frac{P_n}{t^{n-1}}x_1$; $(n\geq 2)$ where $P_n=a_n(r-\alpha)^n$.
Suppose that $\alpha\in S_2$. So we can choose that $x_0=1$, $x_1=\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}$. We can obtain
$x_2=(x_1)^2$, , $x_3=(x_1)^3$, $\ldots$, $x_n=(x_1)^n$, $\ldots$ for all $n\geq 2$. Hence, since

$\left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| < 1$, we observe that $x \in \gamma^*$. This demonstrate that

$$\left\{\alpha \in \mathbb{C} : \left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| < 1\right\} \subseteq \sigma_p(B(r,s,t)^*, \gamma^*).$$

Assume that $\alpha \notin S_2$. So, $|u_1| \leq 1$ and $\alpha \neq r$. We have

$$\frac{x_{n+1}}{x_n} = \left(\frac{r-\alpha}{t}\right)\left(\frac{a_n}{a_{n-1}}\right)\left(\frac{-x_0 + \frac{a_{n+1}}{a_n}x_1}{-x_0 + \frac{a_n}{a_{n-1}}x_1}\right).$$

Part 1: $|u_2| = |u_1| < 1$.

In this part we have $s^2 = 4t(r-\alpha)$. So, we have $a_n = \left(\frac{2n}{-s}\right)\left(\frac{-s}{2(r-\alpha)}\right)^n$. We observe that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = u_2 = u_1 \text{ and } \lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \frac{1}{|u_2|} > 1.$$

So $(x_n) \notin l_1$.

Part 2: $|u_2| = |u_1| = 1$

In this part we have $s^2 = 4t(r-\alpha)$. Suppose that $\alpha \in \sigma_p(B(r,s,t)^*, \gamma^*)$, for $x \neq \theta$ in γ^*. We get $x_n = \left(\frac{-s}{2t}\right)^{n-1}\left((1-n)\left(\frac{-s}{2t}\right)x_0 + x_1\right)$. Since $\lim\limits_{n\to\infty}|x_n| = 0$ then we have $x = \theta$. But this is a contradiction. Hence we must have $\alpha \notin \sigma_p(B(r,s,t)^*, \gamma^*)$.

Part 3: $|u_2| < |u_1| \leq 1$.

In this part we have $s^2 \neq 4t(r-\alpha)$.
Clearly $\lim\limits_{n\to\infty} \frac{a_n}{a_{n-1}} = u_1$. Since $x_0 = 1$, $x_1 = \frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}$, $-x_0 + u_1x_1 = 0$. So, we get $x_n = \frac{1}{{u_1}^n}x_0$. Therefore $(x_n) \notin l_1$. Because $|u_1| \leq 1$ and $\lim\limits_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{1}{u_2}\right| > 1$, so $(x_n) \notin l_1$.
Therefore, we have $\sigma_p(B(r,s,t)^*, \gamma^*) \subseteq \left\{\alpha \in: \left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| < 1\right\}$, from Part 1, 2 and 3. □

Theorem 3.5. $\sigma_r\left(B(r,s,t), \gamma\right) = S_2$.

Proof. Since Theorem 3.4, $B(r,s,t)^* - \alpha I$ is not one to one for all $\alpha \in S_2$. So, $B(r,s,t) - \alpha I$ does not have a dense in γ for all $\alpha \in S_2$ from Lemma 2.3. We have also seen that $(B(r,s,t) - \alpha I)^{-1}$ exist. □

Theorem 3.6. $\sigma_c\left(B(r,s,t), \gamma\right) = \left\{\alpha \in \mathbb{C} : \left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| = 1\right\}$

Proof. The proof is simple. □

Theorem 3.7. *Let $\alpha = r$ and $|t| < |s|$. Then $\alpha \in C_1\sigma\left(B\left(r,s,t\right), \gamma\right)$.*

Proof. Since $\alpha = r$, then $B(r,s,t) - \alpha I$ is in state C_1 or C_2. We can see that

$$(B(r,s,t) - \alpha I)^{-1} = \begin{bmatrix} 0 & \frac{1}{s} & 0 & 0 & \cdots \\ 0 & \frac{-t}{s^2} & \frac{1}{s} & 0 & \cdots \\ 0 & \frac{(-t)^2}{s^3} & \frac{-t}{s^2} & \frac{1}{s} & \cdots \\ 0 & \frac{(-t)^3}{s^4} & \frac{(-t)^2}{s^3} & \frac{-t}{s^2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

So, this gives $(B(r,s,t) - \alpha I)^{-1} \in B(\gamma)$ for $|t| < |s|$. Consequently, $\alpha \in C_1\sigma(B(r,s,t),\gamma)$ for $|t| < |s|$. □

Theorem 3.8. *Let $\alpha = r$ and $|t| \geq |s|$. Then $\alpha \in C_2\sigma(B(r,s,t),\gamma)$.*

Proof. Proof is made similarly by Theorem 3.7. □

4. The Approximate Point Spectrum, Defect Spectrum and Compression Spectrum of The Operator $B(r,s,t)$ on γ

In the following three theorem, we give the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $B(r,s,t)$ on γ and these theorems similar as in Theorems of [4].

Theorem 4.1. $\sigma_{ap}(B(r,s,t),\gamma) = \begin{cases} S_1 \setminus \{r\} & , \ |t| < |s| \\ S_1 & , \ |t| \geq |s| \end{cases}$

Proof. Since from Table 1 determined [10],

$$\sigma_{ap}(B(r,s,t),\gamma) = \sigma(B(rs,t),\gamma) \setminus C_1\sigma(B(r,s,t),\gamma)$$

So

$$\sigma_{ap}(B(r,s,t),\gamma) = \begin{cases} S_1 \setminus \{r\} & , \ |t| < |s| \\ S_1 & , \ |t| \geq |s| \end{cases}$$

□

Theorem 4.2. $\sigma_\delta(B(r,s,t),\gamma) = \left\{\alpha \in \mathbb{C} : \left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| \leq 1\right\}.$

Proof. Since the following equality

$$\sigma_\delta(B(r,s,t),\gamma) = \sigma(B(r,s,t),\gamma) \setminus A_3\sigma(B(r,s,t),\gamma)$$

holds from Table 1 determined [10] and we derive by Theorem 3.2, 3.3 that

$$\sigma_\delta(B(r,s,t),\gamma) = \left\{\alpha \in \mathbb{C} : \left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| \leq 1\right\}.$$

□

Theorem 4.3. $\sigma_{co}(B(r,s,t),\gamma) = \left\{\alpha \in \mathbb{C} : \left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| < 1\right\}.$

Proof. We have by Theorem 3.4 and (e) of Proposition 2.1 that

$$\sigma_{co}(B(r,s,t),\gamma) = \left\{\alpha \in \mathbb{C} : \left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| < 1\right\}.$$

□

Conclusion

In this paper, the spectra of matrix operator on convergent series is established. As a conclusion, we say that our results reduce to the spectrum of $B(r, s)$ which is studied Dutta et al. [6] over γ since $B(r, s, 0) = B(r, s)$. So, this study is more general than fine spectrum of the generalized difference operator $B(r, s)$ over the class of convergent series.

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