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# REFINEMENTS OF CARLEMAN'S INEQUALITY

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***Abstract*.** In this paper, we prove that the inequalities

$$\sum_{n=1}^{\infty}(\prod_{k=1}^{n} a_k)^{\frac{1}{n}} \leq e\sum_{n=1}^{\infty}(1-\frac{a}{n+13a})a_n$$

and

$$\sum_{n=1}^{\infty}[(1+\frac{c}{n+b})(\prod_{k=1}^{n} a_k)^{\frac{1}{n}}] \leq e\sum_{n=1}^{\infty} a_n$$

hold if $a_n \geq 0$ $(n = 1, 2, \cdots)$ with $0 < \sum\limits_{n=1}^{\infty} a_n < +\infty$, where $a = 2.739$, $b = [(4\sqrt{2}-3)e-6]/[(3-2\sqrt{2})e] = 2.6203\cdots$ and $c = (e/2-1)(1+b) = 1.3002\cdots$.

## 1. Introduction

Let $a_n \geq 0$ $(n = 1, 2, \cdots)$ with $0 < \sum\limits_{n=1}^{\infty} a_n < +\infty$, then the well-known Carleman's inequality [1] is given by

$$\sum_{n=1}^{\infty}(\prod_{k=1}^{n} a_k)^{\frac{1}{n}} < e\sum_{n=1}^{\infty} a_n.$$

In the last ninety years, the refinement, improvement, generalization, extension and application for the Carleman's inequality have attracted the attention of many researchers [2-19].

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Alzer [2] gave a very elegant proof of the inequality

$$\sum_{k=1}^{\infty} p_k G_k + \frac{1}{2}\sum_{k=1}^{\infty} \frac{p_k^2}{P_k} G_k < e \sum_{k=1}^{\infty} p_k x_k,$$

where $x_i, p_i > 0$ $(i = 1, 2, \cdots)$, $G_k = \prod_{i=1}^{k} x_i^{p_i/P_k}$, $P_k = \sum_{i=1}^{k} p_i$ and $\sum_{k=1}^{\infty} p_k x_k < +\infty$.

In [20], Yuan established that

$$\sum_{n=1}^{\infty} (\prod_{k=1}^{n} a_k)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} \frac{(1-\beta/n)}{(1+1/n)^{\alpha}} a_n,$$

where $\alpha$, $\beta$ satisfy $0 \le \alpha \le 1/\log 2 - 1$, $0 \le \beta \le 1 - 2/e$, $e\beta + 2^{1+\alpha} = e$, $a_n \ge 0$ $(n = 1, 2, \cdots)$ and $0 < \sum_{n=1}^{\infty} a_n < +\infty$.

Yang and Debnath [3] proved that the inequality

$$\sum_{n=1}^{\infty} (\prod_{k=1}^{n} a_k)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} (1 - \frac{1}{2n+2}) a_n$$

holds if $a_n \ge 0$ $(n = 1, 2, \cdots)$ with $0 < \sum\limits_{n=1}^{\infty} a_n < +\infty$.

In [21], the authors established that

$$\sum_{n=1}^{\infty} (\prod_{k=1}^{n} a_k)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} (1 - \frac{1-\frac{2}{e}}{n}) a_n$$

for $a_n \ge 0$ $(n = 1, 2, \cdots)$ with $0 < \sum\limits_{n=1}^{\infty} a_n < +\infty$.

Yang [15] gave the following generalization of Carleman's inequality:

$$\sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} a_k^{1/k^{\lambda}} \right)^{1/S_n(\lambda)} \le e \sum_{n=1}^{\infty} e^{\lambda n^{\lambda-1} S_n(\lambda)} a_n,$$

where $\lambda \in [0, 1]$, $S_n(\lambda) = \sum_{k=1}^{n} \frac{1}{k^{\lambda}}$, $a_n \ge 0$ $(n = 1, 2, \cdots)$ and $\sum\limits_{n=1}^{\infty} a_n < +\infty$.

In [17], Hu proved that

$$\sum_{n=1}^{\infty} (\prod_{k=1}^{n} a_k)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{\infty} \frac{b_k}{(n+1)^k}\right) a_n$$

if $a_n \ge 0$ $(n = 1, 2, \cdots)$ with $0 < \sum\limits_{n=1}^{\infty} a_n < +\infty$, $b_0 = 1$ and $b_n = \frac{1}{n} \sum\limits_{k=1}^{n} \frac{b_{n-k}}{k+1}$.

In this paper, we present two refinements of Carleman's inequality. Our main results are the following Theorems 1.1 and 1.2.

**Theorem 1.1**. Let $a = 2.739$, $a_n \ge 0$ $(n = 1, 2, \cdots)$ with $0 < \sum\limits_{n=1}^{\infty} a_n < +\infty$, then

$$\sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} a_k \right)^{\frac{1}{n}} \le e \sum_{n=1}^{\infty} \left(1 - \frac{a}{n+13a}\right) a_n.$$

**Theorem 1.2**. Let $b = [(4\sqrt{2}-3)e-6]/[(3-2\sqrt{2})e] = 2.6203\cdots$, $c = (e/2-1)(1+b) = 1.3002\cdots$, $a_n \geq 0$ $(n = 1, 2, \cdots)$ with $0 < \sum\limits_{n=1}^{\infty} a_n < +\infty$, then

$$\sum_{n=1}^{\infty}\left[\left(1+\frac{c}{n+b}\right)(\prod_{k=1}^{n} a_k)^{\frac{1}{n}}\right] \leq e\sum_{n=1}^{\infty} a_n.$$

## 2. Lemmas

In order to establish our main results we need several lemmas, which we present in this section.

**Lemma 2.1** (see [22, Corollary 1.3]). Suppose that $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b]^n \to \mathbb{R}$ has continuous partial derivatives and

$$D_m = \{x = (x_1, x_2, \cdots, x_n) | a \leq \min_{1\leq k\leq n}\{x_k\} < x_m = \max_{1\leq k\leq n}\{x_k\} \leq b\}, m = 1, 2, \cdots, n.$$

If $\frac{\partial f(x)}{\partial x_m} > 0$ holds for all $x \in D_m$ and $m = 1, 2, \cdots, n$, then

$$f(x_1, x_2, \cdots, x_n) \geq f(x_{\min}, x_{\min}, \cdots, x_{\min})$$

for all $x_m \in [a, b]$ $(m = 1, 2, \cdots, n)$, where $x_{\min} = \min\limits_{1\leq k\leq n}\{x_k\}$.

**Lemma 2.2** (Stirling formula). If $n \in \mathbb{N}$ is a positive natural number, then

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{\theta_n}{12n}},$$

where $0 < \theta_n < 1$.

**Lemma 2.3**. Let $n \in \mathbb{N}$ be a positive natural number and $a = 2.739$, then

$$e\left[\frac{n^2 + (1+24a)n + \frac{23}{2}a + 156a^2}{(n+13a)(n+\frac{1}{2})(n+1+13a)(n+\frac{3}{2})}\right] > \frac{4}{n}\left[\frac{n!}{(2n+1)!}\right]^{1/n}. \tag{2.1}$$

*Proof.* If $n = 1$ and 2, the numerical computations lead to

$$e\left[\frac{1 + (1+24a) + \frac{23}{2}a + 156a^2}{(1+13a)(2+13a)\times\frac{15}{4}}\right] = 0.668\ldots, \tag{2.2}$$

$$e\left[\frac{4 + 2(1+24a) + \frac{23}{2}a + 156a^2}{(2+13a)(3+13a)\times\frac{35}{4}}\right] = 0.286\ldots, \tag{2.3}$$

$$\frac{4}{3!} = 0.666\cdots, 2\left(\frac{2!}{5!}\right)^{1/2} = 0.258\cdots. \tag{2.4}$$

Equations (2.2)-(2.4) show that inequality (2.1) holds for $n = 1$ and 2.

We claim that the inequality

$$(2n+1)\sqrt{2+\frac{1}{n}} >$$

$$e^{1+\frac{1}{12n}}\left[1+\frac{(1+2a)n^2 + (13a^2+\frac{57}{2}a+1)n + (\frac{351}{2}a^2+\frac{55}{4}a)}{n^3 + (\frac{3}{2}+24a)n^2 + (\frac{1}{2}+\frac{47}{2}a+156a^2)n + (\frac{23}{4}a+78a^2)}\right]^n \tag{2.5}$$

holds for all $n \geq 3$. Indeed, if $n \geq 625$, then it is not difficult to verify that

$$\sqrt{1+\frac{1}{2n}} > e^{\frac{1}{12n}}. \tag{2.6}$$

Note that

$$1+2a > \frac{(1+2a)n^3+(13a^2+\frac{57}{2}a+1)n^2+(\frac{351}{2}a^2+\frac{55}{4}a)n}{n^3+(\frac{3}{2}+24a)n^2+(\frac{1}{2}+\frac{47}{2}a+156a^2)n+(\frac{23}{4}a+78a^2)}, \tag{2.7}$$

$$(2n+1)\sqrt{2} \geq 1351\sqrt{2} = 1769.181\cdots > e^{2+2a} = 1768.699\cdots. \tag{2.8}$$

It follows from (2.6) and (2.8) that

$$(2n+1)\sqrt{2+\frac{1}{n}} > e^{1+\frac{1}{12n}}e^{1+2a}. \tag{2.9}$$

Equality (2.7) implies that

$$e^{1+2a} > (1+x)^{\frac{1+2a}{x}} > (1+x)^{\frac{1}{x}\frac{(1+2a)n^3+(13a^2+\frac{57}{2}a+1)n^2+(\frac{351}{2}a^2+\frac{55}{4}a)n}{n^3+(\frac{3}{2}+24a)n^2+(\frac{1}{2}+\frac{47}{2}a+156a^2)n+(\frac{23}{4}a+78a^2)}} \tag{2.10}$$

for any $x > 0$.

Let $x = \frac{(1+2a)n^2+(13a^2+\frac{57}{2}a+1)n+(\frac{351}{2}a^2+\frac{55}{4}a)}{n^3+(\frac{3}{2}+24a)n^2+(\frac{1}{2}+\frac{47}{2}a+156a^2)n+(\frac{23}{4}a+78a^2)}$, then (2.5) follows from (2.9) and (2.10).

If $3 \leq n \leq 624$, then we verify that inequality (2.5) is also true by compiling the following VB language:

Private Sub Command-Click ()
Dim $a$ As Double, $b$ As Double
For $n = 3$ to 624
$a = (2*n+1)*Sqr(2+1/n)$
$b = Exp(1+1/(12*n)*(((n+35.607)(n+36.607)*(n+1.5))/$
$((n*n+66.736*n+1201.829376)*(n+0.5))^n$
Text 1. Text = Text1. Text & ”n=” & n & ” a =” & a & ”b=” & b & ”a-b=” & (a-b) & Chr(13) & Chr(10)
Next $n$
End Sub.

From (2.5) we get

$$(2n+1)\sqrt{2+\frac{1}{n}} > e^{1+\frac{1}{12n}}\left[\frac{(n+13a)(n+1+13a)(n+\frac{3}{2})}{(n^2+(1+24a)n+\frac{23}{2}a+156a^2)(n+\frac{1}{2})}\right]^n,$$

$$\left[\frac{n^2+(1+24a)n+\frac{23}{2}a+156a^2}{(n+13a)(n+\frac{1}{2})(n+1+13a)(n+\frac{3}{2})}\right]^n (n+\frac{1}{2})^{2n} > \frac{\sqrt{n}e^{1+\frac{1}{12n}}}{\sqrt{2n+1}(2n+1)},$$

$$e^n\left[\frac{n^2+(1+24a)n+\frac{23}{2}a+156a^2}{(n+13a)(n+\frac{1}{2})(n+1+13a)(n+\frac{3}{2})}\right]^n > \frac{4^n\sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n}}}{n^n\sqrt{2\pi(2n+1)}(\frac{2n+1}{e})^{2n+1}}$$

and

$$e\left[\frac{n^2+(1+24a)n+\frac{23}{2}a+156a^2}{(n+13a)(n+\frac{1}{2})(n+1+13a)(n+\frac{3}{2})}\right] > \frac{4}{n}\left(\frac{\sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n}}}{\sqrt{2\pi(2n+1)}(\frac{2n+1}{e})^{2n+1}}\right)^{\frac{1}{n}}. \tag{2.11}$$

Therefore, inequality (2.1) follows from Lemma 2.2 and (2.11) for all $n \geq 3$.

**Lemma 2.4**. Let $a = 2.739$ and $n \in \mathbb{N}$ be a positive natural number. Then

$$e(1 - \frac{a}{n + 13a})\frac{1}{n + \frac{1}{2}} > \sum_{j=n}^{\infty} \frac{1}{j}(\prod_{k=1}^{j} \frac{1}{k + \frac{1}{2}})^{\frac{1}{j}}.$$

*Proof.* Set $f(n) = e(1 - \frac{a}{n+13a})\frac{1}{n+\frac{1}{2}} - \sum_{j=n}^{\infty} \frac{1}{j}(\prod_{k=1}^{j} \frac{1}{k+\frac{1}{2}})^{\frac{1}{j}}$, $n = 1, 2, \cdots$. Then

$$\lim_{n \to +\infty} f(n) = 0 \tag{2.12}$$

and

$$f(n) - f(n+1)$$
$$= e\left[\frac{n^2 + (1 + 24a)n + \frac{23}{2}a + 156a^2}{(n + 13a)(n + \frac{1}{2})(n + 1 + 13a)(n + \frac{3}{2})}\right] - \frac{4}{n}\left(\frac{n!}{(2n+1)!}\right)^{\frac{1}{n}}. \tag{2.13}$$

From Lemma 2.3 and (2.13) we clearly see that the sequence $\{f(n)\}_{n=1}^{+\infty}$ is strictly decreasing. Therefore, Lemma 2.4 follows from (2.12) and the monotonicity of the sequence $\{f(n)\}_{n=1}^{+\infty}$.

**Lemma 2.5**. Let $n \in \mathbb{N}$ be a positive natural number, $b = [(4\sqrt{2} - 3)e - 6]/[(3 - 2\sqrt{2})e] = 2.6203\cdots$ and $c = (e/2 - 1)(1 + b) = 1.3002\cdots$. Then

$$\frac{e}{n+1} \geq \frac{1}{(n!)^{\frac{1}{n}}}\left(1 + \frac{c}{n + b}\right) \tag{2.14}$$

and

$$\frac{e}{n} > \sum_{k=n}^{\infty}\left[\frac{1}{k(k!)^{\frac{1}{k}}}(1 + \frac{c}{k + b})\right]. \tag{2.15}$$

*Proof.* We first prove inequality (2.14). It is not difficult to verify that inequality (2.14) becomes equality for $n = 1, 2$. If $3 \leq n \leq 15$, then

$$\frac{1}{(n!)^{\frac{1}{n}}}\left(1 + \frac{c}{n+b}\right) < \frac{1}{(n!)^{\frac{1}{n}}}\left(1 + \frac{1.3003}{n + 2.62}\right).$$

Let $A_n = e/(n+1)$ and $B_n = (1 + \frac{1.3002}{2.62+b})/(n!)^{1/n}$, then numerical computations give $(A_3, B_3) = (1.234\cdots, 1.231\cdots)$, $(A_4, B_4) = (1.203\cdots, 1.196\cdots)$, $(A_5, B_5) = (1.180\cdots, 1.170\cdots)$, $(A_6, B_6) = (1.162\cdots, 1.150\cdots)$, $(A_7, B_7) = (1.148\cdots, 1.135\cdots)$, $(A_8, B_8) = (1.136\cdots, 1.122\cdots)$, $(A_9, B_9) = (1.127\cdots, 1.111\cdots)$, $(A_{10}, B_{10}) = (1.119\cdots, 1.103\cdots)$, $(A_{11}, B_{11}) = (1.112\cdots, 1.095\cdots)$, $(A_{12}, B_{12}) = (1.105\cdots, 1.088\cdots)$, $(A_{13}, B_{13}) = (1.100\cdots, 1.083\cdots)$, $(A_{14}, B_{14}) = (1.095\cdots, 1.078\cdots)$, $(A_{15}, B_{15}) = (1.091\cdots, 1.073\cdots)$. Therefore, inequality (2.14) holds for $3 \leq n \leq 15$

If $n \geq 16$, then we have

$$\sqrt{2n\pi} \geq \sqrt{32\pi} = 10.026\cdots, \quad e^{c+1} < e^{2.3002} = 9.976\cdots. \tag{2.16}$$

From (2.16) and $e > (1 + 1/x)^x$ $(x > 0)$ we have

$$\sqrt{2n\pi} > e^{c+1} = ee^c > (1 + \frac{1}{n})^n\left[(1 + \frac{c}{n})^{\frac{n}{c}}\right]^c = (1 + \frac{1}{n})^n(1 + \frac{c}{n})^n,$$

$$(2n\pi)^{\frac{1}{2n}} > \frac{n+1}{n}(1+\frac{c}{n}) > \frac{n+1}{n}(1+\frac{c}{n+b}). \tag{2.17}$$

It follows from Lemma 2.2 and (2.17) that

$$\frac{e}{n+1} > \frac{e}{n(2n\pi)^{\frac{1}{2n}}}(1+\frac{c}{n+b}) = \frac{e^{\frac{\theta_n}{12n^2}}}{(n!)^{\frac{1}{n}}}(1+\frac{c}{n+b}) > \frac{1}{(n!)^{\frac{1}{n}}}(1+\frac{c}{n+b}).$$

Next, we prove inequality (2.15). Let

$$H(n) = \frac{e}{n} - \sum_{k=n}^{\infty}\left[\frac{1}{k(k!)^{\frac{1}{k}}}(1+\frac{c}{k+b})\right]. \tag{2.18}$$

Then

$$\lim_{n\to+\infty} H(n) = 0, \tag{2.19}$$

$$H(n) - H(n+1) = \frac{1}{n}[\frac{e}{n+1} - \frac{1}{(n!)^{\frac{1}{n}}}(1+\frac{c}{n+b})]. \tag{2.20}$$

From (2.14) and (2.19) we clearly see that the sequence $\{H(n)\}_{n=1}^{\infty}$ is strictly decreasing. Therefore, inequality (2.15) follows from (2.18) and (2.19) together with the monotonicity of the sequence $\{H(n)\}_{n=1}^{\infty}$.

**Lemma 2.6**. Let $m \in \mathbb{N}$ be a positive natural number, $a_n \geq 0$ $(n = 1, 2, \cdots, m)$, $B_m = \min_{1\leq n\leq m}\{(n+\frac{1}{2})a_n\}$ and $a = 2.739$. Then

$$\begin{aligned} & e\sum_{n=1}^{m}(1-\frac{a}{n+13a})a_n - \sum_{n=1}^{m}(\prod_{k=1}^{n}a_k)^{\frac{1}{n}} \\ \geq\; & 2B_m[e\sum_{n=1}^{m}\frac{n+12a}{(n+13a)(2n+1)} - \sum_{n=1}^{m}\frac{2(n!)^{\frac{1}{n}}}{((2n+1)!)^{\frac{1}{n}}}]. \end{aligned} \tag{2.21}$$

*Proof*. It is not difficult to verify that inequality (2.21) becomes equality for $m = 1$. Suppose that $m \geq 2$ and $b_n = (n+1/2)a_n$, then inequality (2.21) is equivalent to

$$\begin{aligned} & e\sum_{n=1}^{m}(1-\frac{a}{n+13a})\frac{b_n}{n+\frac{1}{2}} - \sum_{n=1}^{m}(\prod_{k=1}^{n}\frac{b_k}{k+\frac{1}{2}})^{\frac{1}{n}} \\ \geq\; & 2B_m[e\sum_{n=1}^{m}\frac{n+12a}{(n+13a)(2n+1)} - \sum_{n=1}^{m}\frac{2(n!)^{\frac{1}{n}}}{((2n+1)!)^{\frac{1}{n}}}]. \end{aligned} \tag{2.22}$$

Let

$$f: \mathbf{b} = (b_1, b_2, \cdots, b_m) \in (0, +\infty)^m \to e\sum_{n=1}^{m}(1-\frac{a}{n+13a})\frac{b_n}{n+\frac{1}{2}} - \sum_{n=1}^{m}(\prod_{k=1}^{n}\frac{b_k}{k+\frac{1}{2}})^{\frac{1}{n}}$$

and

$$D_i = \{(b_1, b_2, \cdots, b_m) \mid \min_{1\leq k\leq m}\{b_k\} < b_i = \max_{1\leq k\leq m}\{b_k\}\}, i = 1, 2, \cdots, m.$$

Then for any $\mathbf{b} \in D_i$ we get

$$\frac{\partial f(\mathbf{b})}{\partial b_i} = e(1-\frac{a}{i+13a})\frac{1}{i+\frac{1}{2}} - \sum_{n=i}^{m}\frac{1}{nb_i}(\prod_{k=1}^{n}\frac{b_k}{k+\frac{1}{2}})^{\frac{1}{n}}$$

$$> e(1-\frac{a}{i+13a})\frac{1}{i+\frac{1}{2}} - \sum_{n=i}^{m}\frac{1}{n}(\prod_{k=1}^{n}\frac{1}{k+\frac{1}{2}})^{\frac{1}{n}}$$
$$> e(1-\frac{a}{i+13a})\frac{1}{i+\frac{1}{2}} - \sum_{n=i}^{\infty}\frac{1}{n}(\prod_{k=1}^{n}\frac{1}{k+\frac{1}{2}})^{\frac{1}{n}}. \tag{2.23}$$

From Lemmas 2.1 and 2.4 together with (2.23) we get

$$f(b_1, b_2, \cdots, b_m) \geq f(B_m, B_m, \cdots, B_m). \tag{2.24}$$

Therefore, inequality (2.22) follows easily from (2.24).

**Lemma 2.7**. Let $m \in \mathbb{N}$ be a positive natural number, $a_i \geq 0$ $(i = 1, 2, \cdots, m)$, $B_m = \min_{1\leq n\leq m}\{na_n\}$, $b = ([4\sqrt{2}-3)e-6]/[(3-2\sqrt{2})e] = 2.6203\cdots$ and $c = (e/2-1)(1+b) = 1.3002\cdots$. Then

$$e\sum_{n=1}^{m}a_n - \sum_{n=1}^{m}\left[(1+\frac{c}{n+b})(\prod_{k=1}^{n}a_k)^{\frac{1}{n}}\right]$$
$$\geq B_m\left[e\sum_{n=1}^{m}\frac{1}{n} - \sum_{n=1}^{m}\frac{1}{(n!)^{\frac{1}{n}}}(1+\frac{c}{n+b})\right]. \tag{2.25}$$

*Proof.* We clearly see that inequality (2.25) becomes equality for $m = 1$. Suppose that $m \geq 2$ and $b_n = na_n$ $(n = 1, 2, \cdots, m)$, then inequality (2.25) is equivalent to

$$e\sum_{n=1}^{m}\frac{b_n}{n} - \sum_{n=1}^{m}\left[(1+\frac{c}{n+b})(\frac{1}{n!}\prod_{k=1}^{n}b_k)^{\frac{1}{n}}\right]$$
$$\geq B_m\left[e\sum_{n=1}^{m}\frac{1}{n} - \sum_{n=1}^{m}(\frac{1}{(n!)^{1/n}}(1+\frac{c}{n+b}))\right]. \tag{2.26}$$

Let

$$f : \mathbf{b} = (b_1, b_2, \cdots, b_m) \in (0, +\infty)^m \to e\sum_{n=1}^{m}\frac{b_n}{n} - \sum_{n=1}^{m}[(1+\frac{c}{n+b})(\frac{1}{n!}\prod_{k=1}^{n}b_k)^{1/n}].$$

Then for any $\mathbf{b} \in D_i$ $(i = 1, 2, \cdots, m)$, we have

$$\frac{\partial f(\mathbf{b})}{\partial b_i} = \frac{e}{i} - \sum_{n=i}^{m}[\frac{1}{nb_i}(1+\frac{c}{n+b})(\frac{1}{n!}\prod_{k=1}^{n}b_k)^{1/n}]$$
$$> \frac{e}{i} - \sum_{n=i}^{m}[\frac{1}{nb_i}(1+\frac{c}{n+b})(\frac{1}{n!}\prod_{k=1}^{n}b_i)^{\frac{1}{n}}]$$
$$= \frac{e}{i} - \sum_{n=i}^{m}[\frac{1}{n(n!)^{\frac{1}{n}}}(1+\frac{c}{n+b})]$$
$$> \frac{e}{i} - \sum_{n=i}^{\infty}[\frac{1}{n(n!)^{\frac{1}{n}}}(1+\frac{c}{n+b})]. \tag{2.27}$$

It follows from Lemma 2.1 and (2.15) together with (2.27) that

$$f(b_1, b_2, \cdots, b_n) \geq f(B_m, B_m, \cdots, B_m). \tag{2.28}$$

Therefore, inequality (2.26) follows from (2.28).

## 3. Proofs of Theorems 1.1 and 1.2

***Proof of Theorem 1.1***. Let $B_m = \min\limits_{1\leq n\leq m}\{(n+\frac{1}{2})a_n\}$ and

$$f(i) = e\sum_{n=1}^{i}\frac{n+12a}{(n+13a)(2n+1)} - \sum_{n=1}^{i}\frac{2(n!)^{\frac{1}{n}}}{((2n+1)!)^{\frac{1}{n}}}, (i=1,2,\cdots,m).$$

Then

$$f(1) = 0.5049\cdots > 0, \tag{3.1}$$

$$f(i+1)-f(i) = e\frac{i+1+12a}{(i+1+13a)(2i+3)} - \frac{2((i+1)!)^{\frac{1}{i+1}}}{((2i+3)!)^{\frac{1}{i+1}}}. \tag{3.2}$$

It follows from Lemma 2.3 and (3.2) that

$$\begin{aligned} &f(i+1)-f(i) > \\ &e\frac{i+1+12a}{(i+1+13a)(2i+3)} - e\frac{i+1}{2}\frac{(i+1)^2+(1+24a)(i+1)+\frac{23}{2}a+156a^2}{(i+1+13a)(i+\frac{3}{2})(i+2+13a)(i+\frac{5}{2})} \\ &= \frac{e[(\frac{5}{2}+a)(i+1)^2+(4+51a)(i+1)+\frac{3}{2}(1+12a)(1+13a)]}{(2i+3)(i+1+13a)(i+2+13a)(i+\frac{5}{2})} > 0. \end{aligned} \tag{3.3}$$

Lemma 2.6 and (3.1) together with (3.3) imply that

$$e\sum_{n=1}^{m}(1-\frac{a}{n+13a})a_n - \sum_{n=1}^{m}(\prod_{k=1}^{n}a_k)^{\frac{1}{n}} \geq 2B_m f(m) \geq 2B_m f(1) \geq 0. \tag{3.4}$$

Therefore, Theorem 1.1 follows from (3.4) with $m\to+\infty$.

***Proof of Theorem 1.2***. Let $m\in\mathbb{N}$ be a positive natural number and $B_m = \min\limits_{1\leq n\leq m}\{na_n\}$. Then we clearly see that

$$\begin{gathered} m! < (m+1)^m, \quad (m!)^{m+1} < (m!)^m(m+1)^m = ((m+1)!)^m, \\ (m!)^{\frac{1}{m}} < ((m+1)!)^{\frac{1}{m+1}}. \end{gathered} \tag{3.5}$$

From (2.14) and (3.5) we have

$$\begin{aligned} \frac{e}{m+1} &\geq \frac{1}{(m!)^{\frac{1}{m}}}(1+\frac{c}{m+b}) \\ &> \frac{1}{((m+1)!)^{\frac{1}{m+1}}}(1+\frac{c}{m+b}) \\ &> \frac{1}{((m+1)!)^{\frac{1}{m+1}}}(1+\frac{c}{m+1+b}). \end{aligned} \tag{3.6}$$

Let

$$G(m) = e\sum_{n=1}^{m}\frac{1}{n} - \sum_{n=1}^{m}[\frac{1}{(n!)^{\frac{1}{n}}}\cdot(1+\frac{c}{n+b})] \quad (m=1,2,\cdots). \tag{3.7}$$

Then (3.6) and (3.7) lead to

$$G(1) = \frac{e}{2} > 0, \tag{3.8}$$

$$G(m+1)-G(m) = \frac{e}{m+1} - \frac{1}{((m+1)!)^{\frac{1}{m+1}}}(1+\frac{c}{m+1+b}) > 0. \tag{3.9}$$

From Lemma 2.7 and (3.7)-(3.9) we get

$$e\sum_{n=1}^{m} a_n - \sum_{n=1}^{m}\left[(1+\frac{c}{n+b})(\prod_{k=1}^{n} a_k)^{\frac{1}{n}}\right] > \frac{e}{2}B_m \geq 0. \tag{3.10}$$

Therefore, Theorem 1.2 follows from (3.10) with $m \to +\infty$.

## References


[1] T. Carleman, Sur les fonctions quasi-analytiques, Comptes rendus du $V^e$ Congres des Mathematiciens Scandinaves, Helsingfors, 1922, 181-196.

[2] H. Alzer, On Carleman's inequality, Portugal. Math., 1993, **50**(3): 331-334.

[3] B.-C. Yang and L. Debnath, Some inequalities involving the constant $e$, and an application to Carleman's inequality, J. Math. Anal. Appl., 1998, **223**(1): 347-353.

[4] H. Alzer, A refinement of Carleman's inequality, J. Approx. Theory, 1998, **95**(3): 497-499.

[5] P. Yan and G.-Z. Sun, A strengthened Carleman's inequality, J. Math. Anal. Appl., 1999, **240**(1): 290-293.

[6] B.-C. Yang and L. Debnath, On a genealization of Carleman's integral inequality, J. Pure Math., 1999, **16**: 1-7.

[7] X.-J. Yang, On Carleman's inequality, J. Math. Anal. Appl., 2001, **253**(2): 691-694.

[8] J. Pečarić and K. B. Stolarsky, Carleman's inequality: history and new generalizations, Aequationes Math., 2001, **61**(1): 49-62.

[9] Y.-H. Kim, Refinements of Carleman's inequality and its genealizations, Indian J. Pure Appl. Math., 2002, **33**(12): 1775-1784.

[10] G.-S. Wang and L.-J. Wang, The Carlemam inequality and its application to periodic optimal control governed by semilinear parabolic differential equations, J. Optim. Theory Appl., 2003, **118**(2): 429-461.

[11] A. Čižmešija, J. Pečarić and L.-E. Persson, On strengthened weighted Carleman's inequality, Bull. Austral. Math. Soc., 2003, **68**(3): 481-490.

[12] D.-Y. Hwang, Some refinements and genealizations of Carleman's inequality, Int. J. Math. Math. Sci., 2004, **41-44**: 2171-2180.

[13] C.-P. Chen, W.-S. Cheung and F. Qi, Note on weighted Carleman-type inequality, Int. J. Math. Math. Sci., 2005, **3**: 475-481.

[14] B.-C. Yang, On a relation between Carleman's inequality and Van der Corput's inequality, Taiwanese J. Math., 2005, **9**(1): 143-150.

[15] B.-C. Yang, On a Hardy-Carleman's type inequality, Taiwanese J. Math., 2005, **9**(3): 469-475.

[16] J. Cao, D.-W. Niu and F. Qi, A refinement of Carleman's inequality, Adv. Stud. Contemp. Math., 2006, **13**(1): 57-62.

[17] H. Yue, A strengthened Carleman's inequality, Commun. Math. Anal., 2006, **1**(2): 115-119.

[18] H.-P. Liu and L. Zhu, New strengthened Carleman's inequality and Hardy's inequality, J. Ineqnal. Appl., **2007**, Article ID 84101, 7 pages.

[19] Z.-X. Lü, Y.-C. Gao and Y.-X. Wei, Note on the Carleman's inequality and Hardy's inequality, Comput. Math. Appl., 2010, **59**(1): 94-97.

[20] B.-Q. Yuan, Refinements of Carlemean's inequality, JIPAM. J. Inequal. Pure Appl. Math., 2001, **2**(2), Article 21, 4 pages.

[21] B.-C. Yang and D.-C. Li, A strengthened Carleman's inequality, J. Math. Tech., 1998, **14**(1): 130-133 (in Chinese).

[22] X.-M. Zhang and Y.-M. Chu, A new method to study analytic inequalities, J. Inequal. Appl., **2010**, Article ID 698012, 13 pages.

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