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APPLICATIONS OF HAHN-BANACH PRINCIPLE TO THE MOMENT PROBLEM

Octav Olteanu

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Abstract. The first aim of this review paper is to show how Hahn-Banach type results can be applied in several aspects related to the multidimensional real moment problem. Here the main tools are constrained extension theorems for linear functionals or operators. An application using suitable L^1 -polynomial approximation on unbounded subsets is also discussed. One uses approximation by sums of tensor products of positive polynomials in each separate variable. This way, one solves the difficulty created by the fact that there are positive polynomials, which are not writable as sums of squares in several dimensions. One characterizes the existence of a solution in terms of computable quadratic forms, similarly to the one-dimensional case. This is the second aim of the paper. The third aim consists in proving applications related to the connection of the moment problem with the extreme "points" and some inverse problems, similar to those solved in the literature by some other methods. Remarks on the multiplicative solutions of some Markov moment problems are recalled in the end of this work.

1. Introduction

The first part of this paper recalls some results on the extension of linear operators, usually applied in solving moment problems. The main tools are constrained extension theorems for linear functionals or operators. The upper constraint measures the continuity and the norm of the solution. The lower constraint is usually the positivity of the solution on the positive cone of the domain - space. Possible representation by means of a positive scalar or vector measure works for such a solution. Applications to the moment problem are deduced in Section 4. This is the first aim of the paper. On the other hand, we point out the importance of approximation methods in several dimensions, due to the fact that the "form" of positive polynomials on simple unbounded subsets is unknown. Polynomial decomposition and approximation in studying the existence and uniqueness of the solutions of some moment problems have been studied

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in [1] - [5], [7], [8], [11]-[15], [18]-[24], [26]. Finding explicit form of positive polynomials on a semi-algebraic set is important in itself, and especially in the moment problem. For the moment problem and representation of positive polynomials on a multidimensional compact set see [4], [23], [26]. In [4], geometric functional analysis results are applied. Namely, Krein-Milman and Hahn-Banach theorem in finite dimensional spaces are the main tools. Firstly, one discuss the existence of the solution of the moment problem, from which one deduces the decomposition of positive polynomials on a compact subset. The case of non-semi-algebraic compacts is treated as well. Similar results are obtained in [26], partially by using a different proof (moment problem and semi-algebraic geometry), only for semi-algebraic compacts. In order to obtain the decomposition of any positive polynomial on the corresponding compact, a Hahn-Banach separation theorem for convex sets is used in the end. In [23], the author uses the results from [4], [26], also applying algebraic and functional analysis methods. In particular, one obtains new decomposition of special positive polynomials on semi-algebraic compacts. In order to solve the moment problem on some unbounded multidimensional sets, an idea is to approximate nonnegative continuous compactly supported functions of several variables, in L^1 - norm, (on a closed unbounded subset), by means of sums of tensor products of positive polynomials in each separate variable. The analytic expression for such polynomials as sums of squares is well - known cf. [1]. Thus, the difficulty created by the existence of positive polynomials in several variables, which are not writable by means of sums of squares, is treated in Section 3. These facts lead to a characterization of the existence of the solution of the multidimensional moment problem in terms of computable quadratic forms, partially similar to the one-dimensional case. This is the second aim of this work. A similar approximation result used in solving the complex moment problem appears in [13]. The proofs in the real case are very different with respect to those from [13]. Applications of the notion of an extreme "point", inverse problems, truncated moment problems and multiplicative solutions are the other aims of the paper. The background of this work is contained in [1], [6], [10], [16]-[18], [25]. The uniqueness of the solution has been discussed in [1]-[3], [5], [7], [28]. In [28], a probabilistic approach of this subject is pointed out. The Hahn - Banach principle and its generalizations, as well as operators in order structure setting [5], [6], [16]-[18], represent a common characteristic of this work. All the theorems on the moment problem which are proved in this work are using extension - type results for linear operators or separation results. The paper is organized as follows. Section 2 contains the main extension results on linear operators, applied along this work. Section 3 contains solutions for some multidimensional moment problems. Polynomial approximation on unbounded subsets and their applications are discussed and applied. Sometimes, we have considered two dimensional moment problems in order to simplify the notations and to fix the ideas. However, the method can be adapted to work in arbitrary several variables. Most of the results in Section 4 are applications of generalizations of Hahn-Banach theorems to operator-valued moment problems. Section 5 is devoted to the relationship between the truncated moment problem and the full moment problem (see also [27]). It contains also applications of extreme "points" and an inverse problem similar to that discussed in [8]. In section 6 (the last one), some remarks on multiplicative solutions [21] are briefly mentioned.

2. Hahn-Banach type results and their generalizations

Given a closed subset A of $\mathbb{R}^n$ and $(y_j)_{j \in \mathbb{Z}_+^n}$ a sequence of real numbers, the A -moment problem asks whether this sequence can be realized as the moment sequence of some positive Borel measure μ on A . One defines the linear form L on the subspace of polynomials, such that the moment conditions

$$L(\varphi_j) = y_j, \quad \varphi_j(t_1, \dots, t_n) = t_1^{j_1} \dots t_n^{j_n}, \quad j = (j_1, \dots, j_n) \in \mathbb{Z}_+^n \quad (2.1)$$

be accomplished. In other words, the question is to characterize those linear forms $L : \mathbb{R}[t_1, \dots, t_n] \rightarrow \mathbb{R}$ for which there exists a positive Borel measure μ on A whose moments exist and satisfy

$$\int_A f(t) d\mu = L(f), \quad \forall f \in \mathbb{R}[t_1, \dots, t_n] \quad (2.2)$$

If such a representing measure does exist, we say that the A -moment problem is solvable for L . A characterizations of the linear forms for which the A -moment problem is solvable is given by the following classical result.

Theorem 2.1 (Haviland [9]). . *The A -moment problem is solvable for L if and only if $L(p) \geq 0$ for every polynomial p which is nonnegative on A .*

The following results will be also used along this work. For the next result, we give a short proof.

Theorem 2.2 (see [5-6]). . *Let X be an ordered vector space, X_+ its positive cone, and X_0 a vector subspace such that $X = X_0 - X_+$. Let Y be an order complete vector lattice. If $f_0 : X_0 \rightarrow Y$ is linear and positive, then there exists a linear positive extension $f : X \rightarrow Y$ of the operator f_0 .*

Proof. The relation $X = X_0 - X_+$ is equivalent to $X = X_0 + X_+$ using multiplication by (-1) . If $x \in X$, $p_1 \in X_0$, $p_1 \leq x$, $p_2 \in X_0$, $p_2 \geq x$, then $f_0(p_2) \geq f_0(p_1)$. This remark allows introducing the following operators:

$$T(x) = \inf\{f_0(p_2); p_2 \in X_0, p_2 \geq x\},$$

$$\Psi(x) = \sup\{f_0(p_1); p_1 \in X_0, p_1 \leq x\}, \quad x \in X.$$

Then T is sublinear, Ψ is supra-linear ($-\Psi$ is sublinear) and $\Psi \leq T$ on X . By the Hahn-Banach theorem, there exists a linear operator f , such that $\Psi \leq f \leq T$ on X . Observe that $T|_{X_0} = \Psi|_{X_0} = f_0$, so that f is actually an extension of f_0 . On the other hand, Ψ is nonnegative on X_+ , by its definition. The conclusion is: $f|_{X_+} \geq \Psi|_{X_+} \geq 0$, so that f is positive. This concludes the proof. $\square$

Theorem 2.2 is sometimes called lemma of the majorizing subspace.

Remark 2.3. In the particular case $Y = \mathbb{R}$, the Hahn-Banach theorem invoked in the proof of Theorem 2.2 is a separation theorem for two disjoint convex sets.

Remark 2.4. Following [5, p. 4-6], it seems that Theorem 2.1 can be deduced from Theorem 2.2, via the properties of the subspace of polynomials on a closed subset.

Using more general extension-type results for linear operators, with two constraints [16-17], one can prove the following characterizations of the existence of the solutions of some abstract moment problems [18].

Theorem 2.5. *Let X be an ordered vector space, Y an order complete vector lattice, $\{x_j\}_{j \in J} \subset X$, $\{y_j\}_{j \in J} \subset Y$ given families and $F_1, F_2 \in L(X, Y)$ two linear operators. The following statements are equivalent:*

(a) *there is a linear operator $F \in L(X, Y)$ such that*

$$F_1(x) \leq F(x) \leq F_2(x) \quad \forall x \in X_+, \quad F(x_j) = y_j \quad \forall j \in J;$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}$ we have:*

$$\left(\sum_{j \in J_0} \lambda_j x_j = \psi_2 - \psi_1, \quad \psi_1, \psi_2 \in X_+ \right) \Rightarrow \sum_{j \in J_0} \lambda_j y_j \leq F_2(\psi_2) - F_1(\psi_1).$$

Theorem 2.6. *Let X, Y be as in Theorem 2.5, $p : X \rightarrow Y$ be a convex operator, $\{x_j\}_{j \in J} \subset X$, $\{y_j\}_{j \in J} \subset Y$ given families. The following statements are equivalent:*

(a) *there exists a linear positive operator $F : X \rightarrow Y$ such that*

$$F(x_j) = y_j \quad \forall j \in J, \quad F(x) \leq p(x) \quad \forall x \in X;$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}$ we have:*

$$\sum_{j \in J_0} \lambda_j x_j \leq x \Rightarrow \sum_{j \in J_0} \lambda_j y_j \leq p(x).$$

3. Polynomial approximation and the Markov moment problem

In this section, we follow some general ideas in [22]. Most of the results of this section involve polynomial approximation. For the next result we need the following lemma [15], [22].

Lemma 3.1. *Let $A \subset \mathbb{R}^n$ be a closed unbounded subset and ν a positive regular M -determinate Borel measure [25] on A , with finite moments of all orders. Then for any non-negative continuous vanishing at infinity function $\psi \in (C_0(A))_+$, there is a sequence $(p_m)_m$ of polynomials on A , $p_m \geq \psi$, $p_m \rightarrow \psi$ in $L_\nu^1(A)$. We have*

$$\lim \int_A p_m d\nu = \int_A \psi d\nu,$$

the cone P_+ of positive polynomials is dense in $(L_\nu^1(A))_+$ and P is dense in $L_\nu^1(A)$.

Recall that a determinate (M -determinate) measure is uniquely determinate by its moments, or, equivalently, by its values on polynomials. Notice that the proof of Lemma 3.1 is using both theorems 2.1 and 2.2. The following statement holds for any closed unbounded subset $A \subset \mathbb{R}^n$, hence does not depend on the form of positive polynomials on A .

Theorem 3.2. *Let A be a closed unbounded subset of $\mathbb{R}^n$, Y an order complete Banach lattice, $(y_j)_{j \in \mathbb{Z}_+^n}$ a given sequence, ν a positive regular determinate Borel measure on A , with finite moments of all orders. Let $F_2 \in B(L_\nu^1(A), Y)$ be a linear positive bounded operator from $L_\nu^1(A)$ into Y . The following statements are equivalent:*

(a) *there exists a linear operator $F \in B(L_\nu^1(A), Y)$ such that $F(\varphi_j) = y_j$, $j \in \mathbb{Z}_+^n$, F is between 0 and F_2 on the positive cone of $L_\nu^1(A)$, and $\|F\| \leq \|F_2\|$, where φ_j , $j \in \mathbb{Z}_+^n$, are defined in (2.1);*

(b) for any finite subset $J_0 \subset \mathbb{Z}_+^n$ and any $\{a_j\}_{j \in J_0} \subset \mathbb{R}$, we have

$$\sum_{j \in J_0} a_j \varphi_j \geq 0 \quad \text{on } A \Rightarrow 0 \leq \sum_{j \in J_0} a_j y_j \leq \sum_{j \in J_0} a_j F_2(\varphi_j).$$

Proof. Let F_0 be the linear operator defined on the subspace of polynomials, such that the moment conditions similar to those from (2.1) to be accomplished, where F_0 stands for L . Then condition (b) says that for any polynomial p which is nonnegative on A , we have

$$0 \leq F_0(p) \leq F_2(p) \quad (3.1)$$

Hence the implication (a) $\Rightarrow$ (b) is obvious. In order to prove the converse, one applies lemma 3.1. Let ψ be a continuous nonnegative compactly supported function on A , and $(p_m)_m$ a sequence of polynomials given by Lemma 3.1. Then all polynomials p_m , $m \in \mathbb{Z}_+$ are nonnegative on A , so that they verify (3.1). On the other hand, there exists a linear positive extension F of F_0 to the space of all functions in $L_\nu^1(A)$ whose absolute values are dominated by a polynomial. This space contains the subspace of continuous compactly supported functions. Observe also that for any linear positive continuous functional y^* on Y , $y^* \circ F$ can be represented by means of a positive Borel measure on A , for which Fatou lemma works. Using (3.1), one deduces

$$y^*(F(\psi)) \leq \liminf y^*(F(p_m)) \leq \lim y^*(F_2(p_m)) = y^*(F_2(\psi)) \quad (3.2)$$

Assume that $F_2(\psi) - F(\psi) \notin Y_+$. Using a separation argument, it should exist a linear positive continuous functional y^* on Y , such that $y^*(F_2(\psi) - F(\psi)) < 0$, that is

$$y^*(F_2(\psi)) < y^*(F(\psi)).$$

This contradicts (3.2), so that we must have

$$0 \leq F(\psi) \leq F_2(\psi), \quad \forall \psi \in (C_c(A))_+.$$

If φ is an arbitrary continuous compactly supported function, then

$$|F(\varphi)| \leq F(|\varphi|) \leq F_2(|\varphi|).$$

Since the norm on Y is solid, the last inequalities imply

$$\|F(\varphi)\| \leq \|F_2(|\varphi|)\| \leq \|F_2\| \cdot \|\varphi\|_1, \quad \forall \varphi \in C_c(A).$$

Hence $\|F\|$ is dominated by $\|F_2\|$ on a dense subspace of $L_\nu^1(A)$. Now the bounded positive linear operator F , having the norm-property mentioned above, has a unique extension to the whole space $L_\nu^1(A)$, preserving its properties. This concludes the proof. $\square$

Next we give the "scalar version" of Theorem 3.2.

Theorem 3.3. Let A, ν be as in Lemma 3.1, and $(y_j)_{j \in \mathbb{Z}_+^n}$ a multi-sequence of real numbers. The following statements are equivalent:

- (a) there exists $h \in L_\nu^1(A)$, $0 \leq h \leq 1$ on A , such that $\int_A \varphi_j \cdot h d\nu = y_j$, $j \in \mathbb{Z}_+^n$;
- (b) for any finite subset $J_0 \subset \mathbb{Z}_+^n$, and any $\{a_j\}_{j \in J_0} \subset \mathbb{R}$, we have

$$\sum_{j \in J_0} a_j \varphi_j \geq 0 \quad \text{on } A \Rightarrow 0 \leq \sum_{j \in J_0} a_j y_j \leq \sum_{j \in J_0} a_j \int_A \varphi_j d\nu.$$

Note that for particular sets A , for which the form of positive polynomials on A in terms of sums of squares is known, one can give a characterization in terms of "computable" quadratic forms or mappings. For example, if $n = 1$, $A = [0, \infty)$, using the form of positive polynomials on $[0, \infty)$ [1], one obtains:

Theorem 3.4. *Let ν be as in Lemma 3.1, on $[0, \infty)$, $Y, (y_j)_{j \in \mathbb{Z}_+}$, F_2 be as in Theorem 3.2. The following statements are equivalent:*

- (a) *there exists a linear operator $F \in B(L_\nu^1([0, \infty)), Y)$ such that $F(\varphi_j) = y_j$, $j \in \mathbb{Z}_+$, F is between 0 and F_2 on the positive cone of $L_\nu^1([0, \infty))$, and $\|F\| \leq \|F_2\|$, where φ_j , $j \in \mathbb{Z}_+$ are defined in (2.1) for $n = 1$;*
- (b) *for any finite subset $J_0 \subset \mathbb{Z}_+$, and any $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}$, we have*

$$0 \leq \sum_{i,j \in J_0} \lambda_i \lambda_j y_{i+j+k} \leq \sum_{I,j \in J_0} F_2(\varphi_{i+j+k}), \quad k \in \{0, 1\}.$$

Our next goal is to give similar statements to that of Theorem 3.4 in several dimensions, where the form of positive polynomials on $\mathbb{R}^n$, $\mathbb{R}_+^n$ in terms of sums of squares is unknown. We will prove the results for the two-dimensional case, although the method partially works in arbitrary several dimensions. In order to do this, we need the following approximation result.

Lemma 3.5. *The subset of sums of tensor products $p_1 \otimes p_2$ of positive polynomials in separate variables t_1, t_2 is dense in $(C_c(\mathbb{R} \times \mathbb{R}))_+$, the cone of all nonnegative continuous compactly supported functions on $\mathbb{R}^2$ for any measure $\nu = \nu_1 \times \nu_2$, where ν_j , $j = 1, 2$ are positive regular M -determinate Borel measures on $\mathbb{R}$, with finite moments of all natural orders.*

Proof. If K is the support of a function $\psi \in C_c(\mathbb{R}^2)$, $\psi \geq 0$, then

$$K \subset K_1 \times K_2, \quad K_j = pr_j(K), \quad j = 1, 2.$$

Consider a rectangle R_2 containing the above Cartesian product of compacts and apply approximation of ψ on the rectangle by corresponding Bernstein polynomials of two variables, by Luzin's theorem. Next, approximate this continuous function on the rectangle by the corresponding Bernstein polynomials in two variables. Each term of such a polynomial is a tensor product $p_1 \otimes p_2$, of positive polynomials in each variable. Extend each p_j such that it vanishes outside $pr_j(R_2)$, applying then Luzin's theorem once more, this time in one variable, $j = 1, 2$. This procedure does not change the values of p_j on K_j , $j = 1, 2$. One obtains approximation by sums of tensor products of positive continuous compactly supported functions, in each variable. The approximating process holds in L^1 norm and uniformly on K . Now application of lemma 3.1 to $n = 1$, $A = \mathbb{R}$, leads to approximation of each such function in each separate variable by a dominating (positive) polynomial, in the space $L_{\nu_1}^1(\mathbb{R})$ respectively in $L_{\nu_2}^1(\mathbb{R})$. The conclusion follows. $\square$

Theorem 3.6. *Let $\nu = \nu_1 \times \nu_2$ be a product measure on $\mathbb{R}^2$, where ν_k , $k = 1, 2$ are σ -finite regular positive M -determinate Borel measures with finite moments of all natural orders on $\mathbb{R}$. Let $(y_{(j,k)})_{(j,k) \in \mathbb{Z}_+^2}$ be a sequence in Y , $F_2 : X = L_\nu^1(\mathbb{R}^2) \rightarrow Y$ a linear positive bounded operator. The following statements are equivalent:*

(a) there exists a unique linear bounded operator $F \in B(X, Y)$ such that

$$\begin{aligned} F(\varphi_{j,k}) &= y_{(j,k)}, \varphi_{j,k}(t_1, t_2) = t_1^j t_2^k, \forall (j, k) \in \mathbb{Z}_+^2, \\ 0 \leq F(\psi) &\leq F_2(\psi), \psi \in X_+, \|F\| \leq \|F_2\|; \end{aligned}$$

(b) for any finite subset $J_0 \in \mathbb{N}^2$, and any $\{\alpha_{j,k}\}_{(j,k) \in J_0} \subset \mathbb{R}$ the following implication holds:

$$\sum_{(j,k) \in J_0} \alpha_{(j,k)} \varphi_{(j,k)}(t_1, t_2) \geq 0 \quad \forall (t_1, t_2) \in \mathbb{R}^2 \Rightarrow \sum_{(j,k) \in J_0} \alpha_{(j,k)} y_{(j,k)} \in Y_+;$$

for any finite subsets $J_1, J_2 \subset \mathbb{Z}_+$ and all

$$\{a_j\}_{j \in J_1} \subset \mathbb{R}, \quad \{b_m\}_{m \in J_2} \subset \mathbb{R},$$

one has:

$$\sum_{i,j \in J_1} a_i a_j \sum_{m,n \in J_2} b_m b_n y_{(i+j, m+n)} \leq \sum_{i,j \in J_1} a_i a_j \sum_{m,n \in J_2} b_m b_n F_2(\varphi_{i+j, m+n})$$

Proof. The implication (a) $\Rightarrow$ (b) is almost obvious. In order to prove the converse, one uses the density of sums of tensor products of positive polynomials in each one separate variable in the positive cone of $C_c(\mathbb{R}^2)$ (lemma 3.5). Using also the analytic expression of positive polynomials on $\mathbb{R}$ [1], one observes that (b) says that the linear operator verifying the moment conditions is positive on the cone of non-negative polynomials on $\mathbb{R}^2$ and dominated by F_2 on the cone generated by tensor products of positive polynomials in separate variables $t_1 \in \mathbb{R}$, respectively $t_2 \in \mathbb{R}$. In particular, this linear operator has a positive extension F to the space of all integrable functions dominated in absolute value by a polynomial, on $\mathbb{R}^2$ (theorem 2.2). This space contains the space of continuous compactly supported functions. Let ψ be a nonnegative continuous compactly supported function. By the preceding lemma 3.5., one approximates ψ on a rectangle containing $pr_1(\text{support } \psi) \times pr_2(\text{support } \psi)$ by means of Luzin's theorem and the corresponding Bernstein polynomials in two variables. Then one approximates ψ by sums of tensor products of positive polynomials in each separate variables:

$$\sum_{j=0}^{k(m)} p_{m,1,j} \otimes p_{m,2,j} \rightarrow \psi, \quad m \rightarrow \infty,$$

in the space $L^1_\nu(\mathbb{R} \times \mathbb{R})$ (lemma 3.5). Using (b) and applying Fatou's lemma [25], for any linear positive (continuous) functional y^* on Y one obtains:

$$\begin{aligned} 0 \leq y^*(F(\psi)) &\leq \liminf_m (y^* \circ F) \left(\sum_{j=0}^{k(m)} p_{m,1,j} \otimes p_{m,2,j} \right) \leq \\ \lim_m (y^* \circ F_2) \left(\sum_{j=0}^{k(m)} p_{m,1,j} \otimes p_{m,2,j} \right) &= y^*(F_2(\psi)), \\ \psi &\in (C_c(\mathbb{R}^2))_+, \quad \forall y^* \in Y_+^*. \end{aligned} \tag{3.3}$$

Assume that

$$F_2(\psi) - F(\psi) \notin Y_+.$$

Using a separation theorem, it should exist a positive linear continuous functional $y^* \in Y_+^*$ such that

$$y^*(F_2(\psi) - F(\psi)) < 0,$$

that is $y^*(F_2(\psi)) < y^*(F(\psi))$. This relation contradicts (3.3). The conclusion is that we must have

$$F(\psi) \leq F_2(\psi), \quad \psi \in (C_c(\mathbb{R} \times \mathbb{R}))_+.$$

Then for arbitrary $g \in C_c(\mathbb{R} \times \mathbb{R})$ one writes

$$|F(g)| \leq F_2(|g|) \Rightarrow \|F(g)\| \leq \|F_2\| \cdot \|g\|_1.$$

The conclusion is that the operator F is positive and continuous, of norm dominated by F_2 , on a dense subspace of $L_\nu^1(\mathbb{R}^2)$. It has a unique linear extension preserving its properties. This concludes the proof. $\square$

Let

$$\begin{aligned} X &= \{x : \mathbb{R}^2 \rightarrow \mathbb{R}; x \text{ continuous}, \\ &\exists p \in P \text{ s.t. } |x(t_1, t_2)| \leq p(t_1, t_2) \forall (t_1, t_2) \in \mathbb{R}^2\}, \\ X_2 &= \{x \in X; x(t_1, t_2) = x(\pm t_1, \pm t_2), (t_1, t_2) \in \mathbb{R}^2\}, \end{aligned}$$

where P is the vector space of all polynomial functions on $\mathbb{R}^2$. If H is a Hilbert space and $A_1, A_2 \in A(H)$ are two bounded positive commuting selfadjoint operators, one defines

$$Y_1 = Y_1(A_1, A_2) = \{T \in A(H); TA_j = A_jT, j = 1, 2\},$$

$$Y = \{U \in Y_1; TU = UT \forall T \in Y_1\}.$$

Then Y endowed with the usual order relation on selfadjoint operators is an order-complete Banach lattice [6], [12] and a commutative Banach algebra.

Theorem 3.7. (see [15], [20]). *Let $\{B_{(2m, 2n)}\}_{(m, n) \in \mathbb{Z}_+^2} \subset Y(A_1, A_2)$ be a given sequence. The following statements are equivalent:*

(a) *there exists $F \in L(X, Y)$ such that*

$$\begin{aligned} F(x_{(2m, 2n)}) &= B_{(2m, 2n)}, (m, n) \in \mathbb{Z}_+^2, F(x_{(k, l)}) = 0, \forall (k, l) \in (\mathbb{Z}_+)^2 \setminus (2\mathbb{Z}_+)^2, \\ 0 \leq F(x_2) &\leq \int_{S(A_1) \times S(A_2)} x_2(t_1, t_2) dE_{A_1} dE_{A_2} \forall x_2 \in (X_2)_+, \\ |F(\varphi)| &\leq \int_{S(A_1) \times S(A_2)} |\varphi(t_1, t_2)| dE_{A_1} dE_{A_2} \forall \varphi \in X_2; \end{aligned}$$

(b) for all $J_1, J_2 \subset \mathbb{Z}_+$ finite subsets and $\forall \{a_j; j \in J_1\}, \forall \{b_m; m \in J_2\} \subset \mathbb{R}$, we have

$$\begin{aligned}
0 &\leq \sum_{i,j \in J_1} \sum_{m,n \in J_2} a_i a_j b_m b_n B_{(2i+2j, 2m+2n)} \leq \sum_{i,j \in J_1} \sum_{m,n \in J_2} a_i a_j b_m b_n A_1^{2i+2j} A_2^{2m+2n} \\
0 &\leq \sum_{i,j \in J_1} \sum_{m,n \in J_2} a_i a_j b_m b_n B_{(2i+2j+2, 2m+2n)} \leq \sum_{i,j \in J_1} \sum_{m,n \in J_2} a_i a_j b_m b_n A_1^{2i+2j+2} A_2^{2m+2n} \\
0 &\leq \sum_{i,j \in J_1} \sum_{m,n \in J_2} a_i a_j b_m b_n B_{(2i+2j, 2m+2n+2)} \leq \sum_{i,j \in J_1} \sum_{m,n \in J_2} a_i a_j b_m b_n A_1^{2i+2j} A_2^{2m+2n+2} \\
0 &\leq \sum_{i,j \in J_1} \sum_{m,n \in J_2} a_i a_j b_m b_n B_{(2i+2j+2, 2m+2n+2)} \leq \\
&\leq \sum_{i,j \in J_1} \sum_{m,n \in J_2} a_i a_j b_m b_n A_1^{2i+2j+2} A_2^{2m+2n+2}
\end{aligned}$$

The next goal is to state a result on the existence of the solution of the Markov moment problem on a semi-algebraic compact K . Let

$$K = \{t \in \mathbb{R}^n; r_j(t) \geq 0, j = 1, \dots, m\}, \quad r_1, \dots, r_m \in \mathbb{R}[t_1, \dots, t_n], \quad (3.4)$$

and assume that K is compact. Such a compact subset is called a semi-algebraic compact. Let $\sum$ denote the set of all finite sums of elements p^2 and $p^2 r_{j_1}, \dots, r_{j_q}$, where $p \in \mathbb{R}[t_1, \dots, t_n]$ and $j_1, \dots, j_q \in \{1, \dots, m\}$. Notice that $\sum$ is a convex cone in $\mathbb{R}[t_1, \dots, t_n]$ and is closed under multiplication of polynomials.

We recall the real variant of the main result on the moment problem of [26]. Using geometric variant of Hahn-Banach principle, the form of the positive polynomials on K is deduced in [26] (Corollary 3.9 from below; see also [4], [23], [18]).

Theorem 3.8. ([26]). *For a multi-sequence $(y)_{j \in \mathbb{Z}_+^n}$ of real numbers, the following statements are equivalent*

(a) *there exists a positive linear form F on $X = C(K)$ such that the moment conditions*

$$F(\varphi_j) = y_j, \quad j \in \mathbb{Z}_+^n$$

are accomplished (where φ_j are the basic polynomials defined in (2.1));

(b) *for any finite subset $J_0 \subset \mathbb{Z}_+^n$ any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$ and any*

$$p_q(t) = r_{i_1}(t) \dots r_{i_q}(t) = \sum_{k \in F} \alpha_k t^k, \quad \{i_1, \dots, i_q\} \subset \{1, \dots, m\}, \quad (3.5)$$

(where $i_{s_1} \neq i_{s_2}$ for $s_1 \neq s_2$, and $F \subset \mathbb{Z}_+^n$ is a finite subset), we have

$$0 \leq \sum_{i,j \in J_0} \lambda_i \lambda_j y_{i+j}, \quad 0 \leq \sum_{i,j \in J_0} \lambda_i \lambda_j \left(\sum_{k \in F} \alpha_k y_{i+j+k} \right)$$

Corollary 3.9. *If a polynomial function p is positive on K , then $p \in \sum$.*

Using the form of positive polynomials on semi-algebraic compacts given by Corollary 3.9, we prove the following applications to the Markov moment problem for operators [14]. Let Y be an order-complete vector lattice endowed with a linear topology such that the positive cone Y_+ is closed and normal. Let $F_2 \in L_+(C(K), Y)$ be a linear

positive operator (note that any such operator is also bounded). Finally, let $(y_j)_{j \in \mathbb{Z}_+^n}$ be a sequence in Y .

Theorem 3.10 (Theorem 2.2 [14]). . *The following statements are equivalent*

(a) *there exists a linear bounded operator $F \in B(C(K), Y)$ such that*

$$F(\varphi_j) = y_j, \quad j \in \mathbb{Z}_+^n, \quad 0 \leq F(\varphi) \leq F_2(\varphi), \quad \forall \varphi \in (C(K))_+$$

(b) *for any finite subset $J_0 \subset \mathbb{Z}_+^n$, any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, and any polynomial p_q given by (3.5), we have*

$$0 \leq \sum_{i,j \in J_0} \lambda_i \lambda_j y_{i+j} \leq \sum_{i,j \in J_0} \lambda_i \lambda_j F_2(\varphi_{i+j}),$$

$$0 \leq \sum_{i,j \in J_0} \lambda_i \lambda_j \left(\sum_{k \in F} \alpha_k y_{i+j+k} \right) \leq \sum_{i,j \in J_0} \lambda_i \lambda_j \left(\sum_{k \in F} \alpha_k F_2(\varphi_{i+j+k}) \right).$$

Notice that any compact K_1 in $\mathbb{R}^n$ can be imbedded in a semi-algebraic compact K . One deduces the following corollary.

Corollary 3.11. (see [14]). *Let K_1 be an arbitrary compact subset of $\mathbb{R}^n$, and K a semi-algebraic compact such that $K_1 \subset K \subset \mathbb{R}^n$. Let ν be a positive Borel regular measure on K_1 and $(y_j)_{j \in \mathbb{Z}_+^n}$ a sequence of real numbers. The following statements are equivalent*

(a) *there exists a Borel function h on K_1 , such that*

$$0 \leq h(t) \leq 1 \text{ a.e. in } K_1, \quad \int_{K_1} (\varphi_j \cdot h)(t) d\nu = y_j, \quad j \in \mathbb{Z}_+^n;$$

(b) *if K is given by (3.4), with the above notations, we have*

$$0 \leq \sum_{i,j \in J_0} \lambda_i \lambda_j y_{i+j} \leq \sum_{i,j \in J_0} \lambda_i \lambda_j \int_{K_1} \varphi_{i+j}(t) d\nu,$$

$$0 \leq \sum_{i,j \in J_0} \lambda_i \lambda_j \left(\sum_{k \in F} \alpha_k y_{i+j+k} \right) \leq \sum_{i,j \in J_0} \lambda_i \lambda_j \left(\sum_{k \in F} \alpha_k \int_{K_1} \varphi_{i+j+k}(t) d\nu \right).$$

Next we consider the "non-compact case", which can be solved by solving "countable many compact cases". Let S be a Borel subset in $\mathbb{R}^n$ such that $S = \bigcup_{m \in \mathbb{Z}} K_m$, where $K_m \subset \mathbb{R}^n$ is a compact subset $\forall m \in \mathbb{Z}$. Assume that a positive regular Borel measure ν on S is given, such that

$$\nu(K_{m_1} \cap K_{m_2}) = 0 \quad \text{if} \quad m_1 \neq m_2.$$

Theorem 3.12 (Theorem 2.4 [14]). . *Let $\{y_j; j \in \mathbb{Z}_+^n\} \subset \mathbb{R} \setminus \{0\}$. The following statements are equivalent*

(a) *there exists a Borel function h on S such that*

$$\int_S t^j h(t) d\nu = y_j, \quad j \in \mathbb{Z}_+^n, \quad 0 \leq h(t) \leq 1 \quad \text{-- a.e. in } S;$$

(b) for any $m \in \mathbb{Z}$ there exists a Borel function h_m on K_m and $\varepsilon_m \in \mathbb{R}$ such that

$$\int_{K_m} t^j h_m(t) d\nu_m = \varepsilon_m y_j, \quad j \in \mathbb{Z}_+^n, m \in \mathbb{Z}, \quad \sum_{m \in \mathbb{Z}} \varepsilon_m = 1,$$

where $\nu_m(B) := \nu(B)$ for any Borel subset $B \subset K_m$, $m \in \mathbb{Z}$.

4. Extension of linear operators and the moment problem

Here we recall some results in [19], where the reader can find related references comments on the original sources. In the next theorem X will be the space of all absolutely convergent power series

$$\varphi(z) = \sum_{j=0}^{\infty} \alpha_j z^j, \quad |z| < b, \quad X_+ = \{\varphi \in X; \alpha_j \in \mathbb{R}_+ \forall j \in \mathbb{Z}_+\}$$

One assumes that the elements of the space X are continuous in the closed disc $\{|z| \leq b\}$. We denote $\varphi_j(z) = z^j$, $|z| \leq b$, $j \in \mathbb{Z}_+$. Let H be a complex Hilbert space, U_0 a selfadjoint operator acting on H , and $Y = Y(U_0)$ the order complete vector lattice and commutative Banach algebra of selfadjoint operators defined in [6]: if U_0 is a selfadjoint operator from H into H ,

$$\begin{aligned} Y_1 &= Y_1(U_0) = \{T \in A(H); U_0 T = T U_0\}, \\ Y &= \{T \in Y_1; T U = U T, \forall U \in Y_1\}. \end{aligned} \quad (4.1)$$

Theorem 4.1. *Let X, H, U_0, Y, φ_j be as above. Assume that $b > 1$ and let $A \in Y$, $\|A\| < b$. Let $\varepsilon > 0$ and $\{B_j\}_{j \in \mathbb{Z}_+} \subset Y$. The following statements are equivalent:*

(a) *there exists a linear operator F applying X into Y such that*

$$\begin{aligned} F(\varphi_j) &= B_j \quad \forall j \in \mathbb{Z}_+, \quad \varphi(A) - \varepsilon \cdot \varphi(I) \leq F(\varphi) \leq \varphi(A) + \varepsilon \cdot \varphi(I) \quad \forall \varphi \in X_+, \\ \|F(\varphi)\| &\leq \|\varphi\| + \varepsilon \cdot |\varphi(1)|, \quad \varphi \in X; \end{aligned}$$

(b) *we have*

$$A^j - \varepsilon \cdot I \leq B_j \leq A^j + \varepsilon \cdot I, \quad j \in \mathbb{Z}_+.$$

Proof. (a) $\Rightarrow$ (b) is obvious, since from (a) we derive

$$B_j = F(\varphi_j) \in [\varphi_j(A) - \varepsilon \cdot \varphi_j(I), \varphi_j(A) + \varepsilon \cdot \varphi_j(I)] = [A^j - \varepsilon \cdot I, A^j + \varepsilon \cdot I], \quad j \in \mathbb{Z}_+.$$

(b) $\Rightarrow$ (a). One applies theorem 2.3 to $x_j = \varphi_j \in X$, $j \in N$. We have:

$$\begin{aligned} \sum_{j \in F} \lambda_j \varphi_j &= \psi_2 - \psi_1 = \sum_{j \in \mathbb{Z}_+} a_j \varphi_j - \sum_{j \in \mathbb{Z}_+} b_j \varphi_j, \quad a_j, b_j \geq 0 \Rightarrow \\ -(a_j + b_j) &\leq -b_j \leq \lambda_j \leq a_j \leq a_j + b_j \Rightarrow |\lambda_j| \leq a_j + b_j, \quad j \in F \Rightarrow \\ \lambda_j B_j &\leq \lambda_j A^j + \varepsilon \cdot \lambda_j I, \quad \forall j \in F^+, \quad \lambda_j B_j \leq \lambda_j A^j - \varepsilon \cdot \lambda_j I, \quad \forall j \in F^-, \\ F^+ &= \{j \in F; \lambda_j \geq 0\}, \quad F^- = \{j \in F; \lambda_j < 0\}. \end{aligned}$$

The preceding relations yield

$$\begin{aligned}
\sum_{j \in F} \lambda_j B_j &\leq \sum_{j \in F^+} \lambda_j A^j + \varepsilon \cdot \left(\sum_{j \in F^+} \lambda_j \right) \cdot I + \sum_{j \in F^-} \lambda_j A^j - \varepsilon \cdot \left(\sum_{j \in F^-} \lambda_j \right) \cdot I = \\
&\sum_{j \in F} \lambda_j A^j + \varepsilon \cdot \left(\sum_{j \in F} |\lambda_j| \right) \cdot I = \psi_2(A) - \psi_1(A) + \varepsilon \cdot \left(\sum_{j \in F} |\lambda_j| \right) \cdot I \leq \\
&\psi_2(A) - \psi_1(A) + \varepsilon \cdot \left(\sum_{j \in \mathbb{Z}_+} |\lambda_j| \right) \cdot I \leq \psi_2(A) - \psi_1(A) + \varepsilon \cdot \left(\sum_{j \in \mathbb{Z}_+} (a_j + b_j) \right) \cdot I = \\
&\psi_2(A) + \varepsilon \cdot \left(\sum_{j \in N} a_j \right) \cdot I - \left[\psi_1(A) - \varepsilon \cdot \left(\sum_{j \in N} b_j \right) \cdot I \right] = F_2(\psi_2) - F_1(\psi_1).
\end{aligned}$$

Theorem 2.5 implies the existence of a linear operator with respect to the real field

$$F \in L(X, Y), \quad F_1 \leq F \leq F_2 \quad \text{on} \quad X_+, \quad F(\varphi_j) = B_j \quad \forall j \in \mathbb{Z}_+.$$

One obtains

$$\begin{aligned}
|F(\varphi)| &\leq |F(\varphi_+)| + |F(\varphi_-)| \leq \varphi_+(A) + \varepsilon \cdot \varphi_+(I) + \varphi_-(A) + \varepsilon \cdot \varphi_-(I) = \\
|\varphi(A)| + \varepsilon |\varphi(I)| &\leq \|\varphi\| \cdot I + \varepsilon \cdot |\varphi(1)| \cdot I \Rightarrow \|F(\varphi)\| \leq \|\varphi\| + \varepsilon |\varphi(1)|, \quad \forall \varphi \in X.
\end{aligned}$$

This concludes the proof. $\square$

In the next theorems, the notion of a resolvent appears naturally during the proofs.

Theorem 4.2. *Let $b > 1, X, Y, \{\varphi_j\}_j$ be as above. Let*

$$A \in Y, \quad S(A) \subset (0, b), \quad \{B_j\}_{j \in \mathbb{Z}_+} \subset Y, \quad \varepsilon > 0.$$

Assume that

$$0 \leq B_j \leq A^j + \varepsilon \cdot I \quad j \in \mathbb{Z}_+$$

Then there exists a linear positive operator $F \in L_+(X, Y)$, such that

$$\begin{aligned}
F(\varphi_j) &= B_j, \quad \forall j \in \mathbb{Z}_+, \quad |F(\varphi)| \leq \|\varphi\| \cdot \left[(I - b^{-1}A)^{-1} + \varepsilon \frac{b}{b-1} I \right] \Rightarrow \\
\|F\| &\leq \frac{b}{b - \|A\|} + \varepsilon \frac{b}{b-1}.
\end{aligned}$$

Proof. The following implications hold

$$\begin{aligned}
\sum_{j \in F} \lambda_j \varphi_j &\leq \varphi = \sum_{j \in \mathbb{Z}_+} a_j \varphi_j \Rightarrow \lambda_j \leq a_j \leq |a_j| \leq \frac{\|\varphi\|_\infty}{b^j}, \quad j \in F \Rightarrow \\
\sum_{j \in F} \lambda_j B_j &\leq \sum_{j \in F_+} \lambda_j (A^j + \varepsilon \cdot I) \leq \|\varphi\| \cdot \left[\sum_{j \in \mathbb{Z}_+} b^{-j} A^j + \varepsilon \cdot \left(\sum_{j \in \mathbb{Z}_+} b^{-j} \right) \cdot I \right] = \\
\|\varphi\| \cdot \left[(I - b^{-1}A)^{-1} + \varepsilon \cdot \frac{b}{b-1} I \right] &= p(\varphi), \quad \varphi \in X.
\end{aligned}$$

Note that in the preceding relations the Cauchy inequalities for the function φ were used. Application of Theorem 2.6 leads to the existence of a linear positive operator

F from X into Y satisfying the moment conditions, such that $F \leq p$ on X . Now the conclusion follows easily, by using the monotony of the norm on Y :

$$\begin{aligned} \|F\| &\leq \|(I - b^{-1}A)^{-1}\| + \varepsilon \cdot \frac{b}{b-1} \leq 1 + \frac{\|A\|}{b} + \frac{\|A\|^2}{b^2} + \dots + \varepsilon \cdot \frac{b}{b-1} = \\ &= \frac{b}{b - \|A\|} + \varepsilon \cdot \frac{b}{b-1}. \end{aligned}$$

The proof is complete. $\square$

Similar results are valid for the multidimensional moment problem. Let $X_{\bar{\rho}}$ be the space of analytic functions φ in the polydisc $T_{\bar{\rho}} = \prod_{k=1}^n \{|z_k| \leq \rho_k\}$, $\bar{\rho} = (\rho_1, \dots, \rho_n)$.

Whence

$$\begin{aligned} \varphi(z_1, \dots, z_n) &= \sum_{j \in \mathbb{Z}_+^n} c_j z^j = \sum_{j \in \mathbb{Z}_+^n} c_j z_1^{j_1} \dots z_n^{j_n}, \\ j &= (j_1, \dots, j_n), |z_j| \leq \rho_j, j = 1, \dots, n; \varphi_j(z_1, \dots, z_n) = z_1^{j_1} \dots z_n^{j_n}. \end{aligned}$$

The order relation on X is defined by the positive cone

$$X_+ = \{\varphi \in X; \operatorname{Re} c_j \geq 0, \operatorname{Im} c_j \geq 0\}.$$

Consider an arbitrary real or complex Hilbert space H , $A_1, \dots, A_n \in A(H)$ commuting selfadjoint operators ($A_i A_j = A_j A_i, i, j = 1, \dots, n$). We introduce the usual spaces:

$$\begin{aligned} Y_1 &= \{U \in A(H); U A_j = A_j U, j = 1, \dots, n\}, \\ Y &= \{U \in Y_1; U V = V U, \forall V \in Y_1\}, \\ Y_+ &= \{U \in Y; \langle U(h), h \rangle \geq 0, \forall h \in H\}. \end{aligned} \tag{4.2}$$

Theorem 4.3. *With the above notations, assume that*

$$\rho_j > \|A_j\|, \quad j = 1, \dots, n,$$

and let $\{B_j; j \in \mathbb{Z}_+^n\} \subset Y$. Consider the following statements:

(a) there exists an R - linear positive continuous operator $F: X_{\bar{\rho}} \rightarrow Y$ such that

$$F(x_j) = B_j, \quad j \in \mathbb{Z}_+^n, \quad F(\varphi) \leq \|\varphi\|_\infty \prod_{k=1}^n \rho_k (\rho_k I - A_k)^{-1}, \quad \|F\| \leq \prod_{k=1}^n \frac{\rho_k}{\rho_k - \|A_k\|};$$

(b) we have

$$0 \leq B_j \leq A_1^{j_1} \dots A_n^{j_n}, \quad \forall j \in \mathbb{Z}_+^n;$$

(c) we have

$$\begin{aligned} 0 \leq B_j &\leq \prod_{k=1}^n \rho_k^{j_k+1} (\rho_k I - A_k)^{-1}, \\ \|B_j\| &\leq \prod_{k=1}^n \frac{\rho_k^{j_k+1}}{\rho_k - \|A_k\|}, \quad \forall j = (j_1, \dots, j_n) \in \mathbb{Z}_+^n. \end{aligned}$$

Then (b) $\Rightarrow$ (a) $\Rightarrow$ (c).

Proof. (b) $\Rightarrow$ (a) Let $F \subset \mathbb{Z}_+^n$ be a finite subset, and $\{\lambda_j\}_{j \in F} \subset \mathbb{R}$ be such that

$$\sum_{j \in F} \lambda_j \varphi_j \leq \varphi = \sum_{j \in \mathbb{Z}_+^n} c_j \varphi_j.$$

Then the order relation on X and Cauchy's inequalities yield

$$\lambda_j \leq \operatorname{Re} c_j \leq |c_j| \leq \frac{\|\varphi\|_\infty}{\rho_1^{j_1} \dots \rho_n^{j_n}}, \quad j = (j_1, \dots, j_n)$$

Using the hypothesis (b) and the above relations, one obtains

$$\begin{aligned} \sum_{j \in F} \lambda_j B_j &\leq \sum_{j \in F^+} \lambda_j B_j \leq \sum_{j \in F^+} \lambda_j A_1^{j_1} \dots A_n^{j_n} \leq \\ &\leq \|\varphi\|_\infty \left(\sum_{j_1 \in \mathbb{Z}_+^n} \left(\frac{A_1}{\rho_1} \right)^{j_1} \dots \left(\frac{A_n}{\rho_n} \right)^{j_n} \right) = \\ &= \|\varphi\|_\infty \left(\sum_{j_1 \in \mathbb{Z}_+} \left(\frac{A_1}{\rho_1} \right)^{j_1} \right) \dots \left(\sum_{j_n \in \mathbb{Z}_+} \left(\frac{A_n}{\rho_n} \right)^{j_n} \right) = \\ &= \|\varphi\|_\infty \left(I - \frac{A_1}{\rho_1} \right)^{-1} \dots \left(I - \frac{A_n}{\rho_n} \right)^{-1} = \\ &= \|\varphi\|_\infty \prod_{k=1}^n \rho_k (\rho_k I - A_k)^{-1} = p(\varphi), \quad \varphi \in X. \end{aligned}$$

Application of Theorem 2.6 leads to the existence of an $\mathbb{R}$ - linear positive operator F applying X into Y such that

$$F(\varphi_j) = B_j, \quad j \in \mathbb{Z}_+^n, \quad F(\varphi) \leq p(\varphi) = p(-\varphi), \quad \forall \varphi \in X \Rightarrow |F(\varphi)| \leq p(\varphi), \quad \varphi \in X.$$

Using the monotony of the norm on Y , we deduce

$$\begin{aligned} \|F(\varphi)\| &\leq \|\varphi\|_\infty \prod_{k=1}^\infty \left\| \left(I - \frac{A_k}{\rho_k} \right)^{-1} \right\| \leq \\ &\leq \|\varphi\|_\infty \left[\sum_{j_1 \in \mathbb{Z}_+} \left(\frac{\|A_1\|}{\rho_1} \right)^{j_1} \right] \dots \left[\sum_{j_n \in \mathbb{Z}_+} \left(\frac{\|A_n\|}{\rho_n} \right)^{j_n} \right] = \\ &= \|\varphi\|_\infty \prod_{k=1}^\infty \frac{\rho_k}{\rho_k - \|A_k\|}, \quad \forall \varphi \in X. \end{aligned}$$

Thus the implication (b) $\Rightarrow$ (a) is proved.

(a) $\Rightarrow$ (c) The hypothesis on positivity of the solution F and of functions φ_j yield

$$\begin{aligned} B_j = F(\varphi_j) &\leq \|\varphi_j\|_\infty \prod_{k=1}^n \rho_k (\rho_k I - A_k)^{-1} = \prod_{k=1}^n \rho_k^{j_k+1} (\rho_k I - A_k)^{-1} \Rightarrow \\ \|B_j\| = \|F(\varphi_j)\| &\leq \|\varphi_j\|_\infty \prod_{k=1}^n \frac{\rho_k}{\rho_k - \|A_k\|} = \prod_{k=1}^n \frac{\rho_k^{j_k+1}}{\rho_k - \|A_k\|}, \quad j = (j_1, \dots, j_n) \in \mathbb{Z}_+^n. \end{aligned}$$

The proof is complete. $\square$

Applying the Theorem 4.3 to the real Hilbert space $H = L_{dt}^2([0, \pi/2])$, $n = 2$, A_k , $k = 1, 2$ being the multiplication operators with the functions $h_1(t) = \sin t$, $h_2(t) = \cos t$, one obtains

Corollary 4.4. *Let $\{B_j; j \in \mathbb{Z}_+^2\} \subset Y = Y(A_1, A_2)$, $\rho > 1$, $\rho_1 = \rho_2 = \rho$. Consider the following statements:*

(a) *there exists $F \in L_+(X, Y)$ such that*

$$F(\varphi_j) = B_j, \quad j \in \mathbb{Z}_+^2, \quad \int_0^{\pi/2} F(\varphi)(h)(t) \cdot h(t) dt \leq$$

$$\|\varphi\|^2 \rho^2 \int_0^{\pi/2} \frac{h^2(t)}{(\rho - \sin t)(\rho - \cos t)} dt, \quad \forall h \in L_{dt}^2\left(\left[0, \frac{\pi}{2}\right]\right);$$

(b) *we have*

$$0 \leq \int_0^{\pi/2} B_j(h)(t) \cdot h(t) dt \leq \int_0^{\pi/2} \sin^{j_1} t \cdot \cos^{j_2} t \cdot h^2(t) dt, \quad \forall h \in H,$$

$$\forall j = (j_1, j_2) \in \mathbb{Z}_+^2$$

(c) *we have*

$$0 \leq \int_0^{\pi/2} B_{(j_1, j_2)}(h)(t) \cdot h(t) dt \leq \rho^{j_1 + j_2 + 2} \int_0^{\pi/2} \frac{h^2(t)}{(\rho - \sin t)(\rho - \cos t)} dt, \quad \forall h \in H.$$

Then (b) $\Rightarrow$ (a) $\Rightarrow$ (c).

Our next goal is to consider some Markov moment problems in terms of nonnegative sequences with respect to an interval. We recall this well-known notion.

Definition 4.5. A sequence $(u_n)_{n=0}^\infty$ in an ordered vector space Y is said to be nonnegative with respect to the interval $I \subset \mathbb{R}$ if for any $n \in \mathbb{Z}_+$, we have

$$\lambda_0 + \lambda_1 t + \dots + \lambda_n t^n \geq 0 \quad \forall t \in I \Rightarrow \lambda_0 u_0 + \lambda_1 u_1 + \dots + \lambda_n u_n \geq 0 \text{ in } Y$$

It is clear that that for compact intervals and for $Y = \mathbb{R}$, this condition is necessary and sufficient for the existence of a unique positive solution of the moment problem associated to the moments $y_j = u_j \in \mathbb{R}$, $j \in \mathbb{Z}_+$. The continuity with respect to the sup-norm is also obvious. But no information concerning dominating L^1 -norm for the solution holds. Such information would be useful for integral representation of the extension of the solution to the space $L^1(I)$ (see the next results). Let $x_j(t) = t^j$, $j \in \mathbb{N}$, $t \in I \subset \mathbb{R}$.

Theorem 4.6. *Let $b \in (0, \infty)$. Consider the following statements:*

(a) *there exists a unique $h \in L^\infty([0, b])$ such that*

$$0 \leq h(t) \leq 1 \text{ a.e.} \quad \int_0^b t^j h(t) dt = y_j, \quad \forall j \in \mathbb{Z}_+;$$

(b) *the sequence*

$$(1, y_0, 2y_1, \dots, ny_{n-1}, \dots) \quad \text{is nonnegative with respect to } [0, b].$$

Then (b) $\Rightarrow$ (a).

Proof. We have:

$$\begin{aligned}
\sum_{j=0}^n \lambda_j \tau^j &= \varphi_2(\tau) - \varphi_1(\tau), \quad \varphi_1, \varphi_2 \in X_+, \quad X = L^1([0, b]) \Rightarrow \\
\lambda_0 t + \lambda_1 \frac{t^2}{2} + \dots + \lambda_n \frac{t^{n+1}}{n+1} &\leq \int_0^b \varphi_2(\tau) d\tau = F_2(\varphi_2) - F_1(\varphi_1), \quad (F_1 \equiv 0), \\
\forall t \in [0, b] &\Rightarrow \int_0^b \varphi_2 d\tau - \lambda_0 t - \frac{t^2}{2} - \dots - \lambda_n \frac{t^{n+1}}{n+1} \geq 0 \quad \forall t \in [0, b] \Rightarrow \\
\int_0^b \varphi_2(\tau) d\tau - \lambda_0 y_0 - \lambda_1 \frac{2y_1}{2} - \dots - \lambda_n \frac{(n+1)y_n}{n+1} &\geq 0 \\
\Rightarrow \sum_{j=0}^n \lambda_j y_j &\leq F_2(\varphi_2) - F_1(\varphi_1).
\end{aligned}$$

Application of Theorem 2.5, (b) $\Rightarrow$ (a) leads to the existence of a linear functional F on $X = L^1([0, b])$ such that

$$F(x_j) = y_j, \quad j \in \mathbb{Z}_+, \quad 0 = F_1(\varphi) \leq F(\varphi) \leq F_2(\varphi) = \int_a^b \varphi dt, \quad \forall \varphi \in X_+.$$

Using the representation of a linear positive functional on L^1 there exists

$$h \in L^\infty([0, b]), \quad 0 \leq h, \quad F(\varphi) = \int_0^b \varphi \cdot h \, dt, \quad \forall \varphi \in L^1([0, b]).$$

From the last equality written for $\varphi = \chi_B$, $B \subset [0, b]$ being a Borel subset, we infer (via measure theory) that $h \leq 1$ a.e. On the other hand, we have

$$\int_0^b t^j h(t) dt = F(x_j) = y_j, \quad \forall j \in \mathbb{Z}_+.$$

The proof is complete. $\square$

The case when the subset A from Theorem 1 [16] is distanced with respect to the subspace S while the restriction involving B is missing, has a nice geometric meaning and leads to nice results concerning the extension of linear functionals and operators. If V is a convex neighborhood of the origin in a locally convex space, we denote by p_V the gauge attached to V . This section is based on some results of [11].

Theorem 4.7 (Theorem 5 [17], p. 969). . *Let X be a locally convex space, Y an order complete vector lattice with strong order unit u_0 and $S \subset X$ a vector subspace. Let $A \subset X$ be a convex subset with the following qualities:*

- (i) *there exists a neighborhood V of the origin such that $(S + V) \cap A = \Phi$ (A and S are distanced);*
- (ii) *A is bounded.*

Then for any equicontinuous family of linear operators $\{f_j\}_{j \in J} \subset L(S, Y)$ and for any $\tilde{y} \in Y_+ \setminus \{0\}$ there exists an equicontinuous family $\{F_j\}_{j \in J} \subset L(X, Y)$ such that

$$F_j|_S = f_j \text{ and } F_j|_A \geq \tilde{y}, \quad \forall j \in J.$$

Moreover, if V is a neighborhood of the origin such that

$$\begin{aligned} f_j(V \cap S) &\subset [-u_0, u_0], \quad (S + V) \cap A = \Phi, \\ 0 < \alpha &\in R \text{ s.t. } p_V \leq \alpha, \quad \alpha_1 > 0 \text{ s.t. } \tilde{y} \leq \alpha_1 u_0, \end{aligned}$$

then the following relations hold

$$F_j(x) \leq (1 + \alpha + \alpha_1)p_V(x) \cdot u_0, \quad x \in X, \quad j \in J.$$

The last relation of Theorem 4.7 gives a relationship between an upper bound and the lower bound of F_j , $j \in J$. The next result is a consequence of the preceding one, in terms of the moment problem. The topology on Y is the norm topology for which the unit closed ball is the interval $[-u_0, u_0]$.

Theorem 4.8 (see [11]). . Let

$$\begin{aligned} 0 < b &\in R, \quad X = C([0, b]), \quad x_j(t) = t^j, \quad j \in \mathbb{Z}_+, \quad j \geq 1, \quad t \in [0, b], \\ \{\varphi_k\}_{k \in \mathbb{Z}_+} &\subset X, \quad \|\varphi_k\| \leq 1, \quad \varphi_0 \equiv 1, \quad \varphi_k(0) = 1, \quad k \in \mathbb{Z}_+. \end{aligned}$$

Let Y be an order complete vector lattice with strong order unit u_0 , $(y_n)_{n \geq 1}$ a sequence in Y such that $(u_0, y_1, y_2, \dots)$ is positive with respect to $[0, b]$.

Then for any $\alpha_1 \in R_+$, there exists $F \in L(X, Y)$ such that

$$F(x_j) = y_j, \quad j \in \mathbb{Z}_+, \quad j \geq 1, \quad F(\varphi_k) \geq \alpha_1 u_0, \quad k \in \mathbb{Z}_+,$$

$$F(x) \leq (2 + \alpha_1) \cdot \|x\| \cdot u_0, \quad x \in X.$$

Moreover, if $\alpha_1 \geq 1$ and Y is endowed with a linear topology such that its positive cone is closed and normal, then F is continuous and positive.

Proof. Let

$$S = \text{Span} \{x_j; j \in \mathbb{Z}_+, j \geq 1\}, \quad A = \text{conv} \{\varphi_k; k \in \mathbb{Z}_+\}.$$

For all $p \in S$, $a \in A$, we have

$$\|p - a\| \geq |p(0) - a(0)| = 1,$$

so that $d(S, A) \geq 1$ holds. This leads to $(S + B(0, 1)) \cap A = \Phi$, so we can take in Theorem 4.5 $V = B(0, 1)$, $p_V = \|\cdot\|_\infty$. From the hypothesis on $\|\varphi_k\|$, $k \in \mathbb{Z}_+$ we infer that

$$\|\varphi_k\| \leq 1 \quad \forall k \in \mathbb{Z}_+ \Rightarrow \|\cdot\|_A \leq 1 \Rightarrow \alpha := 1,$$

where α is the constant from Theorem 4.7. We define:

$$\tilde{y} := \alpha_1 \cdot u_0, \quad f \left(\sum_{j=1}^n \lambda_j x_j \right) = \sum_{j=1}^n \lambda_j y_j.$$

Let $\sum_{j=1}^n \lambda_j x_j \in S \cap B(0, 1)$. Using the hypothesis on the sequence which is positive with respect to $[0, b]$, this yields

$$\begin{aligned} \sup \left\{ \left| \sum_{j=1}^n \lambda_j t^j \right| ; t \in [0, b] \right\} < 1 \Rightarrow \sum_{j=1}^n \lambda_j t^j + 1 > 0, \quad 1 - \sum_{j=1}^n \lambda_j t^j > 0 \Rightarrow \\ u_0 + \sum_{j=1}^n \lambda_j y_j \geq 0, \quad u_0 - \sum_{j=1}^n \lambda_j y_j \geq 0 \Rightarrow f \left(\sum_{j=1}^n \lambda_j x_j \right) = \\ \sum_{j=1}^n \lambda_j y_j \in [-u_0, u_0] \Rightarrow f(S \cap B(0, 1)) \subset [-u_0, u_0]. \end{aligned}$$

The conclusion is that f verifies the conditions in the statement of Theorem 4.7. Application of the latter theorem, leads to the existence of an extension F of f such that

$$F|_A \geq \alpha_1 y_0, \quad F(x) \leq (2 + \alpha_1) \cdot \|x\| \cdot u_0, \quad x \in X.$$

Whence the first part of the theorem is proved. It remains to prove the continuity and positivity of the extension F under the hypothesis mentioned in the statement. Replacing x by $-x$ it is easy to see that

$$|F(x)| \leq (2 + \alpha_1) \cdot \|x\| \cdot u_0, \quad x \in C([0, b]).$$

From this relations, also using that the positive cone Y_+ is normal, the continuity of F follows. In order to prove that F is positive, it is sufficient to show that $F(p) \geq 0$ for all positive polynomials p . Hypothesis yield:

$$\begin{aligned} p(t) = \lambda_0 + \sum_{j=1}^n \lambda_j t^j \geq 0 \quad \forall t \in [0, b] \Rightarrow \lambda_0 u_0 + \sum_{j=1}^n \lambda_j y_j \geq 0 \Rightarrow \\ f(p) = \lambda_0 f(\varphi_0) + \sum_{j=1}^n \lambda_j y_j \geq \lambda_0(\alpha_1 \cdot u_0 - u_0) = \lambda_0(\alpha_1 - 1) \cdot u_0 \geq 0. \end{aligned}$$

The proof is complete. □

A consequence of Theorem 4.8 is the following one.

Corollary 4.9. *Let H be a Hilbert space, $A \in A(H)$ a positive selfadjoint operator, $Y = Y(A)$ the commutative algebra of selfadjoint operators defined in section 3. Let $b \in \mathbb{R}_+$ be such that the spectrum $S(A) \subset [0, b]$ and $\alpha_1 \geq 1$. Then there exists a nondecreasing mapping $\sigma : [0, b] \rightarrow Y$ such that*

$$\begin{aligned} \int_0^b t^j d\sigma(t) = A^j, \quad j \in \mathbb{Z}_+, \quad j \geq 1, \quad \int_0^b e^{-kt} d\sigma(t) \geq \alpha_1 I, \quad k \in \mathbb{Z}_+, \\ \int_0^b x(t) d\sigma(t) \leq (2 + \alpha_1) \cdot \|x\| \cdot I, \quad x \in C([0, b]). \end{aligned}$$

In particular, for such a mapping σ the following relation holds:

$$\alpha_1 + 1 \leq \|\sigma(b) - \sigma(0)\| + \|\exp(-kA)\|, \quad k \in \mathbb{Z}_+.$$

Proof. We apply Theorem 4.8 to $\varphi_k(t) = e^{-kt}$, $k \in \mathbb{Z}_+$, $t \in [0, b]$, $u_0 = I$. The sequence $(I, A, A^2, \dots, A^n, \dots)$ is positive with respect to $[0, b]$ because of the positivity of the spectral measure attached to A . The hypothesis of Theorem 4.7 being accomplished, there exists a positive continuous solution F satisfying the relations in the statement of the latter theorem. This solution applies norm-bounded subsets into order-bounded subsets of $Y(A)$. Application of [6], p. 272, yields the existence of a "representing" nondecreasing mapping σ such that the vector measure $d\sigma$ represents F . Next we prove the last relation of the statement:

$$\alpha_1 I \leq \int_0^b e^{-kt} d\sigma = \sigma(b) - \sigma(0) + \sum_{m=1}^{\infty} \frac{(-k)^m}{m!} A^m = \sigma(b) - \sigma(0) - I + \exp(-kA).$$

Further computation yield

$$\begin{aligned} (\alpha_1 + 1)I &\leq \sigma(b) - \sigma(0) + \exp(-kA) \Rightarrow \\ &\Rightarrow \alpha_1 + 1 \leq \|\sigma(b) - \sigma(0)\| + \|\exp(-kA)\|, \quad k \in \mathbb{Z}_+. \end{aligned}$$

The proof is complete. $\square$

Next we consider the analogue of theorem 4.8, for a space X of analytic functions. Let $b > 0$ and X be the space of all functions φ which can be represented as an absolutely convergent series $\varphi(z) = \sum_{j=0}^{\infty} \lambda_j z^j$, $|z| < b$, $\lambda_j \in \mathbb{R}$, φ being continuous in the closed disc $|z| \leq b$. Let $\varphi_j(z) = z^j$, $j \in \mathbb{N}$, $Y = L^\infty(\Omega)$, (Ω, μ) being a measure space, $u_0(\omega) = 1$, $\forall \omega \in \Omega$.

Theorem 4.10. *Let $\{\psi_k\}_{k \in \mathbb{Z}_+} \subset X$ be a sequence such that*

$$\|\psi_k\|_\infty \leq M, \quad \psi_k(0) = 1, \quad \forall k \in \mathbb{N}.$$

Assume that $b > 1$ and let $(y_j)_{j \geq 1}$ be a sequence in Y such that

$$\|y_j\| \leq b - 1, \quad j \geq 1, \quad j \in \mathbb{Z}.$$

Then for any $\tilde{y} \in Y_+ \setminus \{0\}$, there exists $F \in L(X, Y)$ such that

$$\begin{aligned} F(x_j) &= y_j, \quad j \geq 1, \quad F(\psi_k) \geq \tilde{y}, \quad k \in \mathbb{N}, \\ F(x) &\leq (1 + M + \|\tilde{y}\|_\infty) \cdot \|x\|_\infty \cdot u_0, \quad x \in X. \end{aligned}$$

Proof. If $S = \text{Span} \{\varphi_j; j \geq 1\}$, $A = \text{conv} \{\psi_k; k \in \mathbb{Z}_+\}$, we have

$$d(S, A) \geq 1, \quad (S + B(0, 1)) \cap A = \Phi, \quad V = B(0, 1), \quad p_V = \|\cdot\|_\infty.$$

Let $\varphi \in B(0, 1) \cap S$, $\varphi = \sum_{j=1}^n \lambda_j \varphi_j$, $\|\varphi\| < 1$. Using the Cauchy inequalities, we derive;

$$|\lambda_j| \leq \|\varphi\|/b^j < 1/b^j, \quad j \in \{1, 2, \dots, n\}.$$

Define $f \left(\sum_{j=1}^n \lambda_j \varphi_j \right) = \sum_{j=1}^n \lambda_j y_j$, $n \in \mathbb{Z}_+$, $n \geq 1$, $\lambda_j \in \mathbb{R}$. Using also the hypothesis on $\|y_j\|$, $j \geq 1$, one obtains:

$$\left| \sum_{j=1}^n \lambda_j y_j \right| \leq \sum_{j=1}^n |\lambda_j| \cdot \|y_j\| \leq \left(\sum_{j=1}^n (1/b^j) \cdot \|y_j\|_\infty \right) \cdot u_0 \leq$$

$$\leq \left(\sum_{j=1}^{\infty} (1/b^j) \right) \cdot (b-1)u_0 = u_0.$$

Whence $\varphi = \sum_{j=1}^n \lambda_j \varphi_j \in S \cap B(0, 1) \Rightarrow f(\varphi) \in [-u_0, u_0]$. Since

$$\| \cdot \|_A \leq M, \tilde{y} \leq \|\tilde{y}\|_{\infty} \cdot u_0,$$

one takes in Theorem 4.7 $\alpha = M$, $\alpha_1 = \|\tilde{y}\|_{\infty}$. Now the conclusion follows via Theorem 4.7, for which all the requirements of the hypothesis are accomplished. $\square$

We give sufficient conditions for the existence of a solution, in a very general setting. The method works in an arbitrary measure space, which may be not involve polynomials. In the particular case of subsets of $\mathbb{R}^n$, the form of positive polynomials is not necessarily used.

Theorem 4.11. *Let (T, ν) be a measure space, where ν is a positive σ -finite measure on T . Let $X = L^1_{\nu}(T)$, $\{x_j\}_{j \in J} \subset X$, $\{y_j\}_{j \in J} \subset \mathbb{R}$. Consider the following statements:*

(a) *there exists $h \in L^{\infty}_{\nu}(T)$ such that*

$$\int_T x_j(t) \cdot h(t) d\nu = y_j, \quad j \in J, \quad -1 \leq h(t) \leq 1 \quad \nu - a.e.;$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$ we have*

$$\sum_{i,j \in J_0} \lambda_i \lambda_j y_i y_j \leq \sum_{i,j \in J_0} \lambda_i \lambda_j \int_T x_i(t) d\nu \cdot \int_T x_j(t) d\nu;$$

(c) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, we have*

$$\sum_{i,j \in J_0} \lambda_i \lambda_j y_i y_j \leq \sum_{i,j \in J_0} |\lambda_i| \cdot |\lambda_j| \cdot \int_T |x_i(t)| d\nu \cdot \int_T |x_j(t)| d\nu.$$

Then (b) $\Rightarrow$ (a) $\Rightarrow$ (c).

Proof. (b) $\Rightarrow$ (a). The following implications hold:

$$\begin{aligned} & \sum_{j \in J_0} \lambda_j x_j = \varphi_2 - \varphi_1, \quad \varphi_k \in X_+, \quad k = 1, 2 \Rightarrow \\ & - \int_T \varphi_1 d\nu - \int_T \varphi_2 d\nu \leq \int_T \varphi_2 d\nu - \int_T \varphi_1 d\nu = \\ & \sum_{j \in J_0} \lambda_j \int_T x_j d\nu \leq \int_T \varphi_2 d\nu + \int_T \varphi_1 d\nu = \\ & \int_T \varphi_2 d\nu - \left(- \int_T \varphi_1 d\nu \right) = F_2(\varphi_2) - F_1(\varphi_1) \Rightarrow \\ & \left| \sum_{j \in J_0} \lambda_j \int_T x_j d\nu \right| \leq F_2(\varphi_2) - F_1(\varphi_1). \end{aligned}$$

On the other hand, the condition (b) can be rewritten as

$$\left(\sum_{j \in J_0} \lambda_j y_j \right)^2 \leq \left(\sum_{j \in J_0} \lambda_j \int_T x_j d\nu \right)^2 \Leftrightarrow \left| \sum_{j \in J_0} \lambda_j y_j \right| \leq \left| \sum_{j \in J_0} \lambda_j \int_T x_j d\nu \right|.$$

From the preceding relation, we conclude

$$\sum_{j \in J_0} \lambda_j y_j \leq \left| \sum_{j \in J_0} \lambda_j y_j \right| \leq \left| \sum_{j \in J_0} \lambda_j \int_T x_j(t) d\nu \right| \leq F_2(\varphi_2) - F_1(\varphi_1).$$

Whence the condition from the statement of Theorem 2.5 are accomplished, so that there exists a linear form F on X such that

$$\begin{aligned} F(x_j) &= y_j; \quad j \in J, \quad - \int_T \varphi d\nu = F_1(\varphi) \leq F(\varphi) \leq F_2(\varphi) = \int_T \varphi d\nu \Rightarrow \\ |F(\varphi)| &\leq \int_T \varphi d\nu \quad \forall \varphi \in X_+ \Rightarrow |F(\varphi)| \leq |F(\varphi^+)| + |F(\varphi^-)| \leq \\ \int_T (\varphi^+ + \varphi^-) d\nu &= \int_T |\varphi| d\nu = \|\varphi\|_1 \Rightarrow \|F\| \leq 1. \end{aligned}$$

Following the representation of continuous linear functionals on L^1 spaces, there exists

$$\begin{aligned} h &\in L^\infty_\nu(T), \quad F(\varphi) = \int_T \varphi(t) h(t) d\nu \quad \forall \varphi \in L^1_\nu(T), \\ \|h\|_\infty &= \|F\| \leq 1 \Rightarrow -1 \leq h(t) \leq 1 \quad \nu - a.e. \end{aligned}$$

We also infer that

$$y_j = F(x_j) = \int_T x_j(t) h(t) d\nu, \quad \forall j \in J.$$

Thus the proof of the implication (b) $\Rightarrow$ (a) is finished. The implication (a) $\Rightarrow$ (c) is almost obvious:

$$\begin{aligned} \left| \sum_{j \in J_0} \lambda_j y_j \right| &= \left| \sum_{j \in J_0} \lambda_j \int_T x_j \cdot h \cdot d\nu \right| \leq \\ &\leq \sum_{j \in J_0} |\lambda_j| \cdot \int_T |x_j(t)| \cdot |h(t)| \cdot d\nu \leq \sum_{j \in J_0} |\lambda_j| \cdot \int_T |x_j(t)| \cdot d\nu. \end{aligned}$$

Taking the squares of the two members, the inequality is preserved. This concludes the proof. $\square$

Corollary 4.12. *Let*

$$T = \{(t_1, t_2) \in \mathbb{R}^2; 0 \leq t_1 < \infty, 0 \leq t_2 \leq \exp(-t_1)\}, \quad \{y_{(j_1, j_2)}\}_{(j_1, j_2) \in \mathbb{Z}_+^2} \subset \mathbb{R}.$$

Consider the following statements

(1) *there exists a Lebesgue measurable function h on T such that*

$$\int_T t_1^{j_1} t_2^{j_2} h(t_1, t_2) dt_1 dt_2 = y_{(j_1, j_2)}, \quad \forall (j_1, j_2) \in \mathbb{Z}_+^2, \quad -1 \leq h(t_1, t_2) \leq 1 \text{ a.e.};$$

(2) for any finite subset $J_0 \subset \mathbb{Z}_+^n$ and any $\{\lambda_{(j_1, j_2)}; (j_1, j_2) \in J_0\} \subset \mathbb{R}$ we have

$$\sum_{(i_1, i_2), (j_1, j_2) \in J_0} \lambda_{(i_1, i_2)} \lambda_{(j_1, j_2)} y_{(j_1, j_2)} \leq \sum_{(i_1, i_2), (j_1, j_2) \in J_0} \lambda_{(i_1, i_2)} \lambda_{(j_1, j_2)} \frac{i_1!}{(i_2 + 1)^{i_1 + 2}} \cdot \frac{j_1!}{(j_2 + 1)^{j_1 + 2}}.$$

Then (b) $\Rightarrow$ (a).

Note that T is a closed unbounded and non-semi-algebraic subset. On the other hand, in [12] theorem 2.3 of the present work is applied in solving complex moment problems.

5. Extreme points and the Markov moment problem

The aim of this Section is to apply some ideas from [8], [10] and [27] in solving truncated moment problems and their connection to the full moment problem. The results of this section represent the content of section 4, [20], where the reader may find the proofs. The following theorem concerns the finite Markov moment problem considered in [8], also using some results from [10], [27]. The general idea is to solve the truncated moment problem via bang-bang principle, then passing through the weak limit in order to solve the full moment problem. Let us denote

$$\psi_j(t) = j \cdot t^{j-1}, \quad t \in [0, b], \quad j \in \mathbb{Z}_+, \quad j \geq 1.$$

Theorem 5.1. *For a given family of numbers $(m_j)_{j=1}^n$, consider the following statements:*

(a) *there exists $h \in L^\infty([0, b])$ such that*

$$0 \leq h(t) \leq 1 \text{ a.e. } m_j = j \int_0^b t^{j-1} h(t) dt, \quad j = 1, 2, \dots, n;$$

(b) *for any family of scalars $(\lambda_j)_{j=0}^n$, we have:*

$$\sum_{j=1}^n \lambda_j m_j \leq \sum_{j=1}^n \lambda_j b^j;$$

(c) *there exists a Borel subset $B \subset [0, b]$ such that*

$$\int_B j \cdot t^{j-1} dt = m_j, \quad j = 1, \dots, n.$$

Then (b) $\Rightarrow$ (a) $\Leftrightarrow$ (c)

Corollary 5.2. *Under the conditions (a) or (c) of Theorem 5.1, there are sequences:*

$$y_{1,n} < x_{1,n} < y_{2,n} < x_{2,n} < \dots < y_{l,n} < x_{l,n} < \dots, \quad n \in \mathbb{Z}_+,$$

such that the following relations hold:

$$m_k = \inf_{n \in \mathbb{N}} \left(\sum_{j=1}^{\infty} (x_{j,n}^k - y_{j,n}^k) \right), \quad k = 1, \dots, n.$$

Remark 5.3. In order to approximate the numbers $x_{j,n}^k, y_{j,n}^k$, one can make use of Fourier approximate expansion of h with respect to the orthonormal sequence attached to the functions kt^{k-1} via Gram-Schmidt algorithm, also using the values of the moments m_k . Thus one obtains a smooth approximation $\tilde{h}$ of h , and the intervals of ends $y_{l,n}, x_{l,n}$ are connected components of the open sets of the form

$$\left\{ \frac{j_l}{2^{p(n)}} < \tilde{h}(t) < \frac{j_l + 1}{2^{p(n)}} \right\}.$$

Remark 5.4. A similar result to that of Theorem 5.1 in several dimensions holds, with the same proof. We state it for the two dimensional case.

Theorem 5.5. Let $(m_{(j_1, j_2)})_{\substack{1 \leq j_1 \leq n_1 \\ 1 \leq j_2 \leq n_2}}$, be a given family of real numbers, and consider the functions

$$\psi_{(j_1, j_2)}(t_1, t_2) = j_1 j_2 t_1^{j_1-1} t_2^{j_2-1}, \quad 1 \leq j_p \leq n_p, \quad p = 1, 2, \quad (t_1, t_2) \in K_2 = [0, b_1] \times [0, b_2].$$

Consider the following statements:

(a) there exists a Borel function h , $0 \leq h(t_1, t_2) \leq 1$ a.e. such that

$$\iint_{K_2} (\psi_{(j_1, j_2)} \cdot h)(t_1, t_2) dt_1 dt_2 = m_{(j_1, j_2)}, \quad 1 \leq j_1 \leq n_1, \quad 1 \leq j_2 \leq n_2;$$

(b) for any family of scalars $\{\lambda_{(j_1, j_2)}\}_{\substack{1 \leq j_1 \leq n_1 \\ 1 \leq j_2 \leq n_2}}$, we have

$$\sum_{\substack{1 \leq j_1 \leq n_1 \\ 1 \leq j_2 \leq n_2}} \lambda_{(j_1, j_2)} m_{(j_1, j_2)} \leq \sum_{\substack{1 \leq j_1 \leq n_1 \\ 1 \leq j_2 \leq n_2}} \lambda_{(j_1, j_2)} b_1^{j_1} b_2^{j_2};$$

(c) there exists a Borel subset $B_2 \subset K_2$ such that

$$\iint_{B_2} \psi_{(j_1, j_2)}(t_1, t_2) dt_1 dt_2 = m_{(j_1, j_2)}, \quad 1 \leq j_p \leq n_p, \quad p = 1, 2.$$

Then (b) $\Rightarrow$ (a) $\Leftrightarrow$ (c)

Corollary 5.6. If one of the conditions (a), (c) of theorem 5.5 is accomplished, there are sequences

$$\begin{aligned} y_{1,n} &< x_{1,n} < \dots < y_{l,n} < x_{l,n} < \dots, \\ v_{1,n} &< u_{1,n} < \dots < v_{l,n} < u_{l,n} < \dots, \quad n \in N, \end{aligned}$$

such that

$$m_{(k_1, k_2)} = \inf_{n \in \mathbb{Z}_+} \left\{ \sum_{l \in N} (x_{l,n}^{k_1} - y_{l,n}^{k_1}) \cdot (u_{l,n}^{k_2} - v_{l,n}^{k_2}) \right\}.$$

The above statements solve the truncated moment problems and sketch an algorithm for determining the numbers $y_{l,n}, x_{l,n}, v_{l,n}, u_{l,n}$. In the last twenty years the truncated moment problem was studied intensively. In [27] the author proves that solving the truncated (not necessarily complex) moment problem for any natural n , yields to the solution of the full moment problem. The idea is of passing through the limit, which is allowed by a weakly compactness argument. The following theorem proposes a similar construction, by using Krein -Milman theorem. We state it firstly in the one dimensional case, although the several dimensional case one proves similarly.

Theorem 5.7. *With the notations from Theorem 5.1, let $(m_k)_{k \geq 1}$ be a sequence of real numbers. Consider the following statements:*

(a) *there exists a Borel function h such that*

$$0 \leq h(t) \leq 1 \text{ a.e. } m_k = k \int_a^b t^{k-1} h(t) dt, \quad k \in \mathbb{Z}_+ \setminus \{0\};$$

(b) *for any natural number $n \geq 1$, and any $\varepsilon > 0$, there exists nonnegative scalars β_j , $j = 1, \dots, n$, and sequences:*

$$y_{j,1} < x_{j,1} < \dots < y_{j,l} < x_{j,l} < \dots, \quad j = 1, \dots, n,$$

such that

$$1 - \varepsilon \leq \sum_{j=1}^n \beta_j \leq 1, \quad m_k = \lim_n \left(\sum_{j=1}^n \beta_j \left(\sum_l (x_{j,l}^k - y_{j,l}^k) \right) \right); \quad k = 1, \dots, n;$$

(c) *for any $n \in \mathbb{Z}_+ \setminus \{0\}$, there exists a Borel subset B_n such that*

$$m_k = k \int_{B_n} t^{k-1} dt, \quad k = 1, \dots, n.$$

(d) *for any natural $n \geq 1$ and any $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}$, the following relation holds true:*

$$\sum_{k=1}^n \lambda_k m_k \leq \sum_{k=1}^n \lambda_k (b^k - a^k)$$

Then $(d) \Rightarrow (a) \Leftrightarrow (b) \Leftrightarrow (c)$.

Remark 5.8. For the full moment problem, the following algorithm holds in determining $y_{j,l}, x_{j,l}$.

Step 1. Approximating the function h .

Let $(e_n)_{n \geq 1}$ be a Hilbert base constructed by the aid of Gram-Schmidt procedure, applied to the system of linearly independent functions

$$\varphi_n(t) = n \cdot t^{n-1}, \quad n \in \mathbb{Z}_+ \setminus \{0\}.$$

Then for each fixed $n \geq 1$, one has:

$$e_n = \sum_{j=1}^n a_j^{(n)} \varphi_j \Rightarrow \langle h, e_n \rangle = \sum_{j=1}^n a_j^{(n)} \langle h, \varphi_j \rangle = \sum_{j=1}^n a_j^{(n)} m_j,$$

where the coefficients $a_j^{(n)}$ are known from the Gram-Schmidt procedure. Hence we can determine each Fourier coefficient of h , that is we can approximate h in L^2 - norm by a sequence of polynomial functions h_n , $n \geq 1$. Then there exists a subsequence $h_{k_n} \rightarrow h$ pointwise almost everywhere in $[a, b]$ [25].

Step 2. For each $n \in \mathbb{Z}_+ \setminus \{0\}$, the subsets

$$\left\{ t; \frac{m_l}{2^p} < h_{k_n}(t) < \frac{m_l + 1}{2^p} \right\}, \quad p, m_l \in \mathbb{Z}_+,$$

are open subsets. Their connected components has as end points approximations of the unknowns $y_{j,l}, x_{j,l}$. From [27], using a weakly compactness standard argument, we

can obtain h as the limit of a subsequence of $(\chi_{B_n})_n$, where B_n are as at point (c) of Theorem 5.7. Considering a suitable open set $D_{k_n} \supset B_{k_n}$, from (a) one obtains:

$$m_k \approx \int_{D_{k_n}} k \cdot t^{k-1} dt \approx \sum_l (x_{n,l}^k - y_{n,l}^k), \quad D_{k_n} = \cup_l]y_{n,l}, x_{n,l}[.$$

□

Note that all results of this section can be adapted to the multidimensional moment problem, with similar proofs.

6. A remark on multiplicative solutions

This section contains the last results in [21], on multiplicative solutions of the moment problem. One of the main tools used along the proofs is theorem 2.6 of the present work. Let T be a nonempty set, X an algebra of functions on T , X_+ the cone of pointwise nonnegative functions on T , which are elements of X . Let Y be an order complete lattice, which is also a commutative Banach algebra. Let:

$$\{p_\alpha\}_{\alpha \in A} \subset X, \quad \{\varphi_\beta\}_{\beta \in B} \subset X, \quad \{u_\alpha\}_{\alpha \in A} \subset Y, \quad \{v_\beta\}_{\beta \in B} \subset Y$$

be such that $S = \text{Span}\{p_\alpha \varphi_\beta\}_{(\alpha, \beta) \in A \times B}$ is a majorizing subspace of X .

Theorem 6.1. *The following statements are equivalent:*

(a) *there exists a linear positive operator $F \in L_+(X, Y)$ such that*

$$F(p_\alpha \varphi_\beta) = u_\alpha v_\beta \quad \forall (\alpha, \beta) \in A \times B,$$

$$x \leq \sum_{(\alpha, \beta)} \gamma_{(\alpha, \beta)} p_\alpha \varphi_\beta \Rightarrow F(x) \leq \sum_{(\alpha, \beta)} \gamma_{(\alpha, \beta)} u_\alpha v_\beta,$$

where all the sums are finite;

(b) *the following implication holds:*

$$\sum_{(\alpha, \beta)} \lambda_{(\alpha, \beta)} p_\alpha \varphi_\beta \leq \sum_{(\alpha', \beta')} \gamma_{(\alpha', \beta')} p_{\alpha'} \varphi_{\beta'} \Rightarrow \sum_{(\alpha, \beta)} \lambda_{(\alpha, \beta)} u_\alpha v_\beta \leq \sum_{(\alpha', \beta')} \gamma_{(\alpha', \beta')} u_{\alpha'} v_{\beta'}$$

Proof. The implication (a) $\Rightarrow$ (b) is obvious. For the converse observe that any element $x \in X$ is dominated by an element from S . Writing this for $-x$ too, we get that any $x \in X$ is between two elements $s_1(x) \leq x \leq s_2(x)$, $s_j(x) \in S$, $j = 1, 2$. We denote:

$$x_{(\alpha, \beta)} = p_\alpha \varphi_\beta, \quad y_{(\alpha, \beta)} = u_\alpha v_\beta, \quad (\alpha, \beta) \in A \times B.$$

Define $p : X \rightarrow Y$,

$$p(x) = \inf \left\{ \sum_{(\alpha', \beta')} \gamma_{(\alpha', \beta')} u_{\alpha'} v_{\beta'}; x \leq \sum_{(\alpha', \beta')} \gamma_{(\alpha', \beta')} p_{\alpha'} \varphi_{\beta'} \right\}, \quad x \in X.$$

If $f : S \rightarrow Y$ is the linear operator verifying the moment conditions, then (b) says that f is linear and positive on S , so that for any $s_1 \leq x$, $s_1 \in S$ we have:

$$f(s_1) \leq p(x).$$

It follows that p is well defined and a standard computation shows that it is a convex operator. Moreover, these remarks and the definition of p lead to the following implication:

$$\begin{aligned} \sum_{(\alpha,\beta)} \lambda_{(\alpha,\beta)} p_\alpha \varphi_\beta &= \sum_{(\alpha,\beta)} \lambda_{(\alpha,\beta)} x_{(\alpha,\beta)} \leq x \Rightarrow \\ \sum_{(\alpha,\beta)} \lambda_{(\alpha,\beta)} u_\alpha v_\beta &= \sum_{(\alpha,\beta)} \lambda_{(\alpha,\beta)} y_{(\alpha,\beta)} \leq p(x). \end{aligned}$$

Hence the condition of theorem 2.6 is accomplished, and the conclusion follows. $\square$

In the following result related to the preceding one, Y will be as above; additionally assume that it is a Banach algebra with multiplication unit u_0 . Examples are the algebras of selfadjoint operators given by (4.2), (9), when $u_0 = I$. In the sequel, such an algebra will be the target space of our operators. Let:

$$X = C([a, b]), \quad p_\alpha(t) = \varphi_\alpha(t) = t^\alpha, \quad \forall \alpha \in A = B = \mathbb{Z}_+,$$

$$\{u_\alpha\}_{\alpha \in \mathbb{Z}_+} = \{v_\alpha\}_{\alpha \in \mathbb{Z}_+} \subset Y, \quad u_0 = v_0 = I.$$

Theorem 6.2 *Under the above assumptions, the following assertions are equivalent:*

(a) *there exists a nondecreasing mapping $\sigma : [a, b] \rightarrow Y$ such that:*

$$\begin{aligned} \int_a^b t^{\alpha+\beta} d\sigma &= u_\alpha v_\beta \quad \forall (\alpha, \beta) \in \mathbb{Z}_+^2, \\ x &\leq \sum_{(\alpha,\beta)} \gamma_{(\alpha,\beta)} p_{\alpha+\beta} \Rightarrow \int_a^b x(t) d\sigma \leq \sum_{(\alpha,\beta)} \gamma_{(\alpha,\beta)} u_\alpha v_\beta, \\ \int_a^b x(t) \cdot y(t) d\sigma &= \int_a^b x(t) d\sigma \cdot \int_a^b y(t) d\sigma \quad \forall x, y \in C([a, b]) \end{aligned}$$

(b) *the following implication holds:*

$$\sum_{(\alpha,\beta)} \lambda_{(\alpha,\beta)} p_{\alpha+\beta} \leq \sum_{(\alpha',\beta'')} \gamma_{(\alpha',\beta'')} p_{\alpha'+\beta''} \Rightarrow \sum_{(\alpha,\beta)} \lambda_{(\alpha,\beta)} u_\alpha v_\beta \leq \sum_{(\alpha',\beta'')} \gamma_{(\alpha',\beta'')} u_{\alpha'} u_{\beta''}$$

$$u_0 = v_0 = I.$$

Proof. The implication (a) $\Rightarrow$ (b) is almost obvious, because the linear operator defined by the vector measure $d\sigma$ is multiplicative and positive on X . To prove the converse, we use the preceding result of theorem 6.1. There exists a linear positive operator $F \in L_+(X, Y)$ such that:

$$F(p_\alpha \varphi_\beta) = F(p_{\alpha+\beta}) = u_\alpha v_\beta = u_\alpha u_\beta, \quad \forall (\alpha, \beta) \in \mathbb{Z}_+^2, \quad F(x) \leq p(x) \quad \forall x \in X.$$

Since F applies topological bounded subsets of $C([a, b])$ into order-bounded subsets of Y , the Theorem [6] p. 272 asserts that there exists a σ representing mapping. Since F is positive, σ is nondecreasing. We have:

$$\begin{aligned} F(p_{\alpha+\beta}) &= \int_a^b t^{\alpha+\beta} d\sigma = u_\alpha u_\beta = (u_\alpha u_0) \cdot (u_\beta u_0) = \\ \int_a^b t^\alpha d\sigma \cdot \int_a^b t^\beta d\sigma &= F(p_\alpha) \cdot F(p_\beta), \quad (\alpha, \beta) \in \mathbb{Z}_+^2. \end{aligned}$$

The conclusion is that on basic monomials F is multiplicative. A straightforward computation shows that F is multiplicative on the subspace of polynomials. From the

density of the latter in $C([a, b])$ and also using the continuity of the product operation on $Y \times Y$, we deduce that F is multiplicative on the space of continuous functions. This concludes the proof. $\square$

Corollary 6.1. *If F is the solution from theorem 6.2 and $B_1 = F(u_1) = \int_a^b t d\sigma$, we must have:*

$$F(p_m) = F(\varphi_m) = u_m = v_m = B_1^m \quad \forall m \in \mathbb{Z}_+$$

so that the measure $d\sigma$ leads to the spectral measure of the operator B_1 .

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(Octav Olteanu) DEPARTMENT OF MATHEMATICS - INFORMATICS, POLITEHNICA UNIVERSITY OF BUCHAREST, BUCHAREST 060042, ROMANIA.

E-mail address: olteanuoctav@yahoo.ie