

MULTIPLIERS FOR THE CONJUGATE SERIES OF A FOURIER SERIES

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Abstract. In this paper, we obtain multipliers for the absolute Riesz summability of the conjugate series of a Fourier series which not only improves some of the results obtained by R.Mohanty [Proc. London Math. Soc., 52(1951), 295-320] and P. Chandra [J. Orissa Math. Soc., 30(No. 2)(2011), 1-12] but also provides some new results.

1. Definitions and Notations

Let $\lambda(\omega)$ be the type and $\eta > 0$ be the order of Riesz mean. Then $\sum_{n=1}^{\infty} a_n$ is said to be summable absolutely by the type $\lambda(\omega)$ and order $\eta > 0$ of Riesz mean, symbolically we write,

$$\sum_{n=1}^{\infty} a_n \in |R, \lambda(\omega), \eta| (\eta > 0) \quad (1.1)$$

if ([7]; Definition B)

$$\int_h^{\infty} \{ \lambda^{(1)}(\omega) / \lambda^{(1+\eta)}(\omega) \} | \sum_{n < \omega} \{ \lambda(\omega) - \lambda(n) \}^{\eta-1} \lambda(n) a_n | d\omega < \infty, \quad (1.2)$$

where h is a suitable positive number ([9], [10]) and

$$\lambda^{(1)}(\omega) = \frac{d}{d\omega} \lambda(\omega). \quad (1.3)$$

Throughout, we use the following notations for a real fixed x and all real values of δ :

$$\psi(t) = \frac{1}{2} \{ f(x+t) - f(x-t) \} \quad (1.4)$$

$$\psi_1(t) = \frac{1}{t} \int_0^t \psi(u) du \quad (1.5)$$

$$\beta(t) = \psi(t) - \psi_1(t) \quad (1.6)$$

$$y_n^\delta = \log^{-\delta}(n+1) \quad (1.7)$$

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$$b_n^\delta(t) = \int_t^\pi \log^\delta \frac{k}{u} d\left(\frac{\cos nu}{nu}\right) \quad (1.8)$$

$$e(\omega) = \exp(\omega^\alpha) (0 < \alpha < 1) \quad (1.9)$$

$$P(\omega, \eta) = \{e(\omega) - e(m)\}^{\eta-1} e(m) \quad y_m^\delta \quad (0 < \eta \leq 1). \quad (1.10)$$

We also write $y^\delta(z)$ for y_z^δ if z is a continuous parameter. Throughout k is a positive constant not necessarily the same at each occurrence and

$$m = [\omega], \quad \text{the integral part of } \omega. \quad (1.11)$$

2. Theorems and Comments

Recently, in 2014, we [6] have obtained multipliers, best possible in certain sense, for the absolute Nörlund summability of Fourier series. In the present paper, we obtain multipliers for the conjugate series by using absolute Riesz summability.

Let $\sum_{n=1}^\infty B_n(x)$ denote the conjugate series of the Fourier series of $f \in L_{2\pi}$ at a point x . Then we propose to prove the following:

Theorem 2.1. *Let $\psi(0+) = 0$ and let , for $\delta < 1$,*

$$(i) \quad \beta(t) \log^{1-\delta} \frac{k}{t} \in BV(0, \pi) \quad \text{and} \quad (ii) \quad t^{-1} \beta(t) \log^{-\delta} \frac{k}{t} \in L(0, \pi). \quad (2.1)$$

Then, for $0 < \alpha < 1$,

$$\sum_{n=1}^\infty \frac{B_n(x)}{\log^\delta(n+1)} \in |R, \exp(\omega^\alpha), \eta| \quad (\eta > 0, \delta < 1). \quad (2.2)$$

Earlier, in 2011, Chandra [5] obtained the following theorem which unifies two known results associated with conjugate series of the Fourier series due to Mohanty [7] and Mohanty and Ray [8]:

Theorem 2.2. *Let δ be real and let*

$$(i) \quad \psi(t) \log^{1-\delta} \frac{k}{t} \in BV(0, \pi) \quad \text{and} \quad (ii) \quad t^{-1} \psi(t) \log^{1-\delta} \frac{k}{t} \in L(0, \pi). \quad (2.3)$$

Then (2.2) hold for all real values of δ .

For the sake of comparing the known results with Theorem 2.1, we mention the following theorem due to Mohanty [7]:

Theorem 2.3. *Let*

$$(i) \quad \psi(t) \log \frac{k}{t} \in BV(0, \pi) \quad \text{and} \quad (ii) \quad t^{-1} \psi(t) \in L(0, \pi). \quad (2.4)$$

Then, for $0 < \alpha < 1$,

$$\sum_{n=1}^\infty B_n(x) \in |R, \exp(\omega), 1|. \quad (2.5)$$

In Lemma 5 of the present paper, it has been shown that the space of the functions characterised by (2.1) with $\psi(0+) = 0$ strictly includes the space characterised by (2.3) for $\delta < 1$. Hence Theorem 2.1 improves Theorem 2.2 for $\delta < 1$ in the sense that the former theorem provides larger space of functions than Theorem 2.2. For $\eta = 1$ and $\delta = 0$, we get the following corollary from Theorem 2.1:

Corollary 2.4. Let $\psi(0+) = 0$ and let

$$(i) \quad \beta(t) \log \frac{k}{t} \in BV(0, \pi) \quad \text{and} \quad (ii) \quad t^{-1} \beta(t) \log \frac{k}{t} \in L(0, \pi). \quad (2.6)$$

Then (2.5) holds

We observe that the above corollary improves Theorem 2.3 as per observation made above. We further remark that the condition $t^{-1} \psi(t) \in L(0, \pi)$ of Theorem 2.3 is stronger than the condition $\psi(0+) = 0$ of the above corollary. For other values of $\delta < 1$, we get some new results from Theorem 2.1.

3. Inequalities

For real number δ , $0 < \alpha < 1$ and $0 < \eta \leq 1$, we state the following inequalities obtained uniformly in $0 < t \leq \pi$ which will be used in the proof of the Theorem 2.1:

$$\sum_{n=1}^{m-1} \{e(\omega) - e(n)\}^{\eta-1} e(n) \quad y_n^\delta n^{-2} = O\{e^\eta(\omega) y^\delta(\omega) \omega^{-1-\alpha}\} \quad (3.1)$$

$$\int_2^\infty \{\omega^{\alpha-1} / e^\eta(\omega)\} P(\omega, \eta) d\omega = O(1) \sum_{n=2}^\infty y_n^\delta n^{-1-\eta(1-\alpha)} \quad (3.2)$$

$$b_n^\delta(t) = O(n^{-1}) t^{-1} \log^\delta \frac{k}{t} \quad (3.3)$$

$$b_n^\delta(t) = O(n^{-2}) t^{-2} \log^\delta \frac{k}{t} + \frac{\cos n\pi}{n\pi} \log^\delta \frac{k}{\pi} - \frac{\cos nt}{nt} \log^\delta \frac{k}{t}. \quad (3.4)$$

For the proof of (3.1) and (3.2), see respectively (3.4) and Lemma 10 of Chandra [4].

Proof. of (3.3). We have

$$\begin{aligned} b_n^\delta(t) &= - \int_t^\pi u^{-1} \log^\delta \frac{k}{u} \sin nu \, du - \frac{1}{n} \int_t^\pi u^{-1} \log^\delta \frac{k}{u} \cos nu \, du \\ &= O(n^{-1}) t^{-1} \log^\delta \frac{k}{t} + \frac{1}{n} \int_t^\pi u^{-\frac{1}{2}} \log^\delta \frac{k}{u} u^{-\frac{3}{2}} \, du \\ &= O(n^{-1}) t^{-1} \log^\delta \frac{k}{t}, \end{aligned}$$

uniformly in $0 < t < \pi$, by the second mean value theorem.

Proof. of (3.4) Integrating by parts and using the second mean value theorem, once again, we get

$$\begin{aligned} b_n^\delta(t) &= \int_t^\pi \log^\delta \frac{k}{u} \frac{d}{du} \left(\frac{\cos nu}{nu} \right) du \\ &= \frac{\cos n\pi}{n\pi} \log^\delta \frac{k}{\pi} - \frac{\cos nt}{nt} \log^\delta \frac{k}{t} + \frac{\delta}{n} \int_t^\pi u^{-2} \log^{\delta-1} \frac{k}{u} \cos nu \, du \\ &= \frac{\cos n\pi}{n\pi} \log^\delta \frac{k}{\pi} - \frac{\cos nt}{nt} \log^\delta \frac{k}{t} + O(n^{-2}) t^{-2} \log^\delta \frac{k}{t}. \end{aligned}$$

This completes the proof. □

4. Lemmas

We shall use the following lemmas in the proof of the theorem:

Lemma 4.1. *Let $(H(t))^{-1}$ be bounded, non-decreasing and that $(H(t))^{-1} \rightarrow 0$ as $t \rightarrow 0+$ and let $\sigma > 0$ be finite. Then in order that*

$$\begin{aligned} (i) \quad & F(t)H(t) \in BV(0, \sigma) \text{ and} \\ (ii) \quad & F(t) \frac{d}{dt} H(t) \in L(0, \sigma) \end{aligned} \quad (4.1)$$

should hold, it is necessary and sufficient that

$$\begin{aligned} (i) \quad & \int_0^\sigma |H(t)| |dF(t)| < \infty \text{ and} \\ (ii) \quad & F(0+) = 0 \end{aligned} \quad (4.2)$$

should hold.

See Chandra [1]; Lemma 7, the case $H(t) = \log \frac{k}{t}$ was first obtained by Mohanty [7].

Lemma 4.2 ([3]). *Let F be L -integrable over $(0, \sigma)$. Then in order that*

$$F(t) \in BV(0, \sigma) \quad (4.3)$$

it is necessary and sufficient that

$$\begin{aligned} (i) \quad & G(t) \in BV(0, \sigma) \text{ and} \\ (ii) \quad & t^{-1}G(t) \in L(0, \sigma), \end{aligned} \quad (4.4)$$

where $G(t) = F(t) - \frac{1}{t} \int_0^t F(u) du$.

Lemma 4.3. *Let $\psi(0+) = 0$ and let $\psi(t) \in BV(0, \pi)$. Then*

$$B_n(x) = -\frac{2}{\pi} \psi_1(\pi) \frac{\cos n\pi}{n} - \frac{2}{\pi} \int_0^\pi \beta(t) t d\left(\frac{\cos nt}{nt}\right), \quad (4.5)$$

where $B_n(x) = \frac{2}{\pi} \int_0^\pi \psi(t) \sin nt dt$.

Proof. Integrating by parts, we get

$$\begin{aligned} B_n(x) &= \frac{2}{n\pi} [\psi(0+) - \psi(\pi) \cos n\pi] + \frac{2}{n\pi} \int_0^\pi \cos nt d\psi(t) \\ &= \frac{2}{n\pi} [-\psi(\pi) \cos n\pi] + \frac{2}{n\pi} \int_0^\pi \cos nt d\psi(t) \\ &= \frac{2}{n\pi} \int_0^\pi \cos nt d\psi(t), \end{aligned}$$

since by hypothesis $\psi(0+) = 0$ and, by periodicity of f , $\psi(\pi) = 0$. Now by using the hypothesis $\psi(t) \in BV(0, \pi)$, we may write

$$\beta(t) = \frac{1}{t} \int_0^t u d\psi(u).$$

Thus, further integrating by parts and using the above definition of $\beta(t)$, we get

$$\begin{aligned} B_n(x) &= \frac{2}{n\pi} \beta(\pi) \cos n\pi - \frac{2}{\pi} \int_0^\pi t \beta(t) d\left(\frac{\cos nt}{nt}\right) \\ &= -\frac{2}{\pi} \psi_1(\pi) \frac{\cos n\pi}{n} - \frac{2}{\pi} \int_0^\pi \beta(t) t d\left(\frac{\cos nt}{nt}\right), \end{aligned}$$

since $\psi(0+) = 0 \Rightarrow \beta(0+) = 0$ and $\beta(\pi) = -\psi_1(\pi)$. This completes the proof of Lemma 4.3. $\square$

Lemma 4.4. *Let, for all real values of δ ,*

$$A = \{f \in L_{2\pi} : \psi(t) \log^{1-\delta} \frac{k}{t} \in BV(0, \pi) \text{ and } t^{-1} \psi(t) \log^{-\delta} \frac{k}{t} \in L(0, \pi)\}$$

and

$$B = \{f \in L_{2\pi} : \psi(t) \log^{1-\delta} \frac{k}{t} \in BV(0, \pi) \text{ and } t^{-1} \psi_1(t) \log^{-\delta} \frac{k}{t} \in L(0, \pi)\}$$

Then the inclusion:

$$A \subset B \text{ is strict} \tag{4.6}$$

Proof. We first remark that the conditions $\psi(t) \log^{1-\delta} \frac{k}{t} \in BV(0, \pi)$ and $t^{-1} \psi(t) \log^{-\delta} \frac{k}{t} \in L(0, \pi)$ are independent for all real values of δ which can be observed by the following examples of $f \in L_{2\pi}$, where

$$\psi(t) \log^{1-\delta} \frac{k}{t} = \left(\log \log \frac{k}{t} \right)^{-1} (0 < t \leq \pi) \tag{4.7}$$

$$\psi(t) \log^{1-\delta} \frac{k}{t} = \begin{cases} t \sin \frac{1}{t} & (0 < t \leq \pi) \\ 0 & t = 0 \end{cases} \tag{4.8}$$

We now prove (4.6). Let $f \in A$. Then $t^{-1} \psi(t) \log^{-\delta} \frac{k}{t} \in L(0, \pi)$ and hence

$$\begin{aligned} \int_0^\pi \frac{|\psi_1(t)|}{t \log^\delta \frac{k}{t}} dt &\leq \int_0^\pi t^{-2} \log^{-\delta} \frac{k}{t} dt \int_0^t |\psi(u)| du \\ &= \int_0^\pi |\psi(u)| du \int_u^\pi t^{-1} \log^{-\delta} \frac{k}{t} dt \\ &\leq \int_0^\pi \frac{|\psi(t)|}{t \log^\delta \frac{k}{t}} dt < \infty, \end{aligned}$$

since $f \in A$. This proves that $f \in B$. To prove that this inclusion is strict, we consider a function $f \in L_{2\pi}$ and point $x = 0$ and define

$$\frac{1}{2} f(t) = \begin{cases} \left\{ \left(2 - \delta / \log \frac{k}{t} \right) t \sin \frac{1}{t} - \cos \frac{1}{t} \right\} \log^\delta \frac{k}{t} & (0 < t \leq \pi) \\ 0 & (-\pi \leq t \leq 0) \end{cases}$$

and by periodicity elsewhere. Then

$$\begin{aligned} \int_0^\pi \frac{|\psi(t)|}{t \log^\delta \frac{k}{t}} dt &= \int_0^\pi t^{-1} \left| \left(2 - \delta / \log \frac{k}{t} \right) t \sin \frac{1}{t} - \cos \frac{1}{t} \right| dt \\ &\geq \int_0^\pi t^{-1} \left| \cos \frac{1}{t} \right| dt - \int_0^\pi \left(2 + \delta / \log \frac{k}{t} \right) dt = \infty \end{aligned}$$

and hence $f \notin A$. However,

$$\begin{aligned} \psi_1(t) &= \frac{1}{t} \int_0^t f(u) du = \frac{1}{t} \int_0^t d \left(u^2 \log^\delta \frac{k}{u} \sin \frac{1}{u} \right) \\ &= t \log^\delta \frac{k}{t} \sin \frac{1}{t} \end{aligned}$$

so that

$$\int_0^\pi \frac{|\psi_1(t)|}{t} \log^{-\delta} \frac{k}{t} dt = \int_0^\pi \left| \sin \frac{1}{t} \right| dt < \infty$$

Consequently, $f \in B$. This proves (4.6). $\square$

Lemma 4.5. *Let, for all real values of δ ,*

$$C = \left\{ f \in L_{2\pi} : \begin{array}{ll} (i) & \beta(t) \log^{1-\delta} \frac{k}{t} \in BV(0, \pi) \\ (ii) & t^{-1} \beta(t) \log^{1-\delta} \frac{k}{t} \in L(0, \pi). \end{array} \right.$$

Then for A , defined in Lemma 4.4, the inclusion:

$$A \subset C \quad \text{is strict.} \quad (4.9)$$

Proof. In view of Lemma 4.4, it is sufficient to show that

$$B \subset C. \quad (4.10)$$

Let $f \in B$. Then $\psi(t) \log^{1-\delta} \frac{k}{t} \in BV(0, \pi)$ and by Lemma 4.2 with $F(t) = \psi(t) \log^{1-\delta} \frac{k}{t}$,

$$G(t) = \left\{ \psi(t) \log^{1-\delta} \frac{k}{t} - \frac{1}{t} \int_0^t \psi(u) \log^{1-\delta} \frac{k}{u} du \right\} \in BV(0, \pi) \quad (4.11)$$

$$\Leftrightarrow \{\psi(t) - \psi_1(t)\} \log^{1-\delta} \frac{k}{t} \in BV(0, \pi)$$

$$= \beta(t) \log^{1-\delta} \frac{k}{t} \in BV(0, \pi),$$

by Theorem 2 of Chandra [3]. This proves that the condition (i) of the space C holds for all real values of δ . Once again by Lemma 4.2,

$$t^{-1} G(t) \in L(0, \pi). \quad (4.12)$$

However,

$$\frac{1}{t} \int_0^t \psi(u) \log^{1-\delta} \frac{k}{u} du = \psi_1(t) \log^{1-\delta} \frac{k}{t} + \frac{1-\delta}{t} \int_0^t \psi_1(u) \log^{-\delta} \frac{k}{u} du. \quad (4.13)$$

Hence, by using (4.13) in the notation used in (4.11), we get

$$t^{-1}G(t) = t^{-1}\beta(t) \log^{1-\delta} \frac{k}{t} - \frac{1-\delta}{t^2} \int_0^t \psi_1(u) \log^{-\delta} \frac{k}{u} du. \quad (4.14)$$

In view of (4.12), the above equation shows that condition (ii) of the space C holds for all real values of δ if and only if

$$\left\{ \frac{1}{t^2} \int_0^t \psi_1(u) \log^{-\delta} \frac{k}{u} du \right\} \in L(0, \pi), \quad (4.15)$$

which holds since $f \in B$ and

$$\int_0^\pi \frac{1}{t^2} \left| \int_0^t \psi_1(u) \log^{-\delta} \frac{k}{u} du \right| dt \leq \int_0^\pi u^{-1} |\psi_1(u)| \log^{-\delta} \frac{k}{u} du < \infty.$$

Consequently, this completes the proof of (4.10). $\square$

Lemma 4.6. ([2]). *Let $0 < \eta \leq 1, \omega > t^{-1}$. Then for all real values of δ ,*

$$\sum_{n=1}^{m-1} \{e(\omega) - e(n)\}^{\eta-1} e(n) y_n^\delta n^{-1} \cos nt = O\left\{t^{-\eta} e^\eta(\omega) y^\delta(\omega) \omega^{-\alpha-\eta(1-\alpha)}\right\},$$

uniformly in $0 < t \leq \pi$.

5. Proof of Theorem 2.1.

In view of the first consistency theorem for the absolute Riesz summability ([9],[10]), it is sufficient to prove the theorem for $0 < \eta \leq 1$. By Lemma 4.1, it follows that the hypothesis (2.1) holds if and only if

$$(i) \int_0^\pi t^{-1} \log \frac{k}{t} |dg(t)| < \infty \quad \text{and} \quad (ii) \quad g(0+) = 0 \quad (5.1)$$

where $g(t) = \frac{t\beta(t)}{\log^\delta \frac{k}{t}}$, Further, (2.1) implies that

$$(i) \quad \beta(t) \in BV(0, \pi) \quad \text{and} \quad (ii) \quad t^{-1}\beta(t) \in L(0, \pi) \quad (5.2)$$

which, by Lemma 4.2, is equivalent to

$$\psi(t) \in BV(0, \pi). \quad (5.3)$$

By (5.3) and the hypothesis $\psi(0+) = 0$, we get from Lemma 3

$$\begin{aligned} B_n(x) &= \frac{2}{\pi} \int_0^\pi \psi(t) \sin nt \, dt \\ &= -\frac{2}{\pi} \psi_1(t) \frac{\cos n\pi}{n} - \frac{2}{\pi} \int_0^\pi \beta(t) t \, d\left(\frac{\cos nt}{nt}\right) \\ &= -\frac{2}{\pi} (p_n + q_n), \text{ say.} \end{aligned}$$

Further, integrating by parts and using (5.1)(ii), we get

$$\begin{aligned} q_n &= \int_0^\pi g(t) \log^\delta \frac{k}{t} \, d\left(\frac{\cos nt}{nt}\right) \\ &= \int_0^\pi b_n^\delta(t) dg(t) \end{aligned} \quad (5.4)$$

Now

$$\sum_{n=1}^\infty B_n(x) y_n^\delta \in |R, e(\omega), \eta| \quad (0 < \eta \leq 1) \quad (5.5)$$

if

$$\sum_{n=1}^\infty p_n y_n^\delta \in |R, e(\omega), \eta| \quad (0 < \eta \leq 1) \quad (5.6)$$

and

$$\sum_{n=1}^\infty q_n y_n^\delta \in |R, e(\omega), \eta| \quad (0 < \eta \leq 1) \quad (5.7)$$

For the proof of (5.6), it is sufficient to show that

$$\mathfrak{X} = \int_e^\infty \left\{ \frac{\omega^{\alpha-1}}{e^\eta(\omega)} \right\} \left| \sum_{n=1}^m \{e(\omega) - e(n)\}^{\eta-1} e(n) y_n^\delta \frac{\cos n\pi}{n} \right| d\omega < \infty. \quad (5.8)$$

By using (1.10), we get

$$\begin{aligned} \mathfrak{X} &= \int_e^\infty \left\{ \frac{\omega^{\alpha-1}}{e^\eta(\omega)} \right\} \left| \sum_{m=1}^{m-1} \{e(\omega) - e(n)\}^{\eta-1} e(n) y_n^\delta \frac{\cos n\pi}{n} \right| d\omega \\ &\quad + O(1) \int_e^\infty \{\omega^{\alpha-1}/e^\eta(\omega)\} P(\omega, \eta) d\omega \end{aligned}$$

$$= \mathfrak{X}_1 + \mathfrak{X}_2, \text{ say.}$$

By (3.2), we get

$$\mathfrak{X}_2 = O(1) \sum_{n=1}^\infty y_n^\delta n^{-1-\eta(1-\alpha)} < \infty,$$

and by applying Abel's lemma , we get

$$\begin{aligned}\mathfrak{X}_1 &= O(1) \int_e^\infty \left\{ \frac{\omega^{\alpha-1}}{e^\eta(\omega)} \right\} \{e(\omega) - e(m-1)\}^{\eta-1} e(m-1) \frac{y^\delta(\omega)}{\omega} d\omega \\ &= O(1) \int_e^\infty \frac{y^\delta(\omega)}{\omega^{1+\eta(1-\alpha)}} d\omega < \infty\end{aligned}$$

since

$$\begin{aligned}\{e(\omega) - e(m-1)\}^{\eta-1} &\leq \{e^{(1)}(m-1)\}^{\eta-1} \\ &= O(1) \omega^{(\alpha-1)(\eta-1)} e^{\eta-1}(m-1).\end{aligned}$$

This completes the proof of (5.8). Now, we observe that (5.7) holds if

$$\int_e^\infty \left\{ \omega^{\alpha-1}/e^\eta(\omega) \right\} \left| \sum_{n<\omega} \{e(\omega) - e(n)\}^{\eta-1} e(n) y_n^\delta \int_0^\pi b_n^\delta(t) dg(t) \right| d\omega \quad (5.9)$$

is convergent. And in view of (5.1)(i) , it is sufficient, for the proof of (5.9), to show that

$$R = \int_e^\infty \left\{ \omega^{\alpha-1}/e^\eta(\omega) \right\} \left| \sum_{n=1}^m \{e(\omega) - e(n)\}^{\eta-1} e(n) y_n^\delta b_n^\delta(t) \right| d\omega \quad (5.10)$$

$$= O(t^{-1}) \log \frac{k}{t}, \quad \text{uniformly in } 0 < t < \pi.$$

We write

$$R = J + L, \quad \text{say} \quad (5.11)$$

where, by (3.3), (1.10) and (3.2), we get

$$L = \int_e^\infty \left\{ \omega^{\alpha-1}/e^\eta(\omega) \right\} \left| \{e(\omega) - e(m)\}^{\eta-1} e(m) y_m^\delta b_m^\delta(t) \right| d\omega \quad (5.12)$$

$$\begin{aligned}&= O(t^{-1}) \log^\delta \frac{k}{t} \int_e^\infty \frac{\omega^{\alpha-1}}{e^\eta(\omega)} P(\omega, \eta) d\omega \\ &= O(t^{-1}) \log^\delta \frac{k}{t} \sum_{n=2}^\infty y_n^\delta n^{-1-\eta(1-\alpha)} \\ &= O(t^{-1}) \log \frac{k}{t}\end{aligned}$$

uniformly in $0 < t < \pi$, since $\delta < 1$. And

$$\begin{aligned}J &= \int_e^\infty \left\{ \omega^{\alpha-1}/e^\eta(\omega) \right\} \left| \sum_{n=1}^{m-1} \{e(\omega) - e(n)\}^{\eta-1} e(n) y_n^\delta b_n^\delta(t) \right| d\omega \\ &= \int_e^T + \int_T^\infty = J_1 + J_2, \quad \text{say}\end{aligned} \quad (5.13)$$

where $T = (\frac{k}{t})^{\frac{1}{1-\alpha}}$ ($0 < \alpha < 1$). By (3.3), once again

$$\begin{aligned}
J_1 &= O(t^{-1}) \log^\delta \frac{k}{t} \int_1^T \left\{ \frac{e^{(1)}\omega}{e^{1+\eta}(\omega)} d\omega \right\} \int_1^\omega \{e(\omega) - e(z)\}^{\eta-1} e(z) \frac{y^\delta(z)}{z} dz \\
&= O(t^{-1}) \log^\delta \frac{k}{t} \int_1^T \left\{ \frac{y^\delta(z)}{ze^\eta(z)} dz \right\} \int_z^T \{e(\omega) - e(z)\}^{\eta-1} e^{(1)}(\omega) d\omega \\
&\quad \text{(by changing the order of integration)} \\
&= O(t^{-1}) \log^\delta \frac{k}{t} \int_2^T z^{-1} \log^{-\delta} z dz \quad (\delta < 1) \\
&= O(t^{-1}) \log^\delta \frac{k}{t},
\end{aligned}$$

uniformly in $0 < t < \pi$. And, by (3.4), we get

$$\begin{aligned}
J_2 &= O(t^{-1}) \log^\delta \frac{k}{t} \int_T^\infty \left\{ \frac{\omega^{\alpha-1}}{e^\eta(\omega)} \right\} \left| \sum_{n=1}^{m-1} \{e(\omega) - e(n)\}^{\eta-1} e(n) y_n^\delta \frac{\cos nt}{n} \right| d\omega \\
&\quad + O(t^{-2}) \log^\delta \frac{k}{t} \int_T^\infty \left\{ \frac{\omega^{\alpha-1}}{e^\eta(\omega)} \right\} \left| \sum_{n=1}^{m-1} \{e(\omega) - e(n)\}^{\eta-1} e(n) y_n^\delta n^{-2} \right| d\omega \\
&\quad + O(1) \int_1^\infty \left\{ \frac{\omega^{\alpha-1}}{e^\eta(\omega)} \right\} \left| \sum_{n=1}^{m-1} \{e(\omega) - e(n)\}^{\eta-1} e(n) y_n^\delta \frac{\cos n\pi}{n} \right| d\omega \\
&= O(t^{-1-\eta}) \log^\delta \frac{k}{t} \int_T^\infty y^\delta(\omega) \omega^{-1-\eta(1-\alpha)} + O(t^{-2}) \log^\delta \frac{k}{t} \int_T^\infty y^\delta(\omega) \omega^{-2} + O(1) \\
&= O(t^{-1})
\end{aligned}$$

uniformly in $0 < t < \pi$, by Lemma 4.6 and (3.1). By using the estimates obtained for J_1 and J_2 in (5.13) and combining (5.9) through (5.13), we get the proof of (5.7). This completes the proof of the theorem. $\square$

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