

BOUNDEDNESS, CONTINUITY AND BOUNDED HOMOMORPHISMS IN TOPOLOGICAL GROUPS

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Abstract. In this study, we define a new boundedness concept for topological groups which is compatible with bounded set definition in topological vector spaces. Therefore we extend some definitions and theorems concerned with boundedness in topological vector spaces to topological groups by using this new (boundedness) concept. Further, we prove some new theorems and propositions in topological groups.

1. Introduction

There exists considerable literature of boundedness concept in the topological groups, topological vector spaces and uniform spaces. This concept has been investigated intensively for three decades and has been used on the uniform topological spaces by C. J. Atkin in [1]. Since topological groups are private and convenient class of uniform topological spaces, we can use this definition for topological groups. The other boundedness definitions in topological groups have been presented by Bruguera and Tkachenko, Hejzman [3], [4]. Furthermore Bonales, Hernandez, Kaplan, Smith and Trigos-Arrieta have studied as connected with Pontryagin duality in topological groups [2], [5], [6], [7], [8], [9], [11], [12].

The question is that inspire to us "Can the boundedness concept in topological vector spaces be used to the topological groups as more general?". Thus, our principal goal is to define a new boundedness concept and then to mention some new results on topological groups. Furthermore we give the definition of absorbing set with introducing the set inflation in groups in this paper. The definition is also the generalization of current available boundedness definition in topological vector spaces to topological groups because every topological vector spaces have an additive topological group structure. Therefore we present a kind of boundedness definition in topological groups. The new

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boundedness definition is not a generalization of existing ones for topological groups. We also examined the order relation between the initial topology and Bohr (weak) topology of a group which topologies compatible with duality. The Bohr topology on a topological abelian group is a topology induced by all continuous characters and it is a group topology [10], [13]. Bounded sets are bounded in all compatible topologies with the duality in topological vector spaces. But we prove that boundedness is not duality invariant in topological groups although it is a duality invariant in topological vector spaces. Lastly, we give an example of a Bohr bounded set which is not bounded in the initial topology.

2. Preliminaries, background and notation

Let G be an abstract group A and B be two subsets of G . Then AB is the set of all elements of xy such that $x \in A$ and $y \in B$. The definition of A^2 and $A^m = A^{m-1}A$ is clear by taking $B = A$ for some $m \in \mathbb{N}$. Further, $A^{-1} = \{a^{-1} : a \in A\}$, $A^{-m} = (A^{-1})^m$ and for the unit element e of G , $A^0 = \{e\}$. Given $x \in A^m$ there exists some $a_1, a_2, \dots, a_m \in A$ such that $x = a_1 a_2 \dots a_m$. If $x^m \in A^m$ and $e \notin A$ then x^n may not be an element of A^m for $n < m$. Hence we define the set $A^{\leq m}$ by $x = a_1 a_2 \dots a_n$ for $m, n \in \mathbb{N}$, $a_1, a_2, \dots, a_n \in A$ and some $n \leq m$. It is clear that $A^m \subseteq A^{\leq m}$ and $A^m = A^{\leq m}$ whenever e is contained by A . The set A is said to be symmetric providing that $A^{-1} = A$ and symmetric hull of A is defined by

$$[A] := \cap \{U : U \text{ is a symmetric set and } A \subseteq U\}.$$

It is clear that $[A]$ is a symmetric set. So $[A] = \cap \{U^{-1} : A^{-1} \subseteq U^{-1}\}$ and $[A] = \cap \{U^{-1} : A \subseteq U^{-1}\}$ then

$$[A] = A^{-1} \cup A.$$

It can easily be seen that $[A^m] \subseteq [A]^m$.

Every topological vector space is an additive group then it is written mU for U^m in this case. A set B is bounded in a TVS if and only if there is a positive integer m such that $B \subseteq mU$ or $\frac{1}{m}B \subseteq U$ for every symmetric neighbourhood U of 0. So we can produce a similar bounded set definition by writing U^m for mU in topological groups. This is actually a refined form of the definition in [14]. But we get some affiliation of boundedness (as in TVSs) for topological group and relations of boundedness and continuity with this way such as bornological group and scalar homomorphism on it.

A connected topological group G is generated by a neighbourhood U of the unit element. That is, every element of G can be represented by finite multiplication of elements of U [10]. A set S is said to be precompact in a topological group if there exists a finite set F for each neighbourhood U of e such that $S \subset FU$. We have known that in a TVS with semimetrizable vector topology (it must be a paranormed space in this case), if a set is bounded then it is metrically bounded i.e. boundedness with respect to the semimetric. But inverse of this proposition is not correct [14].

Let G be a group and $p : G \rightarrow \mathbb{R}$ be a function. p is called an absolute value function on G if satisfies the following properties for each $x, y, a \in G$

- (i) $p(x) \geq 0$,
- (ii) $p(e) = 0$ and $p(x^{-1}) = p(x)$,
- (iii) $p(xy) \leq p(x) + p(y)$,
- (iv) If $p(x_n) \rightarrow 0$ then $p(ax_n a^{-1}) \rightarrow 0$ for every sequence (x_n) .

Last condition is unnecessary for abelian groups. The equality $d(x, y) = p(x^{-1}y)$ defines a semimetric generating group topology on G . d is called a left invariant semimetric if $d(ax, ay) = d(x, y)$ for every $x, y, a \in G$. The topology of a topological group first countable comes from a left invariant semimetric [15].

Definition 2.1. Let G be a topological group and $A \subset G$. The set A is called an absorbing set if there exists a natural number m such that $x \in [A]^{\leq m}$.

Definition 2.2. Let G be a topological group and $A \subset G$. The set A is said to be bounded whenever absorbed by every neighbourhood of the unit element e . This is equivalent to find a natural number m satisfying $A \subseteq [U]^m$ for every neighbourhood U of e .

If a mapping conserves bounded sets between topological groups then this mapping is called as bounded map.

Definition 2.3. Let G be a topological group and $B \subseteq G$. If the set B absorbs every bounded set then B is called a bornivorous.

Let G be a topological group. If every bornivorous in G is a neighbourhood of e then G is called bornological group [10].

Now, we mention that basic definitions for last section. Let S_1 be a group with the known multiplication operation of complex numbers whose absolute value is equal to 1. Together with the Euclid topology this group is an abelian compact metric group. S_1 holds an important place in the duality theory of topological groups. Construction of the duality theory would be as follows for also an group X .

Consider the set of all the homomorphisms from X into the S_1 . Each element of this set is called as a character of X . In other words each homomorphism from X into the S_1 is a character of X . Now let us form an operation over this set. Let $f_1 : X \rightarrow S_1$ and $f_2 : X \rightarrow S_1$ (characters of X) be homomorphisms for $x \in X$

$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

Furthermore let us denote that the unit element of this group,

$$1_X : X \rightarrow S_1, 1_X(x) = 1$$

is also a homomorphism. In this group, the set of the continuous characters be a subgroup and this subgroup is called character group or dual group of X . In the case that X is not abelian character group may become trivial group.

The character group of X is denoted by X^* . The character group induces such a topology over X . If we denote this topology with τ_w , the set $U \subset X$ is called as an open set according to τ_w or $U \in \tau_w$ if and only if $f_1, f_2, \dots, f_n \in X^*$. There exist finite characters $f_1, f_2, \dots, f_n$ and open sets $V_1, V_2, \dots, V_n \subset S_1$ such that $\bigcap_{i=1}^n f_i^{-1}(V_i) \subset U$.

3. Main Results

In this section, we give some theorems and lemmas in topological groups.

Lemma 3.1. *If U is a symmetric neighbourhood of e in a topological group G then $\text{int}(U)$ is an open and symmetric neighbourhood of e .*

Proof. It is clear that $\text{int}(A)$ is open neighbourhood of e . We show only that $\text{int}(A)$ be symmetric. Since $A^{-1} = A$, it is sufficient to show that $(\text{int}(A))^{-1} = \text{int}(A^{-1})$. If $x \in (\text{int}(A))^{-1}$ then $x^{-1} \in \text{int}(A)$ and so there exists a set B such that $x^{-1} \in B$, B is open set and $B \subseteq A$. Because of $B \subseteq A$ if and only if $B^{-1} \subseteq A^{-1}$ and the inverse operation in topological groups is a homomorphism so B^{-1} be open and $x \in \text{int}(A^{-1})$. On the other hand, if $x \in \text{int}(A^{-1})$ then there exists an open set $D \subseteq A^{-1}$ such that $x \in D$. Additionally, D^{-1} is an open set and $D^{-1} \subseteq A$ so $x^{-1} \in D^{-1}$ and $x^{-1} \in \text{int}(A)$. That is $x \in (\text{int}(A))^{-1}$. $\square$

Corollary 3.2. Every symmetric neighbourhood of e includes an open symmetric neighbourhood of e in a topological group G .

Corollary 3.3. A family of open symmetric neighbourhood of e is a local basis in a topological group G .

We express other definition of boundedness in following theorem.

Theorem 3.4. Let G be a topological group. A set B is absorbed in G by every element in a local basis of e if and only if B is bounded.

Proof. Sufficiency is trivial. Let $B \subset G$ and $\mathcal{B}$ be a local basis of neighbourhoods of e . We suppose that B is absorbed by every set U of $\mathcal{B}$. If U is any symmetric neighbourhood of e then there exists $V \in \mathcal{B}$ such that $V \subseteq U$ because $\mathcal{B}$ is a basis.

On the other hand for every $m \in \mathbb{N}$

$$V \subseteq U \iff [V]^m \subseteq [U]^m \text{ and } B \subseteq [V]^m$$

so $B \subseteq [U]^m$. $\square$

According to this theorem we can say that a set B is bounded set whenever it is absorbed by symmetric neighbourhood of G . In this case we can write $B \subseteq U^m$ instead of $B \subseteq [U]^m$ since $[U] = U$.

Theorem 3.5. Union of two bounded sets is bounded in a topological group.

Proof. Let A and B be bounded sets in a topological group X . Since A and B be bounded, for every $U \in N_e$ there exists numbers $m, n \in \mathbb{N}$ such that $A \subseteq U^m$ and $B \subseteq U^n$.

On the other hand, let $x \in U^m \cup U^n$. So $x \in U^m$ or $x \in U^n$. That is $x \in U^m$ if and only if $x = a_1 a_2 \dots a_m$ such that $a_1, a_2, \dots, a_m \in U$ or $x \in U^n$ if and only if $x = b_1 b_2 \dots b_n$ such that $b_1, b_2, \dots, b_n \in U$. Since $m \leq m+n$ and $n \leq m+n$, $x = a_1 a_2 \dots a_m e e \dots e \in U^{m+n}$ for n number e and $x = b_1 b_2 \dots b_n e e \dots e \in U^{m+n}$ for m number e then $x \in U^{m+n}$. Therefore we have $A \cup B \subseteq U^m \cup U^n \subseteq U^{m+n}$ by proving $A \cup B$ is a bounded set. $A \cup B \subseteq U^m \cup U^n \subseteq U^{m+n}$ by proving $A \cup B$ is a bounded set. $\square$

Corollary 3.6. Each subset of a bounded set and intersection of two bounded sets are bounded in a topological group.

Remark 3.7. Since boundedness definition in this work has been defined more primitively. This gets some deficiencies. For example singletons are bounded in topological vector spaces however this is not true in topological groups with this boundedness definition as the next example shows.

Example 3.8. Let us consider the topological group $(\mathbb{Z}, +)$ with the usual metric topology. We can define open sets of the induced topology as the intersections of open intervals of $\mathbb{R}$ with the integer numbers $\mathbb{Z}$. The continuity of the group operation is trivial according to this topology. All of the sets in the topology are both open and closed so $(\mathbb{Z}, +)$ is an extremely disconnected topological spaces. Now let us show that the singleton $\{1\}$ is not bounded. This is true because $\{0\}$ is a neighbourhood of 0. Therefore $\{1\}$ isn't subset of $\{0\}^m$ since $\{0\}^m = \{0\}$ for every positive integer m .

Remark 3.9. Since each singleton is bounded, this property is not exist in TVSs. Hence each finite set is bounded in TVSs. We show that this is caused by the disconnectedness. Note that each TVSs is connected topological spaces.

Now let us see relations between boundedness and connectedness.

Theorem 3.10. *Each singleton is bounded in a connected topological group G .*

Proof. Let $x \in G$ be an arbitrary point and U be a symmetric neighbourhood of e . So there exists $n \leq m$ and $a_1, a_2, \dots, a_n \in U$ for $m \in \mathbb{N}$ such that $x = a_1 a_2 \dots a_m \in U^m$ by above theorem. Therefore $\{x\} \subseteq U$. $\square$

Corollary 3.11. It is easily seen by induction method that each finite set is bounded in a connected topological group G .

Theorem 3.12. *Every precompact set is bounded in a connected topological group.*

Proof. Let G be a connected topological group, S be a precompact set in G , U be any symmetric neighbourhood of e and V be an other symmetric neighbourhood of e such that $VV \subset U$. There exists a finite set F such that $S \subset FV$ by hypothesis. Thus F is bounded since each finite set is bounded. So there exists a number $n \in \mathbb{N}$ such that $F \subseteq V^n$. Then $S \subset FV \subset V^n V \subset V^n V^n = (VV)^n \subset U^n$ and in the end S is a bounded set. $\square$

Corollary 3.13. Each compact set be bounded set in a connected topological group.

Remark 3.14. It isn't possible to give this corollary for every topological group. Remember that the singleton $\{1\}$ is a compact set but isn't bounded.

Corollary 3.15. Every neighbourhood of e is an absorbing set in a connected topological group.

Proposition 3.16. Every subset of extremely disconnected T_1 – topological group isn't bounded. Only the singleton $\{e\}$ is bounded.

Proof. Let $B \subseteq G$ and $B \neq \{e\}$. Then $B = \cup \{x\}$ such that $x \in B$. Since $\{x\}$ be closed in G and G is extremely disconnected, $\{x\}$ is an open set. So $\{e\}$ is a neighbourhood of e .

Now we think that a set $\{x\}$ for $x \in B$ such that $x \neq e$. $\{e\}^m = \{e\}$ for every number $m \in \mathbb{N}$ and thus $\{x\}$ is not absorbed by $\{e\}$. Then B is also not absorbed by $\{e\}$. That is B isn't bounded. $\square$

If G is a semimetric topological group then there exists other well known concept of boundedness which name is metrically boundedness. A set $B \subseteq G$ is metrically bounded

in a semimetric space G whenever

$$\sup_{x,y \in B} \{d(x,y)\} < \infty.$$

Now we discuss that when metrically boundedness concept coincides with the boundedness in a semimetric group. A similar discussion was already done in the topological vector spaces.

Theorem 3.17. *Let G be a semimetric group and $A \subseteq G$ be a bounded set. Then A is metrically bounded.*

Proof. The sphere $D_\varepsilon = \{x : d(x,e) \leq \varepsilon\}$ is a neighbourhood of e for a number $\varepsilon > 0$. We know that d is a left invariant semimetric on G . Since A be bounded then $A \subseteq [D_\varepsilon(e)]^m$ for a number $m \in \mathbb{N}$. Now let us show that $[D_\varepsilon(e)]^m \subseteq D_{m\varepsilon}(e)$. Take some $z \in [D_\varepsilon(e)]^m$ then there exist $a_1, a_2, \dots, a_n \in D_\varepsilon(e)$ such that $z = a_1 a_2 \dots a_n$. Let p be the absolute value function giving with the semimetric d then

$$\begin{aligned} d(z,e) &= p(z^{-1}e) \\ &= p(z^{-1}) \\ &= p(z) \\ &= p(a_1 a_2 \dots a_n) \\ &\leq p(a_1) + p(a_2) + \dots + p(a_n) \\ &= d(a_1, e) + d(a_2, e) + \dots + d(a_n, e) \\ &\leq \varepsilon + \varepsilon + \dots + \varepsilon \\ &= n.\varepsilon \leq m.\varepsilon. \end{aligned}$$

Hence $z \in D_{m\varepsilon}(e)$. This proves that $A \subseteq D_{m\varepsilon}(e)$ i.e. $\sup_{x,y \in A} \{d(x,y)\} \leq 2m\varepsilon$ and A is (metrically) bounded. $\square$

Remark 3.18. Opposite of the theorem usually isn't true by example 3.7. Because the singleton $\{1\}$ is metrically bounded in metric space $(\mathbb{Z}, d)$.

Theorem 3.19. *A set B is bounded if and only if every countable subset of B is bounded in a topological group.*

Proof. Only the sufficiency needs a proof. We assume that every countable subset of B be bounded but not B . In this case there exists a symmetric neighbourhood U of e for every number $m \in \mathbb{N}$ such that B isn't included by U^m .

Now, let us construct $\{x_m\}_{m=1}^\infty$ such that

$$x_1 \in B \setminus U, x_2 \in B \setminus U^2, \dots, x_i \in B \setminus U^i, \dots$$

Obviously the sequence $\{x_m\}_{m=1}^\infty$ is not absorbed by U . That is $\{x_m\}_{m=1}^\infty \subset B$ isn't bounded. This is a contradiction with our hypothesis. $\square$

4. Bounded Transformations

In this section we research that relationship between mappings on topological groups.

Theorem 4.1. *A continuous homomorphism in between topological groups be bounded.*

Proof. Let G and G' be two topological groups, $f : G \rightarrow G'$ be a homomorphism and $S \subset G$ be bounded. Furthermore let e and e' are unit elements of G and G' , respectively. Because of S is bounded there exists a number $n \in \mathbb{N}$ such that $S \subset U^n$ for every symmetric neighbourhood U of e .

If V be a symmetric neighbourhood of e' then $f^{-1}(V)$ be also symmetric neighbourhood of e because f is continuous. Therefore $S \subset \{f^{-1}(V)\}^n$ for a number $n \in \mathbb{N}$. Since $f^{-1}(V) = f^{-1}(V \cup V^{-1}) = f^{-1}(V) \cup f^{-1}(V^{-1})$, $f^{-1}(V)$ is equal to its symmetric hull. If we show that $\{f^{-1}(V)\}^n \subseteq f^{-1}(V^n)$ then we the proof of this theorem is completed. Since $S \subseteq f^{-1}(V^n)$ then $f(S) \subset V^n$. So $f(S)$ is absorbed by the set V . This imply that $f(S)$ be bounded.

Now we proof that above relation. Let take $z \in \{f^{-1}(V)\}^n$. Then it is written as $z = a_1 a_2 \dots a_m$ for $m \leq n$ and $a_1, a_2, \dots, a_m \in f^{-1}(V)$.

In this case we can write $f(z) = f(a_1) f(a_2) \dots f(a_m)$ for each $f(a_i) \in V$. Then $f(z) \in V^n$ and thus $z \in f^{-1}(V^n)$. This completes the proof. $\square$

Now we research when opposite of the theorem is correct. That is we seek an answer the question that boundedness and continuity of homomorphism between which style topological groups are invariant concepts.

Theorem 4.2. *Every bornivorous is neighbourhood of e in a semimetric group G .*

Proof. Let B be a symmetric bornivorous in G . Additionally, we assume that B isn't a neighbourhood of e . Then the set $B^{\leq n}$ isn't also a neighbourhood of e , for every number $n \in \mathbb{N}$.

The open sphere $D_{\frac{1}{n}}(e) = \{x : d(x, e) < \frac{1}{n}\}$ isn't contained by B for every number n since B isn't a neighbourhood of e . So this sphere isn't contained by the sets $B^{\leq n}$. Because they aren't neighbourhood of e , too. Then $\{D_{\frac{1}{n}}(e)\} \setminus B^{\leq n} \neq \emptyset$ for every number n .

Furthermore the sequence $\{x_m\}_{m=1}^{\infty}$ which is constructed the style that

$$x_1 \in \{D_1(e)\} \setminus B, \quad x_2 \in \{D_{\frac{1}{2}}(e)\} \setminus B^{\leq 2}, \dots$$

isn't absorbed by the set B . But the sequence be bounded. Because $\{x_m\}_{m=1}^{\infty}$ is absorbed by neighbourhood $D_1(e)$ of e (in fact the neighbourhood contains all elements of the sequence). This case contradict to the fact that B is a bornivorous. $\square$

It is understand that every neighbourhood of e is a bornivorous and every semimetric group is a bornological group by this theorem.

Lemma 4.3. *Let G and H be two topological groups, $f : G \rightarrow H$ be an injection and bounded homomorphism. If $A \subseteq B$ is a bornivorous then $f(A) \subseteq H$ is a bornivorous.*

Proof. Let S be a bounded set in G . Since the set A be a bornivorous then $S \subseteq A^{\leq n}$ for a number $n \in \mathbb{N}$. Because f is also bounded and $f(S)$. If $f(A^{\leq n}) \subseteq \{f(A)\}^{\leq n}$ then $f(A)$ is a bornivorous in H . Let we take any element $y \in f(A^{\leq n})$. Thus $f(x) = y$ for an element $x \in A^{\leq n}$. $x = a_1 a_2 \dots a_m$ for $m \leq n$ and $a_1, a_2, \dots, a_m \in A$ because $x \in A^{\leq n}$. There exists $c_i \in f(A)$ for each $1 \leq i \leq m$ such that $f(a_i) = c_i$. Furthermore

$$f(a_1 a_2 \dots a_m) = f(a_1) f(a_2) \dots f(a_m) = c_1 c_2 \dots c_m$$

because $y = f(x) = f(a_1 a_2 \dots a_m)$. Since f is a homomorphism, then $y \in \{f(A)\}^{\leq n}$. Thus $f(A) \subseteq H$ is a bornivorous. $\square$

Lemma 4.4. *Let G and H be two topological groups, $f : G \rightarrow H$ be an injection and bounded homomorphism. If $B \subseteq f(G)$ is a bornivorous in H then $f^{-1}(B)$ is also a bornivorous in G .*

Proof. It is easily seen that $B^{\leq n} \subseteq f(G)$ for a number $n \in \mathbb{N}$. Because $f(G)$ is a subgroup of H .

Let S be a bounded set in G . Then $f(S)$ is also bounded set in H . Since B is a bornivorous, B absorbs $f(S)$. That is $f(S) \subseteq B^{\leq n}$ for a number $n \in \mathbb{N}$. This means that $S \subseteq f^{-1}(B^{\leq n})$. So as to complete the proof, we must show that $f^{-1}(B^{\leq n}) \subseteq \{f^{-1}(B)\}^{\leq n}$. Let $x \in f^{-1}(B^{\leq n})$. Thus, we write $x = f^{-1}(y)$ for a number $y \in B^{\leq n}$. Furthermore there exist $a_1, a_2, \dots, a_m \in B$ for $m \leq n$, such that $y = a_1 a_2 \dots a_m$. Also, there exists $c_i \in f^{-1}(B)$ for each a_i , $1 \leq i \leq m$ such that $f(c_i) = a_i$. Thus for $m \leq n$,

$$\begin{aligned} x &= f^{-1}(a_1 a_2 \dots a_m) \\ &= f^{-1}(f(c_1) \dots f(c_m)) \\ &= f^{-1}(f(c_1 \dots c_m)) \\ &= c_1 \cdot c_2 \dots c_m \\ &= f^{-1}(a_1) \dots f^{-1}(a_m). \end{aligned}$$

This means that $x \in f^{-1}(B^{\leq n})$. □

Theorem 4.5. *Let G be a bornological group then every bounded homomorphism f defined from G into topological group H is continuous.*

Proof. Let U be a neighbourhood of e in H . The set U absorbs every bounded set in H by definition of bounded set. That is, the set U is a bornivorous because every neighbourhood of e is a bornivorous. By lemma 4.4. $f^{-1}(U)$ is a bornivorous and G is also a neighbourhood of e . That is, f be continuous at e so f be continuous in everywhere. □

Theorem 4.6. *Let G and G^* be two local compact, second countable topological groups. Then, each homomorphism g from G onto G^* is an open mapping [10].*

Corollary 4.7. *Let G and G^* be two local compact, second countable topological groups, $f : G \rightarrow G^*$ be a continuous, bijective homomorphism. In this case f is a homeomorphism.*

Theorem 4.8. *Let G and G^* be local compact separable metric groups and $f : G \rightarrow G^*$ be a homomorphism with closed graph. Then f is a continuous map.*

Proof. $G \times G^*$ is a locally compact and second countable set because $P_1 : G_f \subseteq G \times G^* \rightarrow G$. Since $G \times G^*$ is locally compact and G_f is closed and any closed subspace of a local compact space is locally compact then G_f is locally compact. Also G_f is second countable. We know that the projection function $P_1 : G_f \rightarrow G$ is continuous. Let the symbols $*$, Δ_G and $\square_G$ denote operations on $G \times G^*$, G and G^* respectively.

The projection function $P_1 : G_f \rightarrow G$ is a homomorphism. Because

$$P_1[(x, y) * (z, t)] = P_1[(x \Delta_G z, y \square_G t)] = x \Delta_G z = P_1(x, y) \Delta_G P_1(z, t).$$

P_1 be injective i.e. the mapping $P_1 : G_f \rightarrow G$ provides the condition of

$$P_1(x, f(x)) = P_1(y, f(y)) \implies x = y$$

Also P_1 is well defined i.e.

$$\begin{aligned} x = y &\implies f(x) = f(y) \\ &\implies (x, f(x)) = (y, f(y)). \end{aligned}$$

Therefore P_1 is a homeomorphism and inverse of P_1 is continuous. Additionally, since

$$P_2 : G_f \subseteq G \times G^* \rightarrow G$$

$$(x, f(x)) \mapsto f(x)$$

is a continuous map then

$$f = P_2 \circ P_1^{-1}$$

is continuous. This completes the proof. $\square$

5. Bohr (Weak) Topology and Boundedness

Proposition 5.1. Let τ_X be an initial topology of own X . Thus τ_X is stronger than τ_w .

Proof. If we take arbitrary open set $A \in \tau_w$ then exist n number of $f_1, f_2, \dots, f_n \in X^*$ and open sets $V_1, V_2, \dots, V_n \subset S_1$ such that $\bigcap_{i=1}^n f_i^{-1}(V_i) \subset A$. $f_i^{-1}(V_i)$ is also open set in X because each V_i is open and each f_i is continuous for $1 \leq i \leq n$.

Since finite intersection of open sets is open then $\bigcap_{i=1}^n f_i^{-1}(V_i) = V \in \tau_X$ Consequently $A \in \tau_X$. $\square$

Proposition 5.2. Let a set $V \subseteq X$ is τ_X bounded. Then the set V is τ_w bounded.

Proof. Since a set $V \subseteq X$ is τ_X bounded, there exists $m \in \mathbb{N}$ for every $A \in N_X(e)$ such that $V \subseteq A^m$. Now we show that the set V is bounded according to τ_w i.e. for every $B \in N_w(e)$, there exists $n \in \mathbb{N}$ such that $V \subseteq B^n$. If $B \in N_w(e)$ then there exists $G \in \tau_w$ for the unit element $e \in G \subset B$. Furthermore $G \in \tau_X$ because $G \in \tau_w$ by above proposition. Also since there exists $G \in \tau_X$ such that $e \in G \subset B$ then $B \in N_X(e)$. For $B \in N_X(e)$ there exists $m \in \mathbb{N}$ such that $V \subseteq B^m$. If we take $n = m$ then there exists $n \in \mathbb{N}$ for every $B \in N_w(e)$ such that $V \subseteq B^n$. Thus V is τ_w bounded. $\square$

Inverse of this proposition is usually false. That is, if a set G is τ_w bounded then G hasn't to be τ_X bounded in every time.

Following example shows that opposite of the proposition is usually false.

Example 5.3. $\mathbb{R}$ with usual topology is a unbounded topological group. But $\mathbb{R}$ is a precompact set in weak topology (Bohr topology). Furthermore $\mathbb{R}$ is a connected set in own Bohr topology because Bohr topology is a constructed topology by continuous homomorphism from $(\mathbb{R}, \mathcal{U})$ to S_1 . Also a set A is connected in own Bohr topology if and only if $f(A)$ is connected in S_1 . We know that connectedness is preserved below continuous mappings. So $f(\mathbb{R})$ is connected in S_1 since $\mathbb{R}$ is a connected set in usual topology. Consequently $\mathbb{R}$ is a bounded set in own Bohr topology. This is an example that a weak bounded set can't be bounded in a topological group although a weak bounded set is a bounded in a topological vector space.

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