Poincare Journal of Analysis & Applications
Vol. 1, No. 2 (2014), 47-54

DOI: 10.46753/pjaa.2014.v01i02.001

SHARP BOUNDS FOR THE CONVEX COMBINATIONS OF ARITHMETIC, LOGARITHMIC AND GEOMETRIC MEANS IN TERMS OF HARMONIC MEAN

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Date of Receiving	:	11. 05. 2014
Date of Revision	:	08. 06. 2014
Date of Acceptance	:	27. 06. 2014

Abstract. In this paper, we find the greatest values r_1 and r_2, and the least values s_1 and s_2 in $(0, 1/2)$ such that the double inequalities $H[r_1a + (1-r_1)b, r_1b + (1-r_1)a] < \alpha A(a,b)+(1-\alpha)L(a,b) < H[s_1a+(1-s_1)b, s_1b+(1-s_1)a]$ and $H[r_2a+(1-r_2)b, r_2b+(1-r_2)a] < \alpha A(a,b)+(1-\alpha)G(a,b) < H[s_2a+(1-s_2)b, s_2b+(1-s_2)a]$ hold for all $a, b > 0$ with $a \neq b$ and any $\alpha \in (0,1)$, where $H(a,b)$, $G(a,b)$, $L(a,b)$ and $A(a,b)$ are the harmonic, geometric, logarithmic and arithmetic means of two positive numbers a and b, respectively.

1. Introduction

For $a, b > 0$ with $a \neq b$, the classical harmonic mean $H(a,b)$, geometric mean $G(a,b)$, arithmetic mean $A(a,b)$, logarithmic mean $L(a,b)$, identric mean $I(a,b)$ are defined by

$$H(a,b) = \frac{2ab}{a+b},\ G(a,b) = \sqrt{ab},\ A(a,b) = \frac{a+b}{2}, \tag{1.1}$$

$$L(a,b) = \frac{a-b}{\log a - \log b},\ I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, \tag{1.2}$$

respectively. It is well known that the inequalities

$$H(a,b) < G(a,b) < L(a,b) < I(a,b) < A(a,b)$$

hold for all $a, b > 0$ with $a \neq b$.

2010 *Mathematics Subject Classification.* 26E60, 26D15.
Key words and phrases. harmonic mean, geometric mean, logarithmic mean, arithmetic mean.
The authors would like to express their deep gratitude to the referee for giving many valuable suggestions. This research was supported by the Natural Science Foundation of China under Grants 11171307 and 61374086, and the Natural Science Foundation of Zhejiang Province under Grant LY13A010004.
Communicated by. Virender
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Recently, the harmonic, geometric, logarithmic and arithmetic means have been the subject of intensive research. In particular, many remarkable inequalities for these means can be found in the literature [1-11].

It might be surprising that the logarithmic mean has applications in physics, economics, and even in meteorology [12-14]. Kahlig and Matkowski [12] study a variant of Jensen's functional equation involving the logarithmic mean, which appears in a heat conduction problem. A representation of the logarithmic mean as an infinite product and an iterative algorithm for computing the logarithmic mean as the common limit of two sequences of special geometric and arithmetic means are given in [15]. In [16, 17] it is shown that the logarithmic mean L can be expressed in terms of Gauss' hypergeometric function ${}_2F_1$. Carlson and Gustafson [17] prove that the reciprocal of the logarithmic mean is strictly totally positive, that is, every $n \times n$ determinant with elements $1/L(a_i, b_i)$, where $0 < a_1 < a_2 < \ldots < a_n$ and $0 < b_1 < b_2 < \ldots < b_n$, is positive for all $n \geq 1$.

In [15, 18-20] the authors present bounds for the logarithmic and identric means in terms of geometric and arithmetic means as follows:

$$G^{2/3}(a,b)A^{1/3}(a,b) < L(a,b) < \frac{2}{3}G(a,b) + \frac{1}{3}A(a,b),$$

$$I(a,b) > \frac{1}{3}G(a,b) + \frac{2}{3}A(a,b),$$

$$G^{1/2}(a,b)A^{1/2}(a,b) < L^{1/2}(a,b)I^{1/2}(a,b)$$

$$< \frac{1}{2}L(a,b) + \frac{1}{2}I(a,b) < \frac{1}{2}G(a,b) + \frac{1}{2}A(a,b)$$

for all $a, b > 0$ with $a \neq b$.

The power mean of order r of the positive real numbers a and b is defined by

$$M_r(a,b) = \left(\frac{a^r + b^r}{2}\right)^{1/r} (r \neq 0), \ M_0(a,b) = \sqrt{ab}.$$

The main properties of the power mean are given in [21]. In particular, the function $r \to M_r(a,b)$ is strictly increasing with respect to $r \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Several authors discussed the relationship of certain means to M_r. The following sharp bounds for L, I, $L^{1/2}I^{1/2}$ and $(L+I)/2$ in terms of the power means are proved in [20, 22-27]:

$$M_0(a,b) < L(a,b) < M_{1/3}(a,b), \ M_{2/3}(a,b) < I(a,b) < M_{\log 2}(a,b),$$

$$M_0(a,b) < L^{1/2}(a,b)I^{1/2}(a,b) < M_{1/2}(a,b), \ \frac{L(a,b) + I(a,b)}{2} < M_{1/2}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

Alzer and Qiu [28] prove that the inequalities

$$\alpha A(a,b) + (1-\alpha)G(a,b) < I(a,b) < \beta A(a,b) + (1-\beta)G(a,b),$$

$$\lambda A(a,b) + (1-\lambda)G(a,b) < L(a,b) < \mu A(a,b) + (1-\mu)G(a,b),$$

$$M_c(a,b) < \frac{L(a,b) + I(a,b)}{2}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 2/3$, $\beta \geq 2/e$, $\lambda \leq 0$, $\mu \geq 1/3$ and $c \leq \log 2/(1 + \log 2)$.

For $p, q, \alpha, \beta \in (0, 1/2)$, Chu et al. [29, 30] prove that the double inequalities

$$H[pa + (1-p)b, pb + (1-p)a] < I(a, b) < H[qa + (1-q)b, qb + (1-q)a]$$

and

$$H[\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a] < P(a, b) < H[\beta a + (1-\beta)b, \beta b + (1-\beta)a]$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $p \leq (1 - \sqrt{1-2/e})/2$, $q \geq (6-\sqrt{6})/12$, $\alpha \leq (1-\sqrt{1-2/\pi})/2$ and $\beta \geq (6-\sqrt{6})/12$, where $P(a, b) = (a-b)/[4 \arctan \sqrt{a/b} - \pi]$ is the first Seiffert mean of a and b.

Chu and Wang [31] find the least values p, q and s in (0,1/2) such that the inequalities

$$H[pa + (1-p)b, pb + (1-p)a] > AG(a, b),$$
$$G[qa + (1-q)b, qb + (1-q)a] > AG(a, b)$$

and

$$L[sa + (1-s)b, sb + (1-s)a] > AG(a, b)$$

hold for all $a, b > 0$ with $a \neq b$, where $AG(a, b)$ is the classical arithmetic-geometric mean of a and b, which is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$ given by

$$a_0 = a, \ b_0 = b,$$
$$a_{n+1} = \frac{a_n + b_n}{2} = A(a_n, b_n), \ b_{n+1} = \sqrt{a_n b_n} = G(a_n, b_n).$$

In [32] the authors prove that if $p, q \in (0, 1/2)$, then the inequalities

$$H[pa + (1-p)b, pb + (1-p)a] > G(a, b) \tag{1.3}$$

and

$$H[qa + (1-q)b, qb + (1-q)a] > L(a, b) \tag{1.4}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $p \geq (2-\sqrt{2})/4$ and $q \geq (3-\sqrt{3})/6$.

For $a, b > 0$ with $a \neq b$ and $x \in [0, 1/2]$. Let

$$f(x) = H[xa + (1-x)b, xb + (1-x)a].$$

Then it is not difficult to verify that the function $f(x)$ is is continuous and strictly increasing on $0, 1/2]$. Note that

$$f(0) = H(a, b) < \alpha A(a, b) + (1-\alpha)L(a, b) < A(a, b) = f(1/2) \tag{1.5}$$

and

$$f(0) = H(a, b) < \alpha A(a, b) + (1-\alpha)G(a, b) < A(a, b) = f(1/2) \tag{1.6}$$

for any $\alpha \in (0, 1)$.

Inspired by inequalities (1.5) and (1.6), it is natural to ask what are the best possible parameters r_1, r_2, s_1 and s_2 in $(0, 1/2)$ such that the double inequalities

$$\begin{aligned} H[r_1 a + (1-r_1)b, r_1 b + (1-r_1)a] &< \alpha A(a, b) + (1-\alpha)L(a, b) \\ &< H[s_1 a + (1-s_1)b, s_1 b + (1-s_1)a] \end{aligned}$$

and

$$\begin{aligned} H[r_2 a + (1-r_2)b, r_2 b + (1-r_2)a] &< \alpha A(a, b) + (1-\alpha)G(a, b) \\ &< H[s_2 a + (1-s_2)b, s_2 b + (1-s_2)a] \end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$ and any $\alpha \in (0, 1)$. Our main results are the following Theorems 1.1 and 1.2.

Theorem 1.1. *If $r_1, s_1 \in (0, 1/2)$ and $\alpha \in (0, 1)$, then the double inequality*

$$H[r_1a + (1 - r_1)b, r_1b + (1 - r_1)a] < \alpha A(a, b) + (1 - \alpha)L(a, b) < H[s_1a + (1 - s_1)b, s_1b + (1 - s_1)a]$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $r_1 \leq (1 - \sqrt{1-\alpha})/2$ and $s_1 \geq (3 - \sqrt{3(1-\alpha)})/6$.

Theorem 1.2. *If $r_2, s_2 \in (0, 1/2)$ and $\alpha \in (0, 1)$, then the double inequality*

$$H[r_2a + (1 - r_2)b, r_2b + (1 - r_2)a] < \alpha A(a, b) + (1 - \alpha)G(a, b) < H[s_2a + (1 - s_2)b, s_2b + (1 - s_2)a]$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $r_2 \leq (1 - \sqrt{1-\alpha})/2$ and $s_2 \geq (2 - \sqrt{2(1-\alpha)})/4$.

Remark 1.1. We clearly see that Theorem 1.1 and 1.2 are the generalizations of inequalities (1.3) and (1.4).

2. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let $\lambda = (1 - \sqrt{1-\alpha})/2$ and $\mu = (3 - \sqrt{3(1-\alpha)})/6$, we first prove that the inequalities

$$\alpha A(a, b) + (1 - \alpha)L(a, b) > H[\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a] \tag{2.1}$$

and

$$\alpha A(a, b) + (1 - \alpha)L(a, b) < H[\mu a + (1 - \mu)b, \mu b + (1 - \mu)a] \tag{2.2}$$

hold for all $a, b > 0$ with $a \neq b$ and any $\alpha \in (0, 1)$.

Since $A(a, b)$, $L(a, b)$ and $H(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b$. Let $t = a/b > 1$ and $p \in (0, 1/2)$, then from (1.1) and (1.2) one has

$$\begin{aligned} H[pa + (1 - p)b, pb + (1 - p)a] &- [\alpha A(a, b) + (1 - \alpha)L(a, b)] \\ &= \frac{h(t)}{(t + 1)^2 \log t} A(a, b), \end{aligned} \tag{2.3}$$

where

$$h(t) = [(4p-4p^2-\alpha)t^2+(8p^2-8p+4-2\alpha)t+(4p-4p^2-\alpha)] \log t-2(1-\alpha)(t^2-1) \tag{2.4}$$

We divide the proof into two cases.

Case 1.1 $p = \lambda = (1 - \sqrt{1-\alpha})/2$. Then (2.4) becomes

$$h(t) = 2(1 - \alpha)(2t \log t - t^2 + 1). \tag{2.5}$$

Therefore, inequality (2.1) follows from and (2.3) and (2.5) together with the elementary inequality $2t \log t + 1 < t^2$.

Case 1.2 $p = \mu = (3 - \sqrt{3(1-\alpha)})/6$. Then (2.4) reduces to

$$h(t) = \frac{2}{3}(1 - \alpha)[(t + 2)^2 \log t - 3t^2 + 3]. \tag{2.6}$$

Therefore, inequality (2.2) follows from (2.3) and (2.6) together with the elementary inequality $(t+2)^2\log t+3>3t^2$.

Next, we prove that $\lambda=(1-\sqrt{1-\alpha})/2$ is the best possible parameter in $(0,1/2)$ such that inequality (2.1) holds for all $a,b>0$ with $a\neq b$ and any $\alpha\in(0,1)$. In fact, if $(1-\sqrt{1-\alpha})/2=\lambda<p<1/2$, then it follows from (2.3) and (2.4) that

$$\begin{aligned}\lim_{t\to+\infty}&[H(pa+(1-p)b,pb+(1-p)a)-\alpha A(a,b)-(1-\alpha)L(a,b)]\\&=[4p(1-p)-\alpha]A(a,b)>0.\end{aligned} \tag{2.7}$$

Inequality (2.7) implies that for any $(1-\sqrt{1-\alpha})/2=\lambda<p<1/2$ there exists large enough T such that $H[pa+(1-p)b,pb+(1-p)a]>\alpha A(a,b)+(1-\alpha)L(a,b)$ for all $a/b\in(T,+\infty)$.

Finally, we prove that $\mu=(3-\sqrt{3(1-\alpha)})/6$ is the best possible parameter in $(0,1/2)$ such that inequality (2.2) holds for all $a,b>0$ with $a\neq b$ and any $\alpha\in(0,1)$. Let $0<\varepsilon<(3-\sqrt{3(1-\alpha)})/6$, $0<t<1$ and $p=(3-\sqrt{3(1-\alpha)})/6-\varepsilon$, then making use of Taylor expansion we have

$$\begin{aligned}&H[p(1+t)+(1-p),p+(1-p)(1+t)]-[\alpha A(1+t,1)+(1-\alpha)L(1+t,1)] \qquad (2.8)\\&=1+\frac{t}{2}+\left(p-p^2-\frac{1}{4}\right)t^2+o\left(t^2\right)-\left[\alpha\left(1+\frac{t}{2}\right)+(1-\alpha)\left(1+\frac{t}{2}-\frac{t^2}{12}+o\left(t^2\right)\right)\right]\\&=-\left(\frac{\sqrt{3(1-\alpha)}}{3}+\varepsilon\right)\varepsilon t^2+o\left(t^2\right).\end{aligned}$$

Equation (2.8) implies that for $0<p<(3-\sqrt{3(1-\alpha)})/6$ there exists $0<\delta_1<1$ such that $H[pa+(1-p)b,pb+(1-p)a]<\alpha A(a,b)+(1-\alpha)L(a,b)$ for all $a/b\in(1,1+\delta_1)$.

Proof of Theorem 1.2. Let $\zeta=(1-\sqrt{1-\alpha})/2$ and $\eta=(2-\sqrt{2(1-\alpha)})/4$, we first prove that the inequalities

$$\alpha A(a,b)+(1-\alpha)G(a,b)>H[\zeta a+(1-\zeta)b,\zeta b+(1-\zeta)a] \tag{2.9}$$

and

$$\alpha A(a,b)+(1-\alpha)G(a,b)<H[\eta a+(1-\eta)b,\eta b+(1-\eta)a] \tag{2.10}$$

hold for all $a,b>0$ with $a\neq b$ and any $\alpha\in(0,1)$.

Since $A(a,b)$, $G(a,b)$ and $H(\zeta a+(1-\zeta)b,\zeta b+(1-\zeta)a)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a>b$. Let $x=\sqrt{a/b}>1$ and $q\in(0,1/2)$, then from (1.1) one has

$$\begin{aligned}H[qa+(1-q)b,&qb+(1-q)a]-[\alpha A(a,b)+(1-\alpha)G(a,b)]\\&=\frac{J(x)}{(x^2+1)^2}A(a,b),\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}J(x)=&(4q-4q^2-\alpha)x^4-2(1-\alpha)x^3+(8q^2-8q+4-2\alpha)x^2\\&-2(1-\alpha)x+(4q-4q^2-\alpha).\end{aligned} \tag{2.12}$$

We divide the proof into two cases.

Case 2.1 $q=\zeta=(1-\sqrt{1-\alpha})/2$. Then (2.12) becomes

$$J(x)=-2(1-\alpha)x(x-1)^2<0 \tag{2.13}$$

for $x > 1$.

Therefore, inequality (2.9) follows from (2.11) and (2.13).

Case 2.2 $q = \eta = (2 - \sqrt{2(1-\alpha)})/4$. Then (2.12) reduces to

$$\begin{aligned} J(x) &= \frac{1-\alpha}{2}x^4 - 2(1-\alpha)x^3 + 3(1-\alpha)x^2 - 2(1-\alpha)x + \frac{1-\alpha}{2} \\ &= \frac{1-\alpha}{2}(x-1)^4 > 0 \end{aligned} \tag{2.14}$$

for $x > 1$.

Therefore, inequality (2.10) follows from (2.11) and (2.14).

Next, we prove that $\zeta = (1 - \sqrt{1-\alpha})/2$ is the best possible parameter in $(0, 1/2)$ such that inequality (2.9) holds for all $a, b > 0$ with $a \neq b$ and any $\alpha \in (0, 1)$. In fact, if $(1 - \sqrt{1-\alpha})/2 = \zeta < q < 1/2$, then it follows from (2.11) and (2.12) that

$$\begin{aligned} &\lim_{x\to+\infty}[H(qa + (1-q)b, qb + (1-q)a) - \alpha A(a, b) - (1-\alpha)G(a, b)] \\ &\qquad = [4q(1-q) - \alpha]A(a, b) > 0. \end{aligned} \tag{2.15}$$

Inequality (2.15) implies that for any $(1-\sqrt{1-\alpha})/2 = \zeta < q < 1/2$ there exists large enough X such that $H[qa + (1-q)b, qb + (1-q)a] > \alpha A(a, b) + (1-\alpha)G(a, b)$ for all $a/b \in (X^2, +\infty)$.

Finally, we prove that $\eta = (2-\sqrt{2(1-\alpha)})/4$ is the best possible parameter in $(0, 1/2)$ such that inequality (2.10) holds for all $a, b > 0$ with $a \neq b$ and any $\alpha \in (0, 1)$. Let $0 < \varepsilon < (2 - \sqrt{2(1-\alpha)})/4$, $0 < x < 1$ and $q = (2 - \sqrt{2(1-\alpha)})/4 - \varepsilon$, then making use of Taylor expansion we have

$$\begin{aligned} &H[q(1+x) + (1-q), q + (1-q)(1+x)] - [\alpha A(1+x, 1) + (1-\alpha)G(1+x, 1)] \qquad (2.16) \\ &= 1 + \frac{x}{2} + \left(q - q^2 - \frac{1}{4}\right)x^2 + o\left(x^2\right) - \left[\alpha\left(1 + \frac{x}{2}\right) + (1-\alpha)\left(1 + \frac{x}{2} - \frac{x^2}{8} + o\left(x^2\right)\right)\right] \\ &\qquad - \left(\frac{\sqrt{2(1-\alpha)}}{2} + \varepsilon\right)\varepsilon x^2 + o\left(x^2\right). \end{aligned}$$

Equation (2.16) implies that for any $0 < q < (2 - \sqrt{2(1-\alpha)})/4$ there exists $0 < \delta_2 < 1$ such that $H[qa + (1-q)b, qb + (1-q)a] < \alpha A(a, b) + (1-\alpha)G(a, b)$ for all $a/b \in (1, (1+\delta_1)^2)$.

References

[1] S.-H. Wu, Generalization and sharpness of the power means inequalitie and their applications, J. Math. Anal. Appl., 2005, **312**(2): 637-652.

[2] C.-P. Chen, Note on monotonicity of the extended mean values, J. Anal. Comput., 2005, **11**(2): 69-71.

[3] H.-N. Shi, S.-H. Wu and F. Qi, An alternative note on the Schur-convexity of the extended mean vaules, Math. Inequal. Appl., 2006, **9**(2): 219-224.

[4] C.-P. Chen and F. Qi, Monotonicity properties and inequalities of functions related to means, Rocky Mountain J. Math., 2006, **36**(3): 857-865.

[5] F. Qi, S.-X. Chen and C.-P. Chen, Monotonicity of ratio between the generalized logarithmic means, Math. Inequal. Appl., 2007, **10**(3): 559-564.

[6] C.-P. Chen, The monotonicity of the ratio between generalized logarithmic means, J. Math. Anal. Appl., 2008, **345**(1): 86-89.

[7] L. Zhu, New inequalities for means in two variables, Math. Inequal. Appl., 2008, **11**(2): 229-235.

[8] L. Zhu, Some inequalities for means in two variables, Math. Inequal. Appl., 2008, **11**(3)" 443-448.

[9] S.-H. Wu and L. Debnath, Inequalities for differences of power means in two variables, Anal. Math., 2011, **37**(2): 151-159.

[10] S.-H. Wu and H.-N. Shi, A realtion of weak majorization and its applications to certain inequalities for means, Math. Slovaca, 2011, **61**(4): 561-570.

[11] W.-D. Jiang, Some sharp inequalities involving reciprocals of the Seiffert and other means, J. Math. Inequal., 2012, **6**(4): 593-599.

[12] P. Kahlig and J. Matkowski, Functional equations involving the logarithmic mean, Z. Angew. Math., 1996, **76**(7): 385-390.

[13] A. O. Pittenger, The logarithmic mean in n variables, Amer. Math. Monthly, 1985, **92**(2): 99-104.

[14] G. Pólya and G. Szegö, Isoperimteric Inequalities in Mathematical Physics, Princeton University Press, Princeton, 1951.

[15] B. C. Carlson, The logarithmic mean, Amer. Math. Monthly, 1972, **79**: 615-618.

[16] B. C. Carlson, Algorithms involving arithmetic and geometric means, Amer. Math. Monthly, 1971, **78**: 496-505.

[17] B. C. Carlson and J. L. Gustafson, Total postivity of mean values and hypergeometric functions, SIAM J. Math. Anal., 1983, **14**(2): 389-395.

[18] E. B. Leach and M. C. Sholander, Extended mean values II, J. Math. Anal. Appl., 1983, **92**(1): 207-223.

[19] J. Sándor, A note on some inequalities for means, Arch. Math., 1991, **56**(5): 471-473.

[20] H. Alzer, Ungleichungen für Mittelwerte, Arch. Math., 1986, **47**(5): 422-426.

[21] P. S. Bullen, D. S. Mitrinović and P. M. Vasić, Means and Their Inequalities, D. Reidel Publishing Co., Dordrecht, 1988.

[22] H. Alzer, Ungleichungen für $(e/a)^a(b/e)^b$, Elem. Math., 1985, **40**: 120-123.

[23] F. Burk, The geometric, logarithmic, and arithmetic mean inequality, Amer. Math. Monthly, 1987, **94**(6): 527-528.

[24] T. P. Lin, The power mean and the logarithmic mean, Amer. Math. Monthly, 1974, **81**: 879-883.

[25] A. O. Pittenger, Inequalities between arithmetic and logarithmic means, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., 1980, **678-715**: 15-18.

[26] A. O. Pittenger, The symmetric, logarithmic and power means, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., 1980, **678-715**: 19-23.

[27] K. B. Stolarsky, The power and generalized logarithmic means, Amer. Math. Monthly, 1980, **87**(7): 545-548.

[28] H. Alzer and S.-L. Qiu, Inequalities for means in two variables, Arch. Math., 2003, **80**(2): 201-215.

[29] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, A sharp double inequality between harmonic and identric means, Abstr. Appl. Anal., **2011**, Article ID 657935, 7 pages.

[30] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, A best-possible double inequality between Seiffert and harmonic means, J. Inequal. Appl., 2011, **2011**: 94, 7 pages.

[31] Y.-M. Chu and M.-K. Wang, Optimal inequalities between harmonic, geometric, logarithmic, and arithmetic-geometric means, J. Appl. Math., **2011**, Article ID 618929, 9 pages.

[32] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, Best possible inequalities among harmonic, geometric, logarithmic and Seiffert mean, Math. Inequal. Appl., 2012, **15**(2): 415-422.

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