Poincare Journal of Analysis & Applications
Vol. 1, No. 2 (2014), 55-61

DOI: 10.46753/pjaa.2014.v01i02.002

NORMAL VE RECTIFYING CURVES IN THE EQUIFORM DIFFERENTIAL GEOMETRY OF G_3

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Date of Receiving	:	30. 06. 2014
Date of Revision	:	12. 08. 2014
Date of Acceptance	:	04. 09. 2014

Abstract. In this study, we define normal and rectifying curves in the equiform differential geometry of the Galilean space. Furthermore, we obtain some characterizations for normal and rectifying curves in the equiform geometry of the Galilean space G.

1. Introduction

A Galilean space may be considered as the limit case of a pseudo-Euclidean space in which the isotropic cone degenerates to a plane. This limit transition corresponds to the limit transition from the special theory of relativity to classical mechanics. On the other hand, Galilean space-time plays an important role in nonrelativistic physics. The fact that the fundamental concepts such as velocity, momentum, kinetic energy, etc. and principles; laws of motion and conservation laws of classical physics are expressed in terms of Galilean space[4]. As it is well known geometry of space is associated with mathematical group. The idea of invariance of geometry under transformation group may imply that, on some spacetimes of maximum symmetry there should be a principle of relativity, which requires the invariance of physical laws without gravity under transformations among inertial systems. Besides, theory of curves and the curves of constant curvature in the equiform differential geometry of the isotropic spaces I_3^1 and I_3^2, and the Galilean space G_3 are described in [1] and [2], respectively. Although the equiform geometry has minor importance related to usual one, the curves that appear here in the equiform geometry, can be seen as generalizations of well-known curves from above mentioned geometries and therefore could have been of research interest.

The purpose of the present paper is to find necessary conditions for normal and rectifying curves in the equiform geometry of the Galilean space G_3.

2010 *Mathematics Subject Classification.* 53A04, 53A35.
Key words and phrases. Normal curve, Rectifying curve, Equiform differential geomertry, Galilean space.
Communicated by. S.K. Kaushik
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2. Preliminaries

The Galilean space G_3 is a Cayley-Klein space equipped with the projective metric of signature $(0,0,+,+)$. The absolute figure of the Galilean space consist of an ordered triple $\{w, f, I\}$, where w is the ideal(absolute) plane, f is the line(absolute line) in w and I is the fixed elliptic involution of points of f.

In the non-homogeneous coordinates the similarity group H_8 has the form

$$\begin{aligned} \bar{x} &= a_{11} + a_{12}x \\ \bar{y} &= a_{21} + a_{22}x + a_{23}y\cos\theta + a_{23}z\sin\theta \\ \bar{z} &= a_{31} + a_{32}x - a_{23}y\sin\theta + a_{23}z\cos\theta \end{aligned} \tag{2.1}$$

where a_{ij} and θ are real numbers [2]. In what follows the real numbers a_{12} and a_{23} will play the special role. In particular, for a_{12}=$a_{23} = 1$, defines the group $B_6 \subset H_8$ of isometries of the Galilean space G_3. The Galilean scalar product can be written as

$$\langle x, y\rangle = \left\{ \begin{array}{ll} x_1x_2, & \text{if } x_1 \neq 0 \text{ or } x_2 \neq 0 \\ y_1y_2 + z_1z_2 & \text{if } \ x_1 = 0 \text{ and } x_2 = 0, \end{array} \right\}$$

where $x = (x_1, y_1, z_1)$ and $y = (x_2, y_2, z_2)$. It leaves invariant the Galilean norm of the vector x defined by

$$\|x\| = \left\{ \begin{array}{ll} x_1, & \text{if } x_1 \neq 0 \\ \sqrt{y_1^2 + z_1^2}, & \text{if } \ x_1 = 0. \end{array} \right\}$$

A curve $\alpha : I \subset \mathbb{R} \to G_3$ of the class C^∞ in the Galilean space G_3 is defined by the parametrization

$$\alpha(s) = (s, y(s), z(s)),$$

where s is a Galilean invariant arc-length of α. Then the curvature $\kappa(s)$ and the torsion $\tau(s)$ are given by, respectively

$$\kappa(s) = \sqrt{\ddot{y}(s)^2 + \ddot{z}(s)^2}, \ \tau(s) = \frac{\det((\dot{\alpha}(s), \ddot{\alpha}(s), \dddot{\alpha}(s))}{\kappa^2(s)}.$$

On the other hand, the Frenet vectors of $\alpha(s)$ in G_3 are defined by

$$\begin{aligned} t(s) &= \dot{\alpha}(s) = ((1, \dot{y}(s), \dot{z}(s)), \\ n(s) &= \frac{1}{\kappa(s)}\ddot{\alpha}(s) = \frac{1}{\kappa(s)}((0, \ddot{y}(s), \ddot{z}(s)), \\ b(s) &= \frac{1}{\kappa(s)}((0, -\ddot{z}(s), \ddot{y}(s)). \end{aligned} \tag{2.2}$$

The vectors t, n, b are called the vector of tangent, principal normal and binormal of α, respectively. For their derivatives the following Frenet formula satisfies [2]

$$\begin{aligned} \dot{t}(s) &= \kappa(s)n(s), \\ \dot{n}(s) &= \tau(s)b(s), \\ \dot{b}(s) &= -\tau(s)n(s). \end{aligned} \tag{2.3}$$

3. Frenet Formulas in Equiform Geometry of G_3

Let $\alpha : I \to G_3$ be a curve in the Galilean space G_3. We define the equiform parameter of α by

$$\sigma := \int \frac{1}{\rho} ds = \int \kappa ds,$$

where $\rho = \frac{1}{\kappa}$ is the radius of curvature of the curve α. Then, we have

$$\frac{ds}{d\sigma} = \rho. \tag{3.4}$$

Let h be a homothety with the center in the origin and the coefficient λ. If we put $\tilde{\alpha} = h(\alpha)$, then it follows

$$\tilde{s} = \lambda s \quad \text{and} \quad \tilde{\rho} = \lambda \rho,$$

where $\tilde{s}$ is the arc-length parameter of $\tilde{\alpha}$ and $\tilde{\rho}$ the radius of curvature of this curve. Therefore, σ is an equiform invariant parameter of α [2].

From now on, we define the Frenet formula of the curve α with respect to the equiform invariant parameter σ in G_3. The vector

$$T = \frac{d\alpha}{d\sigma}$$

is called a tangent vector of the curve α. From 2.2 and 3.4 we get

$$T = \frac{d\alpha}{ds}\frac{ds}{d\sigma} = \rho \cdot \frac{d\alpha}{ds} = \rho \cdot t. \tag{3.5}$$

We define the principal normal vector and the binormal vector by

$$N = \rho \cdot n, \quad B = \rho \cdot b. \tag{3.6}$$

Then, we easily show that $\{T, N, B\}$ are an equiform invariant orthonormal frame of the curve α.

On the other hand, the derivations of these vectors with respect to σ are given by

$$\begin{aligned} T' &= \frac{dT}{d\sigma} = \dot{\rho} T + N, \\ N' &= \frac{dN}{d\sigma} = \dot{\rho} N + \rho\tau B, \\ B' &= \frac{dB}{d\sigma} = \rho\tau N + \dot{\rho} B. \end{aligned} \tag{3.7}$$

Definition 3.1. The function $\mathcal{K} : I \to \mathbb{R}$ defined by

$$\mathcal{K} = \dot{\rho}$$

is called the equiform curvature of the curve α [2].

Definition 3.2. The function $\mathcal{T} : I \to \mathbb{R}$ defined by

$$\mathcal{T} = \rho\tau = \frac{\tau}{\kappa}$$

is called the equiform torsion of the curve α [2].

Thus, the formula analogous to the Frenet formula in the equiform differential geometry of the Galilean space have the following form

$$\begin{aligned} T' &= \mathcal{K}\cdot T + N, \\ N' &= \mathcal{K}\cdot N + \mathcal{T}\cdot B, \\ B' &= -\mathcal{T}\cdot N + \mathcal{K}\cdot B. \end{aligned} \tag{3.8}$$

The equiform parameter $\sigma = \int \kappa(s)ds$ for closed curves is called the total curvature, and it plays an important role in global differential geometry of Euclidean space. Also, the function $\frac{\tau}{\kappa}$ has been already known as a conical curvature and it also has interesting geometric interpretation.

Remark 3.3. Let $\alpha : I \to G_3$ be a curve in the equiform differential geometry of the Galilean space G_3. So, following statements are true (see for details [2]):
i) If $\alpha(s)$ is an isotropic logarithmic spiral in G_3, then $\mathcal{K}$ =const. $\neq 0$ and $\mathcal{T} = 0$.
ii) If $\alpha(s)$ is an circular helix in G_3, then $\mathcal{K}$ =0 and $\mathcal{T}$ =const. $\neq 0$.
iii) If $\alpha(s)$ is an isotropic circle in G_3, then, $\mathcal{K}$ =0 and $\mathcal{T} = 0$.

4. Normal and Rectifying curves in the Equiform Geometry of the Galilean Space

In this section, we firstly give definition of normal and rectifying curves in the equiform differential geometry of G_3 and later we show necessary conditions in case α is a normal curve or rectifying curve in G_3.

Definition 4.1. Let $\alpha : I \to G_3$ be a curve in the Galilean space. If the position vector of α always lies in its normal plane, then it' s called normal curve in G_3. By this definition, position vector of curve α in equiform geometry

$$\alpha(s) = \lambda(s)N(s) + \eta(s)B(s)$$

where $\lambda(s)$ and $\eta(s)$ are orbitrary differentiable functions.

Theorem 4.2. *Let $\alpha : I \to G_3$ be a curve in the equiform differential geometry of the Galilean space G_3 with $\mathcal{K}$, $\mathcal{T} \in \mathbb{R}$. If α is a normal curve, then the following statement hold*

$$\begin{aligned} \lambda(s) &= c_5\cos(\mathcal{T}s) + c_6\sin(\mathcal{T}s) + c_7 s\cos(\mathcal{T}s) + c_8 s\sin(\mathcal{T}s) - \frac{1}{\mathcal{T}^2} \\ \eta(s) &= c_1\cos(\mathcal{T}s) + c_2\sin(\mathcal{T}s) + c_3 s\cos(\mathcal{T}s) + c_4 s\sin(\mathcal{T}s) \end{aligned}$$

where $c_i = const.$ $\{i = 1, ..., 8\}$.

Proof. Let $\alpha : I \to G_3$ be a curve in the equiform differential geometry of the Galilean space G_3. Since α is a normal curve, from definition 4.1 we have

$$\alpha(s) = \lambda(s)N(s) + \eta(s)B(s). \tag{4.9}$$

Differentiating 4.9 with respect to s and using the Frenet formulas, we have

$$T = (\lambda' + \lambda\mathcal{K} - \eta\mathcal{T})N + (\eta' + \lambda\mathcal{T} + \eta\mathcal{K})B \tag{4.10}$$

Again differentiating this equation with respect to s we obtain

$$\begin{aligned} \mathcal{K}T = {} & N\left\{\lambda'' + 2\lambda'\mathcal{K} - 2\eta'\mathcal{T} + \lambda(\mathcal{K}' + \mathcal{K} - \mathcal{T}^2) + \eta(-\mathcal{T}' - 2\mathcal{K}\mathcal{T}) - 1\right\} \\ & -B\left\{\eta'' + 2\lambda'\mathcal{T} + 2\eta'\mathcal{K} + \lambda(\mathcal{T}' + 2\mathcal{K}\mathcal{T}) + \eta(\mathcal{K}' + \mathcal{K}^2 - \mathcal{T}^2)\right\} \end{aligned} \tag{4.11}$$

According to the definition of the linear independent of vectors, we find that

$$\begin{aligned} \mathcal{K} &= 0, & (4.12)\\ \lambda'' + 2\lambda'\mathcal{K} - 2\eta'\mathcal{T} + \lambda(\mathcal{K}' + \mathcal{K} - \mathcal{T}^2) + \eta(-\mathcal{T}' - 2\mathcal{K}\mathcal{T}) - 1 &= 0, & (4.13)\\ \eta'' + 2\lambda'\mathcal{T} + 2\eta'\mathcal{K} + \lambda(\mathcal{T}' + 2\mathcal{K}\mathcal{T}) + \eta(\mathcal{K}' + \mathcal{K}^2 - \mathcal{T}^2) &= 0 & (4.14)\end{aligned}$$

Now, if we substitute 4.12 into 4.13 and 4.14, we get

$$\begin{aligned} \lambda'' - 2\eta'\mathcal{T} - \lambda\mathcal{T}^2 - \eta\mathcal{T}' - 1 &= 0,\\ \eta'' + 2\lambda'\mathcal{T} + \lambda\mathcal{T}' - \eta\mathcal{T}^2 &= 0.\end{aligned}$$

Since $\mathcal{T} \in \mathbb{R}$, we get

$$\begin{aligned} \lambda'' - 2\eta'\mathcal{T} - \lambda\mathcal{T}^2 &= 1,\\ \eta'' + 2\lambda'\mathcal{T} - \eta\mathcal{T}^2 &= 0\end{aligned}$$

If this system of differential equations is solved, then we get

$$\begin{aligned} \lambda(s) &= c_5\cos(\mathcal{T}s) + c_6\sin(\mathcal{T}s) + c_7 s\cos(\mathcal{T}s) + c_8 s\sin(\mathcal{T}s) - \frac{1}{\mathcal{T}^2}\\ \eta(s) &= c_1\cos(\mathcal{T}s) + c_2\sin(\mathcal{T}s) + c_3 s\cos(\mathcal{T}s) + c_4 s\sin(\mathcal{T}s)\end{aligned}$$

where $c_i = const.$ $\{i = 1, ..., 8\}$ and

$$\begin{aligned} c_5 &= -\frac{2\mathcal{T}}{1+\mathcal{T}^2}\left(c_2 + \frac{3+\mathcal{T}^2}{1+\mathcal{T}^2}c_3\right)\\ c_6 &= \frac{2\mathcal{T}}{1+\mathcal{T}^2}\left(c_1 + \frac{1-\mathcal{T}^2}{1+\mathcal{T}^2}c_4\right)\\ c_7 &= -\frac{2\mathcal{T}}{1+\mathcal{T}^2}c_4\\ c_8 &= -\frac{2\mathcal{T}}{1+\mathcal{T}^2}c_3.\end{aligned}$$

Definition 4.3. Let $\alpha : I \to G_3$ be a curve in the equiform differential geometry of the Galilean space G_3. If the position vector of α always lies in its rectifying plane, then it' s called rectifying curve in G_3. By this definition, position vector of curve α in equiform geometry.

$$\alpha(s) = \varpi(s)T(s) + \xi(s)B(s)$$

where $\varpi(s)$ and $\xi(s)$ are orbitrary differentiable functions [3].

Theorem 4.4. *Let $\alpha : I \to G_3$ be a curve in the equiform differential geometry of the Galilean space G_3. Then α is a rectifying curve in the equiform differential geometry of the Galilean space G_3 if and only if one the following four statements hold :*

i) The distance function $h = \|\alpha(s)\|$ satisfies

$$h^2 = \left|s^2 + 2c_1 s + c_1^2 + c_2^2\right|$$

for some orbitrary constants a, c_1, c_2.

ii) The tangential component of the position vector of the curve α is given by

$$\langle\alpha(s), T(s)\rangle = s + c_1,$$

where a is orbitrary constant.

iii) The normal component α^N of the position vector of the curve α has constant length and the distance function h is nonconstant.

iv) The equiform torsion of the curve α, $\mathcal{T}(s) \neq 0$ and the binormal component of the position vector of the curve α is constant i.e., $\langle \alpha(s), B(s)\rangle = c_2$ is constant.

Proof. Let $\alpha : I \to G_3$ be a curve in the equiform differential geometry of the Galilean space G_3. We suppose that α is a rectifiying curve in the equiform differential geometry of the Galilean space G_3. Then, the position vector α satisfies the equation

$$\alpha(s) = \varpi(s)T(s) + \xi(s)B(s). \tag{4.15}$$

Differentiating the equation 4.15 with respect to the arc length function s we have

$$T(\varpi' + \varpi\mathcal{K} - 1) + N(\varpi - \xi\mathcal{T}) + B(\xi' + \xi\mathcal{K}) = 0$$

According to the definition of the linear independent of vectors, we get

$$\begin{aligned} \varpi' + \varpi\mathcal{K} - 1 &= 0 \\ \varpi - \xi\mathcal{T} &= 0 \\ \xi' + \xi\mathcal{K} &= 0 \end{aligned}$$

Because of the solution of above differential equations does not exist, to solve the above equation, let us take $\mathcal{K} = 0$. Then we can be rewritten as the form

$$\begin{aligned} \varpi'(s) - 1 &= 0 \\ \varpi(s) - \xi(s)\mathcal{T}(s) &= 0 \\ \xi'(s) &= 0 \end{aligned}$$

Hence, it follows that

$$\varpi(s) = s + c_1 \tag{4.16}$$

$$\xi(s) = c_2, \tag{4.17}$$

$$\varpi(s) = \xi(s)\mathcal{T}(s). \tag{4.18}$$

Substituting 4.16 and 4.17 into 4.15, we can write the position vector α as follows

$$\alpha(s) = (s + c_1)T(s) + c_2B(s). \tag{4.19}$$

Thus, from 4.19 we easily have

$$h^2 = |\langle \alpha, \alpha\rangle| = \left|s^2 + 2c_1s + c_1^2 + c_2^2\right|.$$

This proves statement (i). Next, from 4.19 we find that tangential component of the position vector as $\langle \alpha(s), T(s)\rangle = s+c_1$. This proves statement (ii). Since α is a rectifying curve, from 4.19 the normal component α^N of the position vector is given by

$$\alpha^N = \xi(s)B(s).$$

Hence,

$$\left\|\alpha^N\right\| = |\xi(s)| = |c_2| \neq 0.$$

Therefore, statement (iii) is proved. And finally, from 4.19 we easily get $\langle \alpha(s), B(s)\rangle = c_2 = const.$ and since the equiform torsion of the curve α, $\mathcal{T}(s) \neq 0$, the statement $(iv$ $)$ is proved.

Conversely, let us suppose that statement (i) or (ii) holds. Then there holds the equation $\langle \alpha(s), T(s)\rangle = s + c_1$, for orbitrary constant a. Differentiating this equation with respect to the arc length function s and using the Frenet formulas,

$$\langle \alpha(s), \mathcal{K}T(s)\rangle + \langle \alpha(s), N(s)\rangle = 0.$$

Since $\mathcal{K} = 0$, we get

$$\langle \alpha(s), N(s)\rangle = 0.$$

Thus, α is a rectifying curve in the equiform geometry of the Galilean space G_3. Next, if statement (iii) holds, let us put $\alpha(s) = \xi(s)T(s) + \alpha^N$. Then we find that

$$\left\langle \alpha^N, \alpha^N \right\rangle = \langle \alpha(s), \alpha(s)\rangle = \langle \alpha(s), T(s)\rangle^2 + c_3. \tag{4.20}$$

where c_3 is orbitrary constant. Differentiating this equation with respect to the arc length function s and using $\mathcal{K} = 0$ we get

$$\langle \alpha(s), T(s)\rangle = (1 + \langle \alpha(s), N(s)\rangle) \langle \alpha(s), T(s)\rangle . \tag{4.21}$$

Since the distance function h is nonconstant, we have $\langle \alpha(s), T(s)\rangle \neq 0$. Furthermore, from 4.21 we obtain $\langle \alpha(s), N(s)\rangle = 0$. Which means that the curve α is a rectifying curve. Finally, if statement (iv) holds, then we have $\langle \alpha(s), B(s)\rangle = c_2 =$ constant. Differentiating this equation with respect to s and using Frenet formulas, we get

$$-\mathcal{K} \langle \alpha(s), B(s)\rangle - \mathcal{T} \langle \alpha(s), N(s)\rangle = 0.$$

Since $\mathcal{K}(s) = 0$ and $\mathcal{T}(s) \neq 0$, we have $\langle \alpha(s), N(s)\rangle = 0$, we easily obtain that the curve α is a rectifying curve.

References

[1] B. J. Pavković, Equiform Geometry of Curves in the Isotropic Spaces I_3^1 and I_3^2, Rad JAZU. (1986), 39-44.

[2] B. J. Pavković, I. Kamenarović, The Equiform Differential Geometry of Curves in the Galilean Space G_3, Glasnik Mat. 22(1987), 449-457.

[3] B. Y. Chen, When does the position vector of a space curve always lie in its rectifying plane?, Amer. Math. Monthly, 110 (2003), 147-152.

[4] I. Yaglom, A simple non-Euclidean geometry and its physical basis, Springer-Verlag, in New York Inc, (1979).

[5] S. Kızıltuğ and Y. Yaylı, Bertrand Curves of AW(k)-Type in the Equiform Geometry of the Galilean Space, Abstract and Applied Analysis, 2014(2014), 1-6.

[6] Z. Erjavec and B. Divjak, The equiform differential geometry of curves in the pseudo-Galilean space, Math. Communications. 13(2008), 321-332.

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