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ON BLOCK TRANSFINITE SEQUENCES IN BANACH SPACES

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Abstract. We introduced the concept of block transfinite sequences in Banach spaces. We prove that every block transfinite sequences with respect to a transfinite basis, more generally, a transfinite basic sequence (minimal transfinite sequence) is basic (respectively, minimal). The existence of a complete minimal transfinite sequence as a super transfinite sequence of block transfinite sequence with respect to a complete minimal transfinite sequence in a Banch space has been established. Finally, it is shown that every block transfinite sequence with respect to a shrinking minimal transfinite sequence is shrinking minimal.

1. Introduction

Bessaga [1] introduced the concept of monotone transfinite basis. Infact, he called it 'monotone basis of type ν'. Subsequently Bessaga [2] replaced the condition of monotonically by a weaker condition of uniform boundedness and introduced the term 'projection basis of type ν'. However, the definition of transfinite basis, we use in the present note is due to Dorembus [4]. Very recently, Jain And Ahmad [5] obtained an inequality characterizing transfinite basic sequences, analogous to the Nikol'skill inequality for bases. Further, Jain and Kaushik studied boundedly complete transfinite bases in [6].

In the present note, we introduced the concept of block transfinite sequences in Banach spaces. We prove that every block transfinite sequences with respect to a transfinite basis, more generally, a transfinite basic sequence (minimal transfinite sequence) is basic (respectively, minimal). The existence of a complete minimal transfinite sequence as a super transfinite sequence of block transfinite sequence with respect to a complete minimal transfinite sequence in a Banch space has been established. Finally, it is shown that every block transfinite sequence with respect to a shrinking minimal transfinite sequence is shrinking minimal.

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2. Preliminaries and Lemmas

Throughout E will denote a Banach space over the scalar field $\mathbb{K}(\mathbb{R} or \mathbb{C})$, E^* the first conjugate space E, ν an infinite ordinal, $(x_\lambda)_{\lambda<\ \nu}$.

A transfinite series $\sum\limits_{\lambda<\ \nu} y_\lambda$ is said to converge to an element $x \in E$, if there exists a unique continuous function S from $[1,\nu)$ into E satisfying $S(1) = 0,\ S(\nu) = x$ and $S(\lambda+1) = S(\lambda) + y_\lambda,\ \lambda <\ \nu$.A TS $(x_\lambda)_{\lambda<\ \nu}$ is said to be a transfinite basis (TB) if for each $x \in E$, there exists a unique TS $(\alpha_\lambda)_{\lambda<\ \nu} \subset \mathbb{K}$ such that $x = \sum\limits_{\lambda<\ \nu} \alpha_\lambda x_\lambda$, where the convergence is in the norm topology of E. A transfinite system $(x_\lambda, f_\lambda)_{\lambda<\ \nu}$ $((x_\lambda)_{\lambda<\ \nu} \subset E, (f_\lambda)_{\lambda<\ \nu} \subset E^*)$ is said to be a biorthogonal transfinite system if $f_\lambda(x_\mu) = \delta_{\lambda\mu},\ (\lambda,\ \mu < \nu)$. A TS $(x_\lambda)_{\lambda<\nu} \subset E$ is complete in E if $[x_\lambda]_{\lambda<\nu} = E$ and is minimal if there exists a TS $(f_\lambda)_{\lambda<\nu} \subset E^*$, called an admissible TS to $(x_\lambda)_{\lambda<\nu}$, such that $(x_\lambda,\ f_\lambda)_{\lambda<\nu}$ is a bioorthogonal transfinite system. An admissible TS to a minimal TS may not be unique. However, we have

Lemma 2.1. *An admissible TS to a minimal TS $(x_\lambda)_{\lambda<\nu} \subset E$ is unique if and only if $[x_\lambda]_{\lambda<\nu} = E$.*

In the case when, an admissible TS to $(x_\lambda)_{\lambda<\nu}$ is unique, it is called the associate transfinite sequence of functionals (a.t.s.f) to $(x_\lambda)_{\lambda<\nu}$.

Lemma 2.2. *[5] Let $(x_\lambda)_{\lambda<\nu}$ be a TS of non-zero elements in E such that $[x_\lambda]_{\lambda<\nu} = E$. Then $(x_\lambda)_{\lambda<\nu}$ is a TB of E if and only if there exists a constants $M \geq 1$ such that*

$$\|\sum_{\chi<\lambda} \alpha_\chi x_\chi\| \leq M\|\sum_{\chi<\mu} \alpha_\chi x_\chi\|$$

for all ordinals $\lambda,\ \mu$ with $\lambda < \mu < \nu$ and arbitrary scalars $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_\mu$ such that $\sum\limits_{\chi<\mu} \alpha_\chi x_\chi$ converges.

The infimum of M, is denoted by $V_{(x_\lambda)_{\lambda<\nu}}$ is said to be the norm of TB $(x_\lambda)_{\lambda<\nu}$.

3. Main Result

Definition 3.1. A TS $(z_\lambda)_{\lambda<\nu} \subset E$ is said to be a block TS with respect to a TS $(x_\lambda)_{\lambda<\nu}$, if it is of the form

$$z_\lambda = \sum_{\chi\in D_\lambda} \alpha_\chi x_\chi \neq 0, \quad (\lambda < \nu),$$

where $(D_\lambda)_{\lambda<\nu}$ is a TS of finite subsets of $[1,\nu)$ such that $[1,\nu) = \bigcup\limits_{\lambda<\nu} D_\lambda$ and $D_\lambda \cap D_\mu = \phi,\ \ \lambda \neq \mu$ and $\{\alpha_\chi\}_{\chi\in \bigcup\limits_{\lambda<\nu} D_\lambda}$ are any scalars.

A block TS $(z_\lambda)_{\lambda<\nu} \subset E$ is said to be a block basic TS with respect to $(x_\lambda)_{\lambda<\nu} \subset E$, if it is a TB of $[z_\lambda]_{\lambda<\nu} \subset E$.

Theorem 3.2. *Every block TS with respect to a basic TS is basic.*

Proof. . Let $(z_\lambda)_{\lambda<\nu} \subset E$ be a block TS with respect to a basic TS $(x_\lambda)_{\lambda<\nu} \subset E$. Let $(\gamma_\lambda)_{\lambda<\nu}$ be any TS of scalars and $\mu,\ \theta$ be any ordinals with $\mu < \theta$. Then, in view of Lemma 2.2, we have

$$\begin{aligned}\Big\|\sum_{\chi<\lambda}\gamma_\chi z_\chi\Big\| &= \Big\|\sum_{\chi<\lambda}\gamma_\chi(\sum_{\eta\in D}\alpha_\eta x_\eta)\Big\| \\ &\leq V_{(x_\lambda)_{\lambda<\nu}}\Big\|\sum_{\chi<\theta}\sum_{\eta\in D}\gamma_\chi\alpha_\eta x_\eta\Big\| \\ &= V_{(x_\lambda)_{\lambda<\nu}}\Big\|\sum_{\chi<\theta}\gamma_\chi z_\chi\Big\|.\end{aligned}$$

Hence, again in view of Lemma 2.2,$(z_\lambda)_{\lambda<\nu}$ is a block basic TS.

Corollary 3.3. Every block TS with respect to a TB is basic.

Theorem 3.4. *Every block TS with respect to a minimal TS is minimal.*

Proof. . Let $(z_\lambda)_{\lambda<\nu}$ be a block TS with respect to a minimal TS $(x_\lambda)_{\lambda<\nu} \subset E$. Then $(z_\lambda)_{\lambda<\nu}$ is of the form

$$z_\lambda = \sum_{\chi\in D_\lambda}\alpha_\chi x_\chi \neq 0, \quad (\lambda<\nu).$$

Since $(x_\lambda)_{\lambda<\nu} \subset E$ is a minimal TS, it follows that $x_\lambda \neq 0, \quad \lambda<\nu$. Thus, there exists an index $\chi \in D_\lambda$ such that $\alpha_\chi \neq$. Let $(f_\lambda)_{\lambda<\nu} \subset E^*$ be an admissible TS to $(x_\lambda)_{\lambda<\nu}$. Define the TS $(g_\lambda)_{\lambda<\nu} \subset E^*$ by

$$g_\chi = \alpha_\chi^{-1} f_\chi, \quad (\chi<\nu).$$

Then $(z_\lambda)_{\lambda<\nu}$ is a minimal TS, since $(z_\lambda,\ g_\lambda)_{\lambda<\nu}$ is a biorthogonal transfinite system.

Theorem 3.5. *Let $(x_\lambda)_{\lambda<\nu}$ be a complete minimal TS in E and let $(y_\lambda)_{\lambda<\nu} \subset E$ be a block TS with respect to $(x_\lambda)_{\lambda<\nu}$. Then, there exists a complete minimal TS $(z_\lambda)_{\lambda<\nu} \subset E$ such that $z_{\alpha(\lambda)} = y_\lambda, \quad \lambda<\nu$ and $\alpha(\lambda) \in D_\lambda$ with $\alpha(\lambda) \geq \chi, \quad \chi \in D_\lambda$ and that*

$$[z_\chi]_{\chi\in D_\lambda} = [x_\chi]_{\chi\in D_\lambda}, \quad (\lambda<\nu).$$

Consequently, for the a.t.s.f $(f_\lambda)_{\lambda<\nu}$ and $(h_\lambda)_{\lambda<\nu}$ in E^* to $(x_\lambda)_{\lambda<\nu}$ and $(z_\lambda)_{\lambda<\nu}$ respectively, we have

$$[h_\chi]_{\chi\in D_\lambda} = [f_\chi]_{\chi\in D_\lambda}, \quad (\lambda<\nu)$$

and

$$[h_\chi] = [f_\chi], \quad (\chi<\nu).$$

Proof. . Since $(x_\lambda)_{\lambda<\nu}$ is a complete minimal TS, $x_\chi \neq 0, \quad (\chi<\nu)$. Thus, there exists an index $\chi \in D_\lambda$ such that $\alpha_\chi \neq 0$. Define $(z_\lambda)_{\lambda<\nu} \subset E$ and $(h_\lambda)_{\lambda<\nu} \subset E^*$ by

$$z_\eta = \begin{cases} x_{\sigma_\lambda(\eta)}, & \eta < \alpha(\lambda) \in D_\lambda \\ y_\lambda, & \eta = \alpha(\lambda) \in D_\lambda \end{cases}$$

and

$$h_\eta = \begin{cases} f_{\sigma_\lambda(\eta)} - (\alpha_{\sigma_\lambda(\eta)} f_\chi)(1/\alpha_\chi), & \eta < \alpha(\lambda) \in D_\lambda \\ \alpha_\chi^{-1} f_\chi, & \eta = \alpha(\lambda) \in D_\lambda \end{cases}$$

where σ_λ is any one to one mapping of $D_\lambda \setminus \{\alpha(\lambda)\}$ onto the set $D_\lambda \setminus \{\chi\}$, $(\lambda < \nu)$ and $(f_\lambda)_{\lambda<\nu} \subset E^*$ is the a.t.s.f to $(x_\lambda)_{\lambda<\nu}$. Then $(z_\lambda,\ h_\lambda)_{\lambda<\nu}$ is a biorthogonal transfinite system such that

$$[z_\chi]_{\chi\in D_\lambda} \subset [x_\chi]_{\chi\in D_\lambda}, \quad (\lambda < \nu).$$

In particular, since $[z_\chi]_{\chi\in D_\lambda}$ is linearly independent, it follows that $[z_\chi]_{\chi\in D_\lambda} = [x_\chi]_{\chi\in D_\lambda}$. Hence, $[z_\lambda]_{\lambda<\nu} = E$. Consequently, by definition of $(h_\lambda)_{\lambda<\nu}$, we have

$$[h_\chi]_{\chi\in D_\lambda} = [f_\chi]_{\chi\in D_\lambda}, \quad (\lambda < \nu).$$

and

$$[h_\chi] = [f_\chi], \quad (\chi < \nu).$$

Definition 3.6. A minimal TS $(x_\lambda)_{\lambda<\nu} \subset E$ with admissible TS $(f_\lambda)_{\lambda<\nu} \subset [x_\lambda]^*_{\lambda<\nu}$ is said to be shrinking if $[f_\lambda]_{\lambda<\nu} = [x_\lambda]^*_{\lambda<\nu}$.

Theorem 3.7. *Evey block TS with respect to a shrinking minimal TS is a shrinking minimal TS.*

Proof. . Let $(x_\lambda)_{\lambda<\nu}$ be a shrinking minimal TS in E. Without loss of generality one may assume that $[x_\lambda]_{\lambda<\nu} = E$, since otherwise, E can be replaced by $[x_\lambda]_{\lambda<\nu}$. Let $(f_\lambda)_{\lambda<\nu} \subset E^*$ be the a.t.s.f to $(x_\lambda)_{\lambda<\nu}$ and let $(y_\lambda)_{\lambda<\nu}$ be a block TS with respect to $(x_\lambda)_{\lambda<\nu}$ of the form

$$y_\lambda = \sum_{\chi\in D_\lambda} \alpha_\chi x_\chi \neq 0, \quad (\lambda < \nu).$$

Then, in view of Theorem 3.5, there exists a complete minimal TS $(z_\lambda)_{\lambda<\nu} \subset E$ with a.t.s.f $(h_\lambda)_{\lambda<\nu} \subset E^*$ such that $z_{\alpha(\lambda)} = y_\lambda,\ \lambda < \nu$ and $\alpha(\lambda) \in D_\lambda$ with $\alpha(\lambda) \geq \chi$, for each $\chi \in D_\lambda$ and

$$[h_\lambda]_{\lambda<\nu} = [f_\lambda]_{\lambda<\nu}.$$

Put $g_\lambda = h_{\alpha(\lambda)},\ \lambda < \nu$ and $\alpha(\lambda)$ as above. Then

$$g_\lambda(y_\mu) = h_{\alpha(\lambda)}(z_{\alpha(\mu)}) = \delta_{\lambda\mu}, \quad (\lambda, \mu < \nu)$$

and

$$h_\lambda(y_\mu) = h_\lambda(z_{\alpha(\mu)}) = 0,\ (\lambda \neq \alpha(\chi),\ \chi < \nu).$$

Therefore, we have

$$[h_\lambda|_{[y_\chi]_{\chi<\lambda}}]_{\lambda<\nu} = [g_\lambda|_{[y_\chi]_{\chi<\lambda}}]_{\lambda<\nu}.$$

Since $E^* = [f_\lambda]_{\lambda<\nu} = [h_\lambda]_{\lambda<\nu}$ it follows that

$$[g_\lambda|_{[y_\chi]_{\chi<\lambda}}]_{\lambda<\nu} = [y_\lambda]^*_{\lambda<\nu}.$$

This completes the proof.

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