Poincare Journal of Analysis & Applications
Vol. 1, No. 2 (2014), 77-92
©Poincare Publishers
DOI: 10.46753/pjaa.2014.v01i02.005

AN APPLICATION OF MODIFIED WEIGHTED NÖRLUND MEANS TO A SERIES ASSOCIATED WITH FOURIER SERIES

PREM CHANDRA

Date of Receiving	:	20. 08. 2014
Date of Revision	:	27. 12. 2014
Date of Acceptance	:	28. 12. 2014

Abstract. In this paper, we apply modified absolute Nörlund means with weights(Das [15]) to study absolute Nörlund summability of Fourier series with multipliers and obtain some necessary and sufficient conditions, imposed upon the generating function of Fourier series, as well as best possible absolute summability multipliers in certain sense. As a consequence, we not only get some new results but some results, which improve earlier results, have also been obtained.

1. DEFINITIONS AND NOTATIONS

Let (p_n) be a sequence of constants such that

$$P_n = p_0 + p_1 + p_2 + \dots + p_n \neq 0, \text{ for } n \geq 0$$

and let $\lambda = (\lambda_n)$ be a given positive sequence. Then modified Nörlund mean with weight $\lambda = (\lambda_n)$ of $\sum_{n=0}^{\infty} \omega_n$ is defined by

$$d_n = \frac{1}{P_n \lambda_n} \sum_{k=0}^{n} p_{n-k} \lambda_k \omega_k. \tag{1.1}$$

If the series $\sum_{n=0}^{\infty} d_n$ converges to s, we say that $\sum_{n=0}^{\infty} \omega_n$ is summable by modified Nörlund mean with weight $\lambda = (\lambda_n)$ or summable (N', p_n, λ_n) to s. And if

$$\sum_{n=0}^{\infty} |d_n| < \infty \tag{1.2}$$

then $\sum_{n=0}^{\infty} \omega_n$ is said to be absolutely summable by modified Nörlund mean of weight $\lambda = (\lambda_n)$ or summable $|N', p_n, \lambda_n|$. For this, see Das [15].

2010 *Mathematics Subject Classification.* 42A24, 42A45, 40F05, 40G05, 40G99.
Key words and phrases. Multipliers, Modified absolute Nörlund summability, Generalized harmonic summability.
Communicated by. L.K. Vashisht

It follows from Das([15]; Lemma 4) that if $p_n > 0$ and (λ_n) is non decreasing then (N', p_n, λ_n) is absolutely conservative provided

$$(i)\ \sum_{n=k}^{\infty} \frac{1}{np_n} = O\left(\frac{1}{\lambda_k}\right)\ (ii)\ (n+1)p_n = O(P_n). \tag{1.3}$$

Further if, (p_n) satisfies Kaluza condition:

$$\mathcal{M} = \{p_n > 0 : p_{n+1}p_{n-1} \geq {p_n}^2 \text{ and } p_{n+1} \leq p_n\}, \tag{1.4}$$

then (Das[14])

$$|N, p_n| \sim |N', p_n, n| \tag{1.5}$$

In 1979, Bosanquet and Das [2] have shown that a Nörlund method is identical with its modified Nörlund mean of weight $\lambda_n = n$ if and only if Nörlund matrix is a Cesàro matrix. In 1980, Das and Mohapatra [17] have studied a special case of (N, p_n) method, which they called the generalized harmonic-Cesàro method (Z, α, β) and is generated by $(p_n) : p_n = A_n^{\alpha-1,\beta}$, determined by the identity

$$(1-z)^{-\alpha-1}\{loga/(1-z)\}^{\beta} = \sum_{n=0}^{\infty} A_n^{\alpha,\beta} z^n (|z| < 1), \tag{1.6}$$

where α, β are real and $a > 2$ is a fixed constant. Thus, in the case $\beta = 0$, the method (Z, α, β) reduces to Cesàro method (C, α) and in the case $\alpha = 0, \beta > 0$, it is equivalent to the generalized harmonic summability method (Das and Mohapatra [16]) which was first studied by Iyenger [22] for positive integer β by considering the following identity:

$$\{\frac{1}{z} log \frac{1}{1-z}\}^{\beta} = \{\sum_{n=0}^{\infty} \frac{z^n}{n+1}\}^{\beta} \equiv \sum_{n=0}^{\infty} c_n^{\beta} z^n \tag{1.7}$$

for $\beta > 0, |z| \leq 1$ and $z \neq 1$, where

$$c_n^{\beta} \sim \frac{\beta (logn)^{\beta-1}}{n} = q_n^{\beta}(suppose). \tag{1.8}$$

We write $|N, q_n^{\beta}|(\beta > 0)$ for the absolute generalized harmonic summability method and $|Z', \alpha, \beta|$ for modified absolute generalized harmonic-Cesàro method associated with the absolute generalized harmonic-Cesàro method $|Z, \alpha, \beta|$, which has been studied by Das and Mohapatra [18].

For real α and β, the following estimates are known ([33],p.192):

$$A_n^{\alpha,\beta} = \sum_{m=0}^{n} A_n^{\alpha-1,\beta} \tag{1.9}$$

$$A_n^{\alpha,\beta} \sim \frac{n^{\alpha}}{\Gamma(\alpha+1)} (logn)^{\beta} (\alpha \neq -1, -2, -3, ...) \tag{1.10}$$

$$A_n^{\alpha,\beta} \sim \beta(-1)^{\alpha-1}\Gamma(|\alpha|) n^{\alpha} (logn)^{\beta-1} (\alpha = -1, -2, -3, ...). \tag{1.11}$$

Let $f \in L_{2\pi}$, the space of all real valued, $2\pi-$ periodic and L-integrable Functions over $[-\pi, \pi]$. Then the Fourier series of f at a point x is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n cosnx + b_n sinnx) \equiv \sum_{n=0}^{\infty} A_n(x), \tag{1.12}$$

where a_n and b_n are the Fourier coefficients of f.
We use the following notations throughout the paper for fixed real number x:

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\} \tag{1.13}$$

$$R(t) = \phi(t) - \phi_1(t), \text{ where } \phi_1(t) = \frac{1}{t}\int_0^t \phi(u)du \tag{1.14}$$

$$H_n(t) = b(t)\frac{sin\ nt}{nt} + \int_t^\pi \frac{sin\ nu}{nu} db(u), b(t) = ty(t^{-1}), \tag{1.15}$$

where $y(t^{-1})$ is defined in $0 < t \leq \pi$. We also write for any sequence (f_n) and $h = 1, 2, 3....$

$$\Delta^0 f_n = f_n, \Delta f_n = f_n - f_{n+1}, \Delta^h f_n = \Delta(\Delta^{h-1} f_n) \tag{1.16}$$

$$\nabla^0 f_n = f_n, \nabla f_n = f_n - f_{n-1}, \nabla^h f_n = \nabla(\nabla^{h-1} f_n). \tag{1.17}$$

$T = \left[\frac{k}{t}\right]$, the integral part, where $0 < t \leq \pi$ and k is a positive constant taken for the convenience in the analysis and not necessarily the same at each occurrence. And for positive integer n, we write y_n for $y(n)$.

2. THEOREMS AND COMMENTS

The main object is to apply modified absolute Nörlund summability method $|N', p_n, \lambda_n|$ with weight $\lambda = (\lambda_n)$ to study absolute summability $|N, p_n|$ -factor problems in Fourier series. We observe that the application of modified Nörlund mean with weight $\lambda = (\lambda_n)$ not only helps us to obtain simple conditions on p_n but it also provide simpler proof. In this paper, we study, as special cases, the following series:

$$\sum_{n=1}^{\infty} n^\alpha A_n(x), \ \sum_{n=1}^{\infty} \frac{n^\alpha A_n(x)}{log(n+1)} \text{ and } \sum_{n=1}^{\infty} \frac{A_n(x)}{log^\delta(n+1)}$$

and investigate some new results. In some of the cases, we are in a position to investigate best possible conditions imposed upon the generating functions of the Fourier series and best possible absolute summability factors in certain sense. In this context, we first state the following known result:
Theorem A. Let, for $0 \leq \alpha < 1$,

$$\int_0^\pi t^{-\alpha}|d\phi(t)| < \infty. \tag{2.18}$$

Then

$$\sum_{n=1}^{\infty} n^\alpha A_n(x) \in |C, \eta| \ (0 \leq \alpha < \eta < 1). \tag{2.19}$$

The case $\alpha = 0$ of Theorem A was first obtained by Bosanquet [1] and later on it was generalized by a number of workers such as Pati([30], [31]), Kanno [23], Dikshit [21] and Kuttner and Sahney [24] who replaced $|C, \eta|$ - method by absolute Nörlund summability method and matrix summability method, respectively.
The case $0 < \alpha < 1$ of Theorem A was first obtained by Mohanty [27] and was generalized by using $|N, p_n|$ and absolute matrix summability method by Kanno [23] and

Chandra and Barve [11], respectively. We observe that retaining the condition (2.18), various conditions on the generating sequence $p_n$of $|N, p_n|$-method have been obtained to study $\sum_{n=1}^{\infty} n^{\alpha} A_n(x)$ but we did not see any one who obtained a different set of condition or conditions than the one given in (2.18) to ensure absolute summability of $\sum_{n=1}^{\infty} n^{\alpha} A_n(x), 0 \leq \alpha < 1$ by a method weaker than $|C, \eta|$ $(0 < \alpha < \eta < 1)$. In this paper, we make such an effort by using absolute generalized harmonic-Cesàro summability method (Das and Mohapatra [17]) and modified absolute Nörlund summability method with weights(Das [15]). To achieve this goal, we first prove

Theorem 2.1. *Let $(p_n) \in \mathcal{M}$ and let monotonic function $y(t^{-1})$, defined in $0 < t \leq \pi$, satisfy the following conditions:*

$$t^c y(t^{-1}) \text{ increases with } t \text{ for some } c \text{ in } 0 < c < 1 \tag{2.20}$$

$$t\frac{d}{dt} y(t^{-1}) = O\{y(t^{-1})\}, \ t \to 0+ \tag{2.21}$$

$$t^{-1}\frac{d}{dt} y(t^{-1}) = O\{y(t^{-1})\} \text{ decreases as } t \text{ increases} \tag{2.22}$$

If, for $g(t) = R(t)/y(t^{-1})$,

$$\int_0^{\pi} log\frac{k}{t} \ |dg(t)| < \infty. \tag{2.23}$$

Then for the truth of

$$\sum_{n=1}^{\infty} A_n(x) y_n^{-1} \in |N, p_n| \tag{2.24}$$

it is sufficient that

$$g(0+) = 0 \tag{2.25}$$

should hold provided

$$y_T P_T \sum_{n=T}^{\infty} \frac{1}{n P_n y_n} = O(log\frac{k}{t}), \tag{2.26}$$

uniformly in $0 < t < \pi$.

We first use Theorem 1 to obtain the following new and interesting result under stronger condition than (2.18) for $0 < \alpha < 1$:

Theorem 2.2. *Let, for $0 < \alpha < 1$ and $\beta > 1, p_n = A_n^{\alpha-1,\beta}$ and let $g(t) = t^{-\alpha} R(t)$. Then, if (2.26) holds, it is necessary and sufficient, for*

$$\sum_{n=1}^{\infty} n^{\alpha} A_n(x) \in |Z, \alpha, \beta|, \tag{2.27}$$

that (2.25) should hold. Further (2.23) cannot be replaced

$$\int_0^{\pi} log^{\sigma}\frac{k}{t} |dg(t)| < \infty \ (0 < \sigma < 1) \tag{2.28}$$

and multipliers (n^{α}) in (2.27) cannot be replaced by

$$(n^{\alpha} log^{\epsilon}(n+1)) \text{ for } 0 < \alpha < 1 \text{ and } \epsilon > 0. \tag{2.29}$$

Also the result (2.27) under the hypotheses (2.23) and (2.25) is incomparable with Theorem A for $0 < \alpha < 1$.

We now give the following new result under conditions weaker than (2.18) for $0 < \alpha < 1$:

Theorem 2.3. *Let, for*

$$g(t) = t^{-\alpha} log^{-1}\frac{k}{t}R(t) \ (0 < \alpha < 1), \tag{2.30}$$

(2.23) hold. Then in order that

$$\sum_{n=1}^{\infty} \frac{n^{\alpha} A_n(x)}{log(n+1} \in |Z, \alpha, \beta| \ (0 < \alpha < 1, \beta > 0) \tag{2.31}$$

should hold, it is necessary and sufficient that (2.25) should hold. Also (2.23) cannot be replaced by (2.28) and multipliers $(n^{\alpha} log^{-1}(n+1))$ *in (2.31) cannot be replaced by*

$$(n^{\alpha} log^{\epsilon-1}(n+1)) \ (\epsilon > 0). \tag{2.32}$$

We now obtain the following result for the absolute generalized harmonic summability method $|N, q_n^{\beta}|$, *where* $q_n^{\beta} = \frac{\beta(logn)^{\beta-1}}{n}(\beta > 0)$.

Theorem 2.4. *Let* $\beta > 0$ *and* δ *be real such that* $\beta + \delta > 1$ *and let, for*

$$g(t) = R(t) log^{-\delta}\frac{k}{t}, \tag{2.33}$$

(2.23) hold. Then in order that

$$\sum_{n=1}^{\infty} \frac{A_n(x)}{log^{\delta}(n+1} \in |N, q_n^{\beta}| \ (\beta > 0) \tag{2.34}$$

should hold it is necessary and sufficient that (2.25) should hold. Further, (2.23) is best possible in the sense that it cannot be replaced by the weaker condition (2.28). Also the multipliers $\frac{1}{log^{\delta}(n+1)}$ *in (2.34) cannot be replaced by*

$$\left(log^{\epsilon}(n+1) log^{\delta}(n+1)\right) \ (\epsilon > 0). \tag{2.35}$$

Remark. Theorem 2.4 for $\delta \leq 2$ was earlier proved in [10].

3. ESTIMATES

Let $y(t^{-1})$ satisfy (2.20), (2.21) and (2.22) and let (p_n) be non-negative and non-increasing. Then we use the following estimates for $0 < t < \pi$ in the proof of the theorems:

$$H_n(t) = b(t)\frac{sin\ nt}{nt} + O\left\{b\left(\frac{1}{n}\right)\right\} \tag{3.36}$$

$$H_n(t) = b(t)\frac{sin\ nt}{nt} + O\{n^{-2}t^{-2}\ b(t)\} \tag{3.37}$$

$$\sum_{m=1}^{n} y_m^{-1}p_{n-m} = O\{P_n y_n^{-1}\}. \tag{3.38}$$

For the proof of (3.36) and (3.37) proceed as in Chandra [7].

Proof. It is known that (y_n) is monotonic, therefore if (y_n) is non-increasing then the proof of (3.38) is trivial. And if (y_n) is non-decreasing, we write, for $N = [n/2]$,

$$\begin{aligned}\sum_{m=1}^{n} y_m^{-1}p_{n-m} &= \sum_{m=1}^{N} y_m^{-1}p_{n-m} + \sum_{m=N+1}^{n} y_m^{-1}p_{n-m}\\ &\le p_{n-N}\sum_{m=1}^{N} y_m^{-1} + y_N^{-1}\sum_{m=0}^{n} p_{n-m}\\ &= O\{P_n y_n^{-1}\},\end{aligned}$$

by using (2.20). □

4. LEMMAS

We use the following lemmas in the proof of the theorems:

Lemma 4.1. *(Das[15]; Theorem 4). Suppose $\lambda_n > 0$ and increasing to ∞ with n. Let h be a non-negative integer and let*

$$(i)\ \nabla^h p_n \in \mathcal{M}\ \text{and}\ (ii)\ n^{h+1}\lambda_n\Delta^{h+1}\left(\frac{1}{\lambda_n}\right) \tag{4.39}$$

In the case $h = 0$, further assume that

$$\sum_{n=k}^{\infty}\frac{1}{n\lambda_n} = O\left(\lambda_k\right). \tag{4.40}$$

Then

$$|N', p_n, \lambda_n| \implies |N, p_n|. \tag{4.41}$$

Lemma 4.2. *Let $p_n \ge 0$ and let (p_n) be non-increasing. If $y(t^{-1})$ satisfies (2.20), then*

$$|\sum_{m=1}^{n} my_m^{-1}p_{n-m}exp(imt)| = O\{ny_n^{-1}P_T\},\ \text{uniformly in}\ 0 < t < \pi. \tag{4.42}$$

By (2.20), it follows that (my_m^{-1}) is increasing. Therefore for the proof of Lemma 4.2, we first apply Abel's lemma for (my_m^{-1}) and then use Lemma 5.11 of McFadden [26].

Lemma 4.3. *For real β and $\alpha > -1$, the following inclusion is strict:*

$$|Z, \alpha, \beta| \subset |C, \eta|\ (\eta > \alpha). \tag{4.43}$$

For it's proof, see Das and Mohapatra ([16],[17],[19]).

Lemma 4.4. *Let r be real and $0 < \alpha < 1$. Then there exist $B_n^{(1)}, B_n^{(2)}, B_n^{(3)}$ and a positive constant d, depending upon α, such that*

$$\int_0^{\pi} \frac{sin\ nt}{nt} t^{\alpha} log^r \frac{k}{t} dt = B_n^{(1)} + B_n^{(2)} + B_n^{(3)}, \tag{4.44}$$

where $B_n^{(1)} > 0$ and $B_n^{(2)} \geq 0$ and that

$$B_n^{(1)} \sim d\ n^{-1-\alpha} log^r n \quad \text{(for real } r\text{)} \tag{4.45}$$

and

$$B_n^{(3)} = O\{n^{-1-\alpha} log^{r-1} n\} \quad \text{(for real } r \neq 0\text{)} \tag{4.46}$$

$$B_n^{(3)} = O\{n^{-2}\} \quad \text{(for real r=0)}. \tag{4.47}$$

Proof. Denote the integral in (4.44) by I and split up into sub-integrals $\int_0^{\frac{\pi}{2n}}$ and $\int_{\frac{\pi}{2n}}^{\pi}$.
Then $I = \int_0^{\frac{\pi}{2n}} + \int_{\frac{\pi}{2n}}^{\pi} = B_n^{(1)} + J$, say, where $B_n^{(1)} > 0$.
By using the transformation: $nt = \theta$, we get
$B_n^{(1)} = n^{-1-\alpha} log^r n \int_0^{\frac{\pi}{2}} s_n(\theta) d\theta$,
where

$$s_n(\theta) = \{1 + log^{-1} n log \frac{k}{\theta}\}^r \theta^{\alpha-1} sin\theta = O(\theta^{\alpha})$$

,$\theta^{\alpha} \in L[0, \frac{\pi}{2}]$ and $s_n(\theta) \to \theta^{\alpha-1} sin\theta$, as $n \to \infty$.
Therefore, by Lebesgue dominated convergence theorem

$$B_n^{(1)} \to n^{-1-\alpha} log^r n \int_0^{\frac{\pi}{2}} \theta^{\alpha-1} s_n(\theta) d\theta \text{ as } n \to \infty$$

.
Hence we get (4.45), where

$$d = \int_0^{\frac{\pi}{2}} \theta^{\alpha-1} s_n(\theta) d\theta > 0. \tag{4.48}$$

Integrating by parts and denoting the first integral by $B_n^{(2)}$, we get
$J = \frac{\alpha-1}{n^2} \int_{\frac{\pi}{2n}}^{\pi} t^{\alpha-2} log^r \frac{k}{t} cos\ nt\ dt - n^{-2} \pi^{\alpha-1} log^r \frac{k}{\pi} cos\ n\pi - \frac{r}{n^2} \int_{\frac{\pi}{2n}}^{\pi} t^{\alpha-2} log^{r-1} \frac{k}{t} cos\ nt\ dt$
$= B_n^{(2)} + B_n^{(3)}$, say,
where for real r

$$\begin{aligned} B_n^{(3)} &= O(n^{-2}) - \frac{r}{n^2} \int_{\frac{\pi}{2n}}^{\pi} t^{\alpha-2} log^{r-1} \frac{k}{t} cos\ nt\ dt \\ &= O(n^{-2}) + O\{|r| n^{-1-\alpha} log^{r-1} n\}, \end{aligned} \tag{4.49}$$

by using the second mean value theorem. From (4.49), we get (4.46) and (4.47). Once again applying the second mean value theorem, we get

$$B_n^{(2)} = (1-\alpha)(log \frac{2kn}{\pi})^r (1 - sin\ n\theta') \geq 0, \tag{4.50}$$

where $\frac{\pi}{2n} < \theta' < \pi$. This completes the proof of the lemma. □

Lemma 4.5. *Let $g(t) = t^{-\alpha}R(t)$. Then there exists an $f \in L_{2\pi}$ for which (2.25) and (2.28) hold but the series in (2.27) at x=0 diverges properly.*

Proof. We take an even function $f \in L_{2\pi}$. Then for $x = 0, \phi(t) = f(t)$.
We now define:

$$f(t) = t^{\alpha} log^{-1}\frac{k}{t}, \text{ for } 0 < \alpha < 1 \text{ and } 0 \leq \alpha \leq \pi. \tag{4.51}$$

Then by using (1.14) and integrating by parts, we get

$$\begin{aligned} R(t) &= t^{\alpha} log^{-1}\frac{k}{t} - \frac{1}{t}\int_0^t u^{\alpha} log^{-1}\frac{k}{u} du \\ &= \frac{1}{1+\alpha}\left\{\alpha t^{\alpha} log^{-1}\frac{k}{t} + \frac{1}{t}\int_0^t u^{\alpha} log^{-2}\frac{k}{u} du\right\}. \end{aligned}$$

Hence
$g(t) = t^{-\alpha}R(t) = \frac{1}{1+\alpha}\left\{\alpha log^{-1}\frac{k}{t} + t^{-\alpha-1}\int_0^t u^{\alpha} log^{-2}\frac{k}{u} du\right\}$
and $g(0+) = 0$, i.e. (2.25) holds. By using the value of $g(t)$, as calculated above, it may be shown that (2.28) holds.
We now show that $\sum_{n=1}^{\infty} n^{\alpha} A_n(0)$ diverges to $-\infty$. We have

$$\begin{aligned} A_n(0) &= \frac{2}{\pi}\int_0^{\pi} t^{\alpha} log^{-1}\frac{k}{t} cos\ nt\ dt \\ &= -\frac{2}{\pi}\alpha\int_0^{\pi}\frac{sin\ nt}{nt} t^{\alpha} log^{-1}\frac{k}{t} dt - \frac{2}{\pi}\int_0^{\pi}\frac{sin\ nt}{nt} t^{\alpha} log^{-2}\frac{k}{t} dt \\ &= -\frac{2}{\pi}\alpha\ u_n - \frac{2}{\pi} v_n, \text{ say.} \end{aligned} \tag{4.52}$$

Then ([8]; Lemma 6)

$$v_n = O(1) n^{-1-\alpha} log^{-2} n \tag{4.53}$$

and, by Lemma 4.4

$$u_n = B_n^{(1)} + B_n^{(2)} + B_n^{(3)}, \tag{4.54}$$

where $B_n^{(1)} \geq 0$,

$$B_n^{(1)} \sim d\ n^{-1-\alpha} log^{-1} n, \tag{4.55}$$

d is a positive constant depending upon α, and

$$B_n^{(3)} = O\{n^{-1-\alpha} log^{-2} n\}. \tag{4.56}$$

However, by (4.53) and (4.56), it follows that

$$\sum_{n=1}^{\infty} |n^{\alpha}(\nu_n + B_n^{(3)})| < \infty. \tag{4.57}$$

And, in view of (4.52) through (4.57) except (4.55), it is sufficient for the proof of the lemma to show that

$$\sum_{n=1}^{\infty} n^{\alpha} B_n^{(1)} \text{ diverges to } +\infty, \tag{4.58}$$

which follows by using (4.55). This completes the proof of the lemma. □

Lemma 4.6. *Let $\phi(0+) = 0$. Then, for R(t) as defined in (1.14), the following hold for $\alpha > 0$:*

$$t^{-\alpha}\phi(t) \in BV[0,\pi] \iff t^{-\alpha}R(t) \in BV[0,\pi] \tag{4.59}$$

$$t^{-\alpha-1}\phi(t) \in L[0,\pi] \iff t^{-\alpha-1}R(t) \in L[0,\pi]. \tag{4.60}$$

Proof. For the proof of (4.59), see Chandra ([5]; Lemma 4). And for the proof of (4.60), we ([5],p.76) use the following inversion formula:

$$\phi(t) = R(t) + \int_0^t u^{-1}R(u)\ du \tag{4.61}$$

to get

$$\begin{aligned} \int_0^\pi t^{-\alpha-1}|\phi(t)|\ dt &\le \left(1+\frac{1}{\alpha}\right)\int_0^\pi t^{-\alpha-1}|R(t)|\ dt \qquad (4.62)\\ &\le \left(1+\frac{2}{\alpha}\right)\int_0^\pi t^{-\alpha-1}|\phi(t)|\ dt, \end{aligned}$$

which gives (4.60). □

5. PROOF OF THE THEOREMS

PROOF OF THEOREM 2.1 We([4];Theorem 1) have

$$A_n(x) = \frac{2}{\pi}\int_0^\pi R(t)(cosnt - \frac{sin\ nt}{nt})\ dt.$$

Then, for $g(t) = R(t)/y(t^{-1})$ and $b(t) = ty(t^{-1})$, we have

$$A_n(x) = \frac{2}{\pi}\int_0^\pi g(t)b(t)\frac{d}{dt}\left(\frac{sin\ nt}{nt})\right)\ dt.$$

$$A_n(x) = \frac{2}{\pi}g(0+)\int_0^\pi b(t)\frac{d}{dt}\left(\frac{sin\ nt}{nt})\right)\ dt + \frac{2}{\pi}\int_0^\pi dg(t)\int_t^\pi b(u)\frac{d}{dt}\left(\frac{sin\ nt}{nt})\right)\ du.$$

By using (2.20) and the integration by parts, we get

$$\begin{aligned} A_n(x) &= -\frac{2}{\pi}g(0+)\int_0^\pi \left(\frac{sin\ nt}{nt})\right)db(t)\ dt - \frac{2}{\pi}\int_0^\pi H_n(t)dg(t)\\ &= -\frac{2}{\pi}(\alpha_n+\beta_n),\ \text{say}. \end{aligned}$$

By (2.25), $\alpha_n = 0$, for all $n \ge 1$ therefore by applying Lemma 4.1 for $h = 0$ and $\lambda_n = n^2$, it will be sufficient for the proof of (2.24) to prove that

$$\sum_{n=1}^\infty \beta_n y_n^{-1} \in |N', p_n, n^2|, \tag{5.63}$$

whenever (2.26) holds. However, by (1.1) and (1.2), (5.63) holds if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n^2 P_n} \left| \sum_{m=1}^{n} m^2 p_{n-m} y_m^{-1} \int_0^{\pi} H_m(t) dg(t) \right| < \infty. \tag{5.64}$$

Now, in view of (2.23), it is sufficient for the proof of (5.64) to show that

$$\sum = \sum_{n=1}^{\infty} \frac{1}{n^2 P_n} \left| \sum_{m=1}^{n} m^2 p_{n-m} y_m^{-1} H_m(t) \right| = O\left(log \frac{k}{t} \right), \tag{5.65}$$

uniformly in $0 < t < \pi$. We write

$$\sum = \sum_{n=1}^{T} + \sum_{T+1}^{\infty} = \sum_{1} + \sum_{2}, \text{ say.} \tag{5.66}$$

By using (3.36) and $|sinmt| \leq mt$, we get

$$\begin{aligned} \sum_{1} &= O(1) b(t) \sum_{n=1}^{T} \frac{1}{n^2 P_n} \sum_{m=1}^{n} m^2 p_{n-m} y_m^{-1} + O(1) \sum_{n=1}^{T} \frac{1}{n^2 P_n} \sum_{m=1}^{n} m p_{n-m} \\ &= O\left\{ log \left(\frac{k}{t} \right) \right\}, \end{aligned} \tag{5.67}$$

uniformly in $0 < t < \pi$, by (2.20). And, by(3.37), (3.38) and Lemma 4.2,

$$\begin{aligned} \sum_{2} &= y(t^{-1}) \sum_{n=T+1}^{\infty} \frac{1}{n^2 P_n} \left| \sum_{m=1}^{n} m p_{n-m} y_m^{-1} sin\ mt \right| + O(1) t^{-1} y(t^{-1}) \sum_{n=T+1}^{\infty} \frac{1}{n^2 P_n} \sum_{m=1}^{n} p_{n-m} y_m^{-1} \\ &= O(1) y_T P_T \sum_{n=T}^{\infty} \frac{1}{n P_n y_n} + O(T y_T) \sum_{n=T}^{\infty} n^{-2} y_n^{-1} \\ &= O\left\{ log \left(\frac{k}{t} \right) \right\} \end{aligned} \tag{5.68}$$

uniformly in $0 < t < \pi$, by (2.26) and (2.20).
Collecting (5.65) through (5.68), we get (5.64) and this completes the proof.

PROOF OF THEOREM 2.2. In Theorem 1, suppose for $0 < \alpha < 1$

$$(i)\ y(t^{-1}) = t^{\alpha} \text{ and } (ii)\ p_n = \frac{n^{\alpha-1}}{\Gamma(\alpha)} (logn)^{\beta} (\beta > 1). \tag{5.69}$$

Then the conditions (2.20), (2.21) and (2.22) are satisfied and $y_n = n^{-\alpha}, y_T = T^{-\alpha}$.
Also for large n,

$$(p_n) \in \mathcal{M} \text{ and } A_n^{\alpha-1,\beta} \sim p_n, \tag{5.70}$$

so that we can have

$$|Z, \alpha, \beta| \sim |N, p_n|. \tag{5.71}$$

Therefore, for the proof of (2.27), we only require to verify (2.26), where $y(t^{-1})$ and p_n are defined by (5.69). We observe that (see (1.9))

$$P_T = O\{T^{\alpha} (logT)^{\beta}\}$$

and since (p_n) is decreasing,

$$\frac{1}{P_n} < \frac{\Gamma(\alpha)}{n^{\alpha}(logn)^{\beta}}. \tag{5.72}$$

Now using the estimates obtained for $y_n, y_T, P_T\ and P_n^{-1}$ in (2.26), we get the required proof of (2.27).

(i) The condition (2.25) , i.e. $g(0+) = 0$ is necessary. For $g(t) = t^{-\alpha}R(t)$,
we have from Theorem 2.1,

$$\alpha^n = \frac{2}{\pi}(1+\alpha)g(0+)\int_0^{\pi} \frac{sin\ nt}{nt} t^{\alpha}\ dt \tag{5.73}$$

and by Lemma 4,

$$\int_0^{\pi} \frac{sin\ nt}{nt} t^{\alpha}\ dt = B_n^{(1)} + B_n^{(2)} + O(n^{-2}) \tag{5.74}$$

where $B_n^{(2)} \geq$ and $B_n^{(1)} > 0$ is such that for positive constant $d = d(\alpha)$,

$$B_n^{(1)} \sim dn^{-1-\alpha}. \tag{5.75}$$

Suppose $g(0+) \neq 0$, then for

$$\sum_{n=1}^{\infty} n^{\alpha}\alpha_n \in |Z, \alpha, \beta|\ (0 < \alpha < 1, \beta > 1) \tag{5.76}$$

it is necessary that

$$\sum_{n=1}^{\infty} n^{\alpha} B_n^{(1)} \in |Z, \alpha, \beta|\ (0 < \alpha < 1, \beta > 1) \tag{5.77}$$

which, by (5.51) , does not hold. Thus for the truth of (5.52), it is necessary that (2.25) should hold.

(ii) The proof that (2.23) cannot be replaced by (2.28) follows from Lemma 4.5.

(iii) For the proof that multipliers (n^{α}) in (2.27) cannot be replaced by the sequence of multipliers given in (2.29), we consider an even function $f \in L_{2\pi}$ and the point $x = 0$. We further define :

$$f(t) = t^{\alpha} log^{-1-\epsilon}\frac{k}{t}\ (0 \leq t \leq \pi),$$

where $\epsilon > 0$, and proceed as in Lemma 4.5 to prove that

$$\sum_{n=1}^{\infty} n^{\alpha} log^{\epsilon}(n+1) A_n(0) = -\infty,$$

which gives the required conclusion.

(iv) We now show that Theorem 2.2 is incomparable with Theorem A for $0 < \alpha < 1$.

(a) It follows from Lemma 4.3 that the summability method $|C, \eta|$ $(\eta > \alpha)$ used in

Theorem A is stronger than $|Z, \alpha, \beta|$.
(b) We observe that for Fourier series the condition (2.18) is equivalent to

$$(i) \int_0^\pi t^{-\alpha}|d\phi(t)| < \infty \text{ and } (ii)\ \phi(0+) = 0. \tag{5.78}$$

However ([3];Lemma 7), (5.78) and (2.23) with (2.25) are equivalent to

$$(i)\ t^{-\alpha}\phi(t) \in BV[0,\pi] \text{ and } (ii)\ t^{-\alpha-1}\phi(t) \in L[0,\pi]. \tag{5.79}$$

and

$$(i)\ t^{-\alpha}R(t)log\frac{k}{t} \in BV[0,\pi] \text{ and } (ii)\ t^{-\alpha-1}R(t) \in L[0,\pi], \tag{5.80}$$

respectively. Now , it is clear from Lemma 4.6 that (5.80), i.e. (2.23) with (2.25) is stronger than (2.18). In view of (a) and (b) , the result (2.27) of Theorem 2.2 is incomparable with (2.19) of Theorem A for $0 < \alpha < 1$.
This completes the proof of Theorem 2.2.

PROOF OF THEOREM 2.3. Letting $y(t^{-1}) = t^\alpha log\frac{k}{t}$ and $p_n = A_n^{\alpha-1,\beta}$ for $0 < \alpha < 1$ and $\beta > 0$ in Theorem 2.1 and proceeding as in the proof of Theorem 2.2, we can get a proof of the theorem.

PROOF OF THEOREM 2.4.

(i) In Theorem 2.1, let $p_n = q_n^\beta = \frac{\beta(log(n+1))^\beta}{n+1}$ $(\beta > 0)$ and $y(t^{-1}) = log^\delta\frac{k}{t}$ for real values of δ. Then we observe that conditions (2.20) through (2.22) and (2.26) are satisfied. Hence sufficiency of (2.25) follows from Theorem 2.1.

(ii) For the necessity of (2.25), see Chandra([7], pp. 218-219).

(iii) For the proof that (2.23) cannot be replaced by (2.28), see Lemma 7 of Chandra [7].

(iv) It follows from Lemma 8 of Chandra [8] that there exists $f \in L_{2\pi}$ for which (2.23) and (2.25) hold for all real values of δ but the series

$$\sum_{n=1}^\infty \frac{A_n(x)}{log^{\delta-\epsilon}(n+1)} \quad (0 < \epsilon < 1) \tag{5.81}$$

at $x = 0$ diverges properly to $-\infty$. Hence the series in (5.19) is not summable by any absolutely conservative method.
This completes the proof of Theorem 2.4.

6. COROLLARIES

In this section, we consider some corollaries of Theorem 2.4 and compare with some existing results. For this, we first observe that for $g(t) = R(t)log^{-\delta}\frac{k}{t}$, where δ is real, (2.23) with (2.25) is equivalent([28]) to

$$(i)\ R(t)log^{1-\delta}\frac{k}{t} \in BV[0,\pi] \text{ and } (ii)\ t^{-1}log^{-\delta}\frac{k}{t}R(t) \in L[0,\pi], \tag{6.82}$$

for all real values of δ. For $\delta = 0$, we get the following result from Theorem 2.4:

Corollary 1. Let $0 < \sigma \leq 1$.Then

$$(i) \int_0^{\pi} \left(log\frac{k}{t}\right)^{\sigma} |dR(t)| < \infty \quad (ii)\ R(0+) = 0 \tag{6.83}$$

is sufficient for

$$\sum_{n=1}^{\infty} A_n(x) \in |N, q_n^{\beta}|\ (\beta > 1) \tag{6.84}$$

if and only if $\sigma = 1$. Further, $R(0+) = 0$ is necessary for (6.84). Also

$$\sum_{n=1}^{\infty} A_n(x) log^{\epsilon}(n+1) \notin |N, q_n^{\beta}|\ (\beta > 1),\ \text{for } \epsilon > 0$$

REMARK 1. In Theorem A, it has been shown that

$$\sum_{n=1}^{\infty} A_n(x) \in |C, \eta|\ (\eta > 0) \tag{6.85}$$

and it follows from Lemma 4.3 that the inclusion :

$$|N, q_n^{\beta}| \in |C, \eta|\ (\eta > 0, \beta > 0) \tag{6.86}$$

is strict. Hence the summability method $|N, q_n^{\beta}|$ $(\beta > 0)$ used in Corollary 1 is weaker than the one used in Theorem A for $\alpha = 0$. However, (6.85) was obtained in Theorem A under the condition : $\phi(t) \in [0, \pi]$, which is equivalent([6]; Corollary 2) to

$$(i)\ R(t) \in BV[0, \pi]\ (ii)\ t^{-1}R(t) \in L[0, 2\pi]. \tag{6.87}$$

Hence, it follows from (6.82) with $\delta = 0$ that (6.83) with $\sigma = 1$ is stronger than (6.87). Therefore Corollary 1 is incomparable with Theorem A for $\alpha = 0$. Thus Corollary 1 yields a new result.
Among the other results, Kanno [23] obtained the following results:

Theorem B. If, for $\delta > 0$,

$$\int_0^{\pi} log^{\delta}\frac{k}{t}|d\phi(t)| < \infty. \tag{6.88}$$

Then

$$\sum_{n=1}^{\infty} A_n(x) log^{\delta}(n+1) \in |C, \eta|\ (0 < \eta < 1). \tag{6.89}$$

The case $\delta = 1$ is due to Mohanty [29].

Theorem C. Let $0 < \beta \leq 1$ and $0 < \delta \leq 1$ and let

$$\int_0^{\pi} log^{1-\delta}\frac{k}{t}|d\phi(t)| < \infty. \tag{6.90}$$

Then

$$\sum_{n=1}^{\infty} A_n(x)log^{-\delta}(n+1) \in |N, q_n^{\beta}| \ (\beta+\delta>1). \tag{6.91}$$

The case $\delta = 1 = \beta$ is due to Varshney [32] and the case : $\beta = 1, 0 < \delta \leq 1$ of Theorem C is due to Lal [25].
In context of Theorem B, we deduce the following result from Theorem 2.4 which provides necessary and sufficient conditions:

Corollary 2. Let $0 < \sigma \leq 1, \delta > 0$ and $\beta > 0$. Then, for $g(t) = R(t)log^{\delta}\frac{k}{t}$,

$$(i) \int_0^{\pi} \left(log\frac{k}{t}\right)^{\sigma} |dg(t)| < \infty. \ (ii) g(0+) = 0 \tag{6.92}$$

is sufficient for

$$\sum_{n=1}^{\infty} A_n(x)log^{\delta}(n+1) \in |N, q_n^{\beta}| \ (\beta+\delta>1). \tag{6.93}$$

if and only if $\sigma = 1$. Also $g(0+) = 0$ is necessary for (6.93) and if $\epsilon > 0$

$$\sum_{n=1}^{\infty} A_n(x)log^{\delta+\epsilon}(n+1) \notin |N, q_n^{\beta}| \ (\beta+\delta>1). \tag{6.94}$$

REMARK 2. It is clear from (6.86) that the summability method $|N, q_n^{\beta}|$ $(\beta > 0)$, used in Corollary 2, is weaker than the one used in Theorem B.
And, on the otherhand, condition (6.92) with $\delta = 1$ is stronger than (6.88).
Hence, Corollary 2 is incomparable with Theorem B.

REMARK 3.In connection with Theorem C, we make the following observations:
(i) The condition (6.90) for $\delta = 1$ reduces to $\phi(t) \in [0, \pi]$ which holds if and only if (6.87) holds and is stronger than (6.82) with $\delta = 1$. Hence, Theorem 2.4 for $\delta = 1$ improves Theorem C.
(ii) For the case $0 < \delta < 1$ of Theorem C, we may assume that $\phi(0+) = 0$ in (6.90) without loss of generality. Then (6.90) with $\phi(0+) = 0$ holds if and only if ([28])

$$(i) \ \phi(t)log^{1-\delta}\frac{k}{t} \in BV[0, \pi] \text{ and } t^{-1}log^{-\delta}\frac{k}{t}\phi(t) \in L[0, \pi] \tag{6.95}$$

However, it is known ([9]) that (6.95)(i) is stronger than (2.23) with (2.25), where $g(t) = R(t)log^{-\delta}\frac{k}{t}$. Hence, once again, Theorem 2.4 not only yields a better result than Theorem C but it also provides best possible conditions, imposed upon the generating functions of Fourier series, and multipliers for the summability method.

Finally, we consider the following result due to Chandra and Mohapatra [12]:

Theorem D. Let $\delta >, 0$ and let

$$\phi(t)log^{-\delta}\frac{k}{t} \in BV[0, \pi]. \tag{6.96}$$

Then

$$\sum_{n=1}^{\infty} \frac{A_n(x)}{(log(n+1))^{1+\delta}} \in |C, 0|. \tag{6.97}$$

This is an improved version of a theorem due to Chang [13].
We obtain the following result from Theorems 2.1 and 2.4, which improves Theorem D:

Corollary 3. Let $\delta > 0$ and let, for $g(t) = R(t)log^{-1-\delta}\frac{k}{t}$,

$$\int_0^{\pi} log\frac{k}{t}|dg(t)| < \infty. \tag{6.98}$$

Then in order that (6.97) should hold, it is necessary and sufficient that

$$g(0+) = 0. \tag{6.99}$$

Also (6.98) cannot be replaced by

$$\int_0^{\pi} \left(log\frac{k}{t}\right)^{\sigma} |dg(t)| < \infty \ (0 < \sigma < 1). \tag{6.100}$$

Further, in (6.97), $(1+\delta)$ cannot be replaced by $1+\delta-\epsilon$ for $\epsilon > 0$.
Proof. For the proof of its sufficiency part we use Theorem 2.1 for

$$p_n = r^n \ (0 < r < 1), n \geq 0 \text{ and } y(t^{-1}) = log^{1+\delta}\frac{k}{t} \ (\delta > 0). \tag{6.101}$$

Then $P_n = O(1)$ and the conditions (2.20) through (2.22) and (2.26) are satisfied. Hence sufficiency of (6.99) for (6.97) follows from Theorem 2.1 by appealing to an ineffectiveness result on absolute Nörlund summability of Dikshit [20]. For the remaining part of the proof, one may proceed as in Theorem 2.4.
Now, for the proof that (6.98) with (6.99) is weaker than (6.96) for $\delta > 0$, see Chandra([9],[10]).

References

[1] L. S. Bosanquet, Note on the absolute summability(C) of a Fourier series, J. London Math. Soc.,11(1936), 11-15.

[2] L. S. Bosanquet and G. Das , Absolute Summability Factors for Nörlund Means , Proc. London Math. Soc.(3), 38 (1979), 1-52.

[3] P. Chandra , On the absolute Riesz summability of Fourier series, its factored conjugate series and their derived series,Rend. Mat(2), Serie VI, 3(1970), 291-311.

[4] P. Chandra , Necessary and sufficient Tauberian conditions for absolute convergence of Fourier series and conjugate series , Period. Math. Hung. , 5(2)(1974), 121-130 .

[5] P. Chandra , On the $|E, q|$-summability of a Fourier series and its conjugate series , Riv. Mat. Univ. Parma(4), 3(1977), 65-78.

[6] P. Chandra , On a class of functions of bounded variation , Jñānābha, 8(1978), 17-24.

[7] P. Chandra , Multipliers for the absolute Euler summability of Fourier series, Proc. Indian Acad. Sci.(Math. Sci.) ,11 (No.2) (2001), 203-219 .

[8] P. Chandra , Multipliers for Absolutely summable Fourier Expansion, Jour. Orissa Math. Soc., 27(Pati's memorial volume) (2008), 127-146.

[9] P. Chandra , Factors for the absolute harmonuic summability of a Fourier series , Jour. Orissa Math. Soc., 31(1)(2012), 1-22.

[10] P. Chandra , Absolute generalized harmonic summability of a factored Fourier series, Indian Jour. Math.,(Special Vol. Dedicated to Prof. B.E.Rhoades) 56(1)(2014), 33-48.
[11] P. Chandra and S. Barve , Application of absolute matrix summability to Fourier series and allied series, Jour. Univ. Kuwait(Science) 7(1980), 11-24.
[12] P. Chandra and R.N. Mohapatra , Absolute Nörlund summability factors of Fourier series , Tamkang J. Math. , 10(2)(1979), 245-252 .
[13] M.T. Cheng , Summability factors of a Fourier series at a point, Duke Math. J., 15(1948), 29-36.
[14] G. DAS , Tauberians theorems for absolute Nörlund summability factors for Fourier series(3) , Proc. London Math. Soc. , 19 (1969), 357-384 .
[15] G. Das , Modified Means, Indian Jour. Pure and applied Math.,20(3)(1989), 347-385.
[16] G. DAS and P.C. Mohapatra , Necessary and sufficient conditions for absolute Nörlund summability of Fourier series , Proc. London Math. Soc.(3) , 41 (1980), 217-253 .
[17] G. Das and P.C. Mohapatra , On a generalized harmonic-Cesàro method of summation and its application to Fourier series, Indian Jour. Math. , 22(1980), 33-47.
[18] G. Das and P.C. Mohapatra , Generalized harmonic-Cesàro method of summation and its modified means, Indian Jour. Math., 24(1982), 145-163.
[19] Das G and Mohapatra P C, Inclusion theorems for absolute harmonic-Cesàro method of summation ,Indian Jour. Math., 33(1991), 155-164.
[20] G.D. Dikshit , A note on Riesz and Nörlund means, Rend. Circolo Mat. Palermo, 18(1969),49-61.
[21] G.D. Dikshit , Absolute Nörlund summability factors for Fourier series, Pacific Jour. Math., 63(2)(1976), 371-379.
[22] K.S.K. Iyengar , A tauberian theorem and its application to convergence of Fourier series, Proc. Indian Acad. Sci. Sect. A ,18(1943), 81-87.
[23] K. Kanno , On the absolute Nörlund summability of the Factored Fourier series, Tôhoku Math. J. 21(1969), 434-447.
[24] B. Kuttner and B.N. Sahney , On the absolute matrix summability of Fourier series, Pacific J. Math. ,43(1972),407-419.
[25] S.N. Lal , On the absolute harmonic summability factors of a Fourier series , Publ. Inst. Math., 9(23)(1969), 13 – 28 .
[26] L. McFadden , Absolute Nörlund summability ,Duke Math. J. , 9 (1942), 188-207 .
[27] R. Mohanty , The absolute Cesàro summability of some series associated with a Fourier series and its allied series, J. London Math.Soc.,25(1950),63-67.
[28] R. Mohanty , On the absolute Riesz summability of Fourier series andallied series, Proc. London Math. Soc., 52(1951), 295-320 .
[29] R. Mohanty , The absolute Cesàro summability of a series associated with a Fourier series, Bull. Calcutta Math. Soc. 44(1952),152-154.
[30] T. Pati , On the absolute Nörlund summability of a Fourier series, J. London Math. Soc., 34(1959), 153-160.
[31] T. Pati , Addendum to “On the absolute Nörlund summability of a Fourier series, J. London Math. Soc., 37(1962), p. 256.
[32] O.P. Varshney , On the absolute harmonic summability of a series realated to a Fourier series , Proc . Amer .Math.Soc. 10(1959),784-789.
[33] A. Zygmund , Trigonometric series, Vol. I (1968), Cambridge University Press.

(Prem Chandra) C-315, Vivekanand Nagar, Ujjain - 456010, India
E-mail address: premchandraujnn@yahoo.in