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Derivatives with respect to horizontal and vertical lifts of the Cheeger-Gromoll metric ${}^{CG}g$ on Cotangent Bundle

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Abstract. In this paper, we define the Cheeger-Gromoll metric in the cotangent bundle T^*M^n, which is completely determined by its action on vector fields of type X^H and ω^V. Later, we obtain the covarient and Lie derivatives applied to the Cheeger-Gromoll metric with respect to the horizontal and vertical lifts of vector and kovector fields, respectively.

1. Introduction

Cheeger-Gromoll metric was defined by Cheeger and Gromoll in [3] and the explicit formula for this metric was given by Musso and Tricerri in [15]. The Levi-Civita connection of ${}^{CG}g$ and its Riemannian curvature tensor are calculated by Sekizawa in [20] and corrected by Gudmundsson and Kappos in [11]. In [19] Salimov and Kazimova studied geodesics of the Cheeger-Gromoll metric on the tangent bundle. The similar metric in theoritical physics has been obtained by Tamm (Nobel Laureate in Physics for the year 1958, see [21]). The geometry of the tangent bundle equipped with Cheeger-Gromoll metric is well known and intensively studied (see for example [9, 10, 11, 14, 18, 19]). In [1] Ağca and Salimov investigate curvature properties and geodesics on the cotangent bundle with respect to the Cheeger-Gromoll metric.

The tangent bundles of differentiable manifolds are very important in many areas of mathematics and physics. Cotangent bundle is dual of the tangent bundle. Because of this duality, some of the geometric results are similar to each other. The most significant difference between them is construction of lifts (see [22] for more details). In this paper, we define the Cheeger-Gromoll metric in the cotangent bundle T^*M^n, which is completely determined by its action on vector fields of type X^H and ω^V. Later, we obtain the covarient and Lie derivatives applied to the Cheeger-Gromoll metric with respect to the horizontal and vertical lifts of vector and kovector fields, respectively.

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Let (M^n, g) be n−dimensional Riemannian manifold T^*M^n be the cotangent bundle of M^n and π the natural projection $T^*M^n \to M^n$. A system of local coordinates $(U, x^i), i = 1, ..., n$ in M^n induces on T^*M^n a system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)$, $\bar{i} := n + i = n + 1, ..., 2n$, where $x^{\bar{i}} = p_i$ are the components of the covector p in each cotangent space $T_x^*M^n$, $x \in U$ with respect to the natural coframe $\{dx^i\}$, $i = 1, ..., n$.

We denote $\Im_s^r(M^n)$ the set of all tensor fields of type (r, s) on M^n and by $\Im_s^r(T^*M^n)$ the corresponding set on the cotangent bundle T^*M^n. During this paper, manifolds, tensor fields and connections are always supposed to be differentiable of class C^∞.

The local expressions a vector and a covector (1-form) field $X \in \Im_0^1(M^n)$ and $\omega \in \Im_1^0(M^n)$ are $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ in $U \subset M^n$, respectively. Then the complete and horizontal lifts $X^C, X^H \in \Im_0^1(T^*M^n)$ of $X \in \Im_0^1(M^n)$ and the vertical lift $\omega^V \in \Im_0^1(T^*M^n)$ of $\omega \in \Im_1^0(M^n)$ are given, with respect to the natural frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\}$, by

$$X^C = X^i \frac{\partial}{\partial x^i} - \sum_i p_h \partial_i X^h \frac{\partial}{\partial x^{\bar{i}}}, \tag{1.1}$$

$$X^H = X^i \frac{\partial}{\partial x^i} + \sum_i p_h \Gamma_{ij}^h X^j \frac{\partial}{\partial x^{\bar{i}}}, \tag{1.2}$$

$$\omega^V = \sum_i \omega_i \frac{\partial}{\partial x^{\bar{i}}} \tag{1.3}$$

where Γ_{ij}^h are the components of the Levi-Civita connection ∇_g on M^n [2, 16](see [22] for more details).

Definition 1.1. A Cheeger-Gromoll metric ${}^{CG}g$ is defined on T^*M^n by the following three equations [1, 16]

$${}^{CG}g(X^H, Y^H) = (g(X, Y))^V = g(X, Y) \circ \pi, \tag{1.4}$$

$${}^{CG}g(\omega^V, Y^H) = 0, \tag{1.5}$$

$${}^{CG}g(\omega^V, \theta^V) = \frac{1}{1 + r^2}(g^{-1}(\omega, \theta) + g^{-1}(\omega, p)g^{-1}(\theta, p)) \tag{1.6}$$

for any $X, Y \in \Im_0^1(M^n)$ and $\omega, \theta \in \Im_1^0(M^n)$, where $r^2 = g^{-1}(p, p) = g^{ij} p_i p_j$.

Since any tensor field of type $(0, 2)$ on T^*M^n is completely determined by its action on vector fields of type X^H and ω^V, it follows that ${}^{CG}g$ is completely determined by its equations (1.4), (1.5) and (1.6).

We now see, from (1.1) and (1.2), that the complete lift X^C of $X \in \Im_0^1(M^n)$ is expressed by

$$X^C = X^H - (p(\nabla X))^V, \tag{1.7}$$

where $p(\nabla X) = p_i(\nabla_h X^i)dx^h$.

Using (1.4), (1.5), (1.6) and (1.7), we have

$$\begin{aligned}{}^{CG}g(X^C,Y^C) &= (g(X,Y))^V\\ &+\frac{1}{1+r^2}(g^{-1}(p(\nabla X),p(\nabla Y))+g^{-1}(p(\nabla X),p).g^{-1}(p(\nabla Y),p),\end{aligned} \tag{1.8}$$

where $g^{-1}(p(\nabla X),p(\nabla Y))=g^{ij}(p_\iota\nabla_i X^\iota)(p_k\nabla_j Y^k)$, $g^{-1}(p(\nabla X),p)=g^{ij}p_i(p(\nabla X))_j$.

Since the tensor field ${}^{CG}g\in \Im_2^0(T^*M^n)$ is completely determined also by its action on vector fields type X^C and Y^C (see [22], p.237), we have an alternative characterization of ${}^{CG}g$ on T^*M^n : ${}^{CG}g$ is completely determined by the condition (1.8). Similarly, we get the following results

$$\begin{aligned}{}^{CG}g(X^C,\omega^V) &= {}^{CG}g(X^H-(p(\nabla X))^V,\omega^V)\\ &= -{}^{CG}g(p(\nabla X))^V,\omega^V)\\ &= -\frac{1}{1+r^2}(g^{-1}(p(\nabla X),\omega)+g^{-1}(p(\nabla X),p)g^{-1}(\omega,p)\end{aligned}$$

$$\begin{aligned}{}^{CG}g(\omega^V,X^C) &= {}^{CG}g(\omega^V,X^H-(p(\nabla X))^V)\\ &= {}^{CG}g(\omega^V,X^H)-{}^{CG}g(\omega^V,(p(\nabla X))^V)\\ &= -\frac{1}{1+r^2}(g^{-1}(\omega,p(\nabla X))+g^{-1}(\omega,p)g^{-1}(p(\nabla X),p))\end{aligned}$$

$$ {}^{CG}g(\omega^V,\theta^V) = \frac{1}{1+r^2}(g^{-1}(\omega,\theta)+g^{-1}(\omega,p)g^{-1}(\theta,p))$$

Theorem 1.2. *Let M^n be a Riemannian manifold with metric g, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle T^*M^n of M^n satisfies the following* [1]

i) $[\omega^V,\theta^V]=0$,
ii) $[X^H,\omega^V]=(\nabla_X\omega)^V$,
iii) $[X^H,Y^H]=[X,Y]^H+\gamma R(X,Y)=[X,Y]^H+(pR(X,Y))^V$
all $X,Y\in\Im_0^1(M^n)$ and $\omega,\theta\in\Im_1^0(M^n)$ (See [22], p.238, p.277 for more details).

2. Main Results

Definition 2.1. Let M^n be an n–dimensional diferentiable manifold. Differantial transformation of algebra $T(M^n)$, defined by

$$D=\nabla_X : T(M^n)\to T(M^n),\ X\in\Im_0^1(M^n)$$

is called as covariant derivation with respect to vector field X if

$$\begin{aligned}\nabla_{fX+gY}t &= f\nabla_X t+g\nabla_Y t,\\ \nabla_X f &= Xf,\end{aligned} \tag{2.9}$$

where $\forall f,g\in\Im_0^0(M^n)$, $\forall X,Y\in\Im_0^1(M^n)$, $\forall t\in\Im(M^n)$ (see [13], p.123).

On the other hand, a transformation defined by

$$\nabla:\Im_0^1(M^n)\times\Im_0^1(M^n)\to\Im_0^1(M^n)$$

is called as an affin connection (see for details [13, 17]).

Proposition 2.2. Covarient differentiation with respect to the horizontal lift ∇^H of a symetric affine connection ∇ in M^n to $T^*(M^n)$ satisfies

$$\begin{aligned} \nabla^H_{X^H} Y^H &= (\nabla_X Y)^H,\ \nabla^H_{\theta^V}\omega^V = 0, \\ \nabla^H_{X^H}\omega^V &= (\nabla_X \omega)^V,\ \nabla^H_{\theta^V} Y^H = 0, \end{aligned} \tag{2.10}$$

for any $X, Y \in \Im^1_0(M^n)$, $\theta, \omega \in \Im^0_1(M^n)$ [22].

Theorem 2.3. *Let ${}^{CG}g$ be Cheeger-Gromoll metric, is defined by (1.4), (1.5), (1.6) and the horizontal lift ∇^H of a symetric affine connection ∇ in M^n to $T^*(M^n)$. From proposition (2.2), we get the following results*

$$\begin{aligned}
i)\ (\nabla^H_{\omega^V}{}^{CG}g)\big(\theta^V,\xi^V\big) &= 0, \\
ii)\ (\nabla^H_{X^H}{}^{CG}g)(\theta^V,\xi^V) &= (\nabla_X(\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p))))^V \\
&\quad -(\frac{1}{1+r^2}(g^{-1}(\nabla_X\theta,\xi)+g^{-1}((\nabla_X\theta),p)g^{-1}(\xi,p)))^V \\
&\quad -(\frac{1}{1+r^2}(g^{-1}(\theta,(\nabla_X\xi))+g^{-1}(\theta,p)g^{-1}((\nabla_X\xi),p)))^V, \\
iii)\ (\nabla^H_{\omega^V}{}^{CG}g)(\theta^V,Z^H) &= 0, \\
iv)\ (\nabla^H_{X^H}{}^{CG}g)(\theta^V,Z^H) &= 0, \\
v)\ (\nabla^H_{\omega^V}{}^{CG}g)(Y^H,\xi^V) &= 0, \\
vi)\ (\nabla^H_{X^H}{}^{CG}g)(Y^H,\xi^V) &= 0, \\
vii)\ (\nabla^H_{\omega^V}{}^{CG}g)(Y^H,Z^H) &= 0, \\
viii)\ (\nabla^H_{X^H}{}^{CG}g)(Y^H,Z^H) &= (\nabla_X g(Y,Z))^V,
\end{aligned}$$

where the complete and horizontal lifts $X^C, X^H \in \Im^1_0(T^*M^n)$ of $X \in \Im^1_0(M^n)$ and the vertical lift $\omega^V \in \Im^1_0(T^*M^n)$ of $\omega \in \Im^0_1(M^n)$ defined by (1.1),(1.2),(1.3), respectively.

Proof. $i)$

$$\begin{aligned}
(\nabla^H_{\omega^V}{}^{CG}g)(\theta^V,\xi^V) &= \nabla^H_{\omega^V}{}^{CG}g(\theta^V,\xi^V) - {}^{CG}g(\nabla^H_{\omega^V}\theta^V,\xi^V) - {}^{CG}g(\theta^V,\nabla^H_{\omega^V}\xi^V) \\
&= \nabla^H_{\omega^V}\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p)) \\
&= \nabla^H_{\omega^V}(\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p)))^V \\
&= \omega^V(\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p)))^V \\
&= 0
\end{aligned}$$

$ii)$

$$\begin{aligned}
(\nabla^H_{X^H}\ {}^{CG}g)(\theta^V,\xi^V) &= \nabla^H_{X^H}\ {}^{CG}g(\theta^V,\xi^V) - {}^{CG}g(\nabla^H_{X^H}\theta^V,\xi^V) - {}^{CG}g(\theta^V,\nabla^H_{X^H}\xi^V)\\
&= \nabla^H_{X^H}(\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p)))^V\\
&\quad -{}^{CG}g((\nabla_X\theta)^V,\xi^V) - {}^{CG}g(\theta^V,(\nabla_X\xi)^V)\\
&= (\nabla_X(\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p))))^V\\
&\quad -(\frac{1}{1+r^2}(g^{-1}(\nabla_X\theta,\xi)+g^{-1}((\nabla_X\theta),p)g^{-1}(\xi,p)))^V\\
&\quad -(\frac{1}{1+r^2}(g^{-1}(\theta,(\nabla_X\xi))+g^{-1}(\theta,p)g^{-1}((\nabla_X\xi),p)))^V
\end{aligned}$$

$iii)$

$$\begin{aligned}
(\nabla^H_{\omega^V}\ {}^{CG}g)(\theta^V,Z^H) &= \nabla^H_{\omega^V}\ {}^{CG}g(\theta^V,Z^H) - {}^{CG}g(\nabla^H_{\omega^V}\theta^V,Z^H) - {}^{CG}g(\theta^V,\nabla^H_{\omega^V}Z^H)\\
&= 0
\end{aligned}$$

$iv)$

$$\begin{aligned}
(\nabla^H_{X^H}\ {}^{CG}g)(\theta^V,Z^H) &= \nabla^H_{X^H}\ {}^{CG}g(\theta^V,Z^H) - {}^{CG}g(\nabla^H_{X^H}\theta^V,Z^H) - {}^{CG}g(\theta^V,\nabla^H_{X^H}Z^H)\\
&= -{}^{CG}g((\nabla_X\theta)^V,Z^H) - {}^{CG}g(\theta^V,(\nabla_X Z)^H)\\
&= 0
\end{aligned}$$

$v)$

$$\begin{aligned}
(\nabla^H_{\omega^V}\ {}^{CG}g)(Y^H,\xi^V) &= \nabla^H_{\omega^V}\ {}^{CG}g(Y^H,\xi^V) - {}^{CG}g(\nabla^H_{\omega^V}Y^H,\xi^V) - {}^{CG}g(Y^H,\nabla^H_{\omega^V}\xi^V)\\
&= 0
\end{aligned}$$

$vi)$

$$\begin{aligned}
(\nabla^H_{X^H}\ {}^{CG}g)(Y^H,\xi^V) &= \nabla^H_{X^H}\ {}^{CG}g(Y^H,\xi^V) - {}^{CG}g(\nabla^H_{X^H}Y^H,\xi^V) - {}^{CG}g(Y^H,\nabla^H_{X^H}\xi^V)\\
&= -{}^{CG}g((\nabla_X Y)^H,\xi^V) - {}^{CG}g(Y^H,(\nabla_X\xi)^V)\\
&= 0
\end{aligned}$$

$vii)$

$$\begin{aligned}
(\nabla^H_{\omega^V}\ {}^{CG}g)(Y^H,Z^H) &= \nabla^H_{\omega^V}\ {}^{CG}g(Y^H,Z^H) - {}^{CG}g(\nabla^H_{\omega^V}Y^H,Z^H) - {}^{CG}g(Y^H,\nabla^H_{\omega^V}Z^H)\\
&= \nabla^H_{\omega^V}(g(Y,Z))^V\\
&= \omega^V(g(Y,Z))^V\\
&= 0
\end{aligned}$$

$viii)$

$$\begin{aligned}
(\nabla^H_{X^H}\ {}^{CG}g)(Y^H,Z^H) &= \nabla^H_{X^H}\ {}^{CG}g(Y^H,Z^H) - {}^{CG}g(\nabla^H_{X^H}Y^H,Z^H) - {}^{CG}g(Y^H,\nabla^H_{X^H}Z^H)\\
&= \nabla^H_{X^H}(g(Y,Z))^V - {}^{CG}g((\nabla_X Y)^H,Z^H) - {}^{CG}g(Y^H,(\nabla_X Z)^H)\\
&= X^H(g(Y,Z))^V - g((\nabla_X Y),Z)^V - g(Y,(\nabla_X Z))^V\\
&= (\nabla_X g(Y,Z))^V - g((\nabla_X Y),Z)^V - g(Y,(\nabla_X Z))^V\\
&= (\nabla_X g(Y,Z))^V
\end{aligned}$$

□

Definition 2.4. Let M^n be an n–dimensional diferentiable manifold. Differential transformation $D = L_X$ is called as Lie derivation with respect to vector field $X \in \Im^1_0(M^n)$ if

$$\begin{aligned} L_X f &= Xf, \forall f \in \Im^0_0(M^n), \\ L_X Y &= [X,Y], \forall X,Y \in \Im^1_0(M^n). \end{aligned} \tag{2.11}$$

$[X,Y]$ is called by Lie bracked. The Lie derivative $L_X F$ of a tensor field F of type $(1,1)$ with respect to a vector field X is defined by [4, 5, 6, 7, 8, 12, 22]

$$(L_X F)Y = [X, FY] - F[X,Y]. \tag{2.12}$$

Theorem 2.5. *Let ${}^{CG}g$ be Cheeger-Gromoll metric, is defined by (1.4), (1.5), (1.6) and L_X the operator Lie derivation with respect to X. From Theorem (1.2) and Definition (2.4), we get the following results*

$$\begin{aligned}
i)\ (L_{\omega^V}\ {}^{CG}g)(\theta^V,\xi^V) &= 0, \\
ii)\ (L_{X^H}\ {}^{CG}g)(\theta^V,\xi^V) &= (\nabla_X(\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p)))^V \\
&\quad -(\frac{1}{1+r^2}(g^{-1}((\nabla_X\theta),\xi)+g^{-1}((\nabla_X\theta),p)g^{-1}(\xi,p)))^V \\
&\quad -(\frac{1}{1+r^2}(g^{-1}(\theta,(\nabla_X\xi))+g^{-1}(\theta,p)g^{-1}((\nabla_X\xi),p)))^V, \\
iii)\ (L_{\omega^V}\ {}^{CG}g)(\theta^V,Z^H) &= \frac{1}{1+r^2}(g^{-1}(\theta,(\nabla_Z\omega))+g^{-1}(\theta,p)g^{-1}((\nabla_Z\omega),p)), \\
iv)\ (L_{X^H}\ {}^{CG}g)(\theta^V,Z^H) &= -(\frac{1}{1+r^2}(g^{-1}(\theta,pR(X,Z))+g^{-1}(\theta,p)g^{-1}(pR(X,Z),p)), \\
v)\ (L_{\omega^V}\ {}^{CG}g)(Y^H,\xi^V) &= \frac{1}{1+r^2}(g^{-1}((\nabla_Y\omega),\xi)+g^{-1}((\nabla_Y\omega),p)g^{-1}(\xi,p)), \\
vi)\ (L_{X^H}\ {}^{CG}g)(Y^H,\xi^V) &= -\frac{1}{1+r^2}(g^{-1}(pR(X,Y),\xi)+g^{-1}(pR(X,Y),p)g^{-1}(\xi,p)), \\
vii)\ (L_{\omega^V}\ {}^{CG}g)(Y^H,Z^H) &= 0, \\
viii)\ (L_{X^H}\ {}^{CG}g)(Y^H,Z^H) &= ((L_X g)(Y,Z))^V,
\end{aligned}$$

where the horizontal lifts $X^H \in \Im^1_0(T^*M^n)$ of $X \in \Im^1_0(M^n)$ and the vertical lift $\omega^V \in \Im^1_0(T^*M^n)$ of $\omega \in \Im^0_1(M^n)$ defined by (1.2) and (1.3), respectively.

Proof. $i)$

$$\begin{aligned}
(L_{\omega^V}\, {}^{CG}g)(\theta^V,\xi^V) &= L_{\omega^V}\, {}^{CG}g(\theta^V,\xi^V) - {}^{CG}g(L_{\omega^V}\theta^V,\xi^V) - {}^{CG}g(\theta^V,L_{\omega^V}\xi^V) \\
&= L_{\omega^V}(\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p)))^V \\
&= \omega^V(\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p)))^V \\
&= 0
\end{aligned}$$

$ii)$

$$\begin{aligned}
(L_{X^H}\, {}^{CG}g)(\theta^V,\xi^V) &= L_{X^H}\, {}^{CG}g(\theta^V,\xi^V) - {}^{CG}g(L_{X^H}\theta^V,\xi^V) - {}^{CG}g(\theta^V,L_{X^H}\xi^V) \\
&= X^H(\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p)))^V \\
&\quad - {}^{CG}g((\nabla_X\theta)^V,\xi^V) - {}^{CG}g(\theta^V,(\nabla_X\xi)^V) \\
&= (\nabla_X(\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p)))^V \\
&\quad -(\frac{1}{1+r^2}(g^{-1}((\nabla_X\theta),\xi)+g^{-1}((\nabla_X\theta),p)g^{-1}(\xi,p)))^V \\
&\quad -(\frac{1}{1+r^2}(g^{-1}(\theta,(\nabla_X\xi))+g^{-1}(\theta,p)g^{-1}((\nabla_X\xi),p)))^V
\end{aligned}$$

$iii)$

$$\begin{aligned}
(L_{\omega^V}\, {}^{CG}g)(\theta^V,Z^H) &= L_{\omega^V}\, {}^{CG}g(\theta^V,Z^H) - {}^{CG}g(L_{\omega^V}\theta^V,Z^H) - {}^{CG}g(\theta^V,L_{\omega^V}Z^H) \\
&= {}^{CG}g(\theta^V,(\nabla_Z\omega)^V) \\
&= \frac{1}{1+r^2}(g^{-1}(\theta,(\nabla_Z\omega))+g^{-1}(\theta,p)g^{-1}((\nabla_Z\omega),p))
\end{aligned}$$

$iv)$

$$\begin{aligned}
(L_{X^H}\, {}^{CG}g)(\theta^V,Z^H) &= L_{X^H}\, {}^{CG}g(\theta^V,Z^H) - {}^{CG}g(L_{X^H}\theta^V,Z^H) - {}^{CG}g(\theta^V,L_{X^H}Z^H) \\
&= - {}^{CG}g((\nabla_X\theta)^V,Z^H) - {}^{CG}g(\theta^V,[X,Z]^H+(pR(X,Z))^V) \\
&= - {}^{CG}g(\theta^V,(pR(X,Z))^V) \\
&= -\frac{1}{1+r^2}(g^{-1}(\theta,pR(X,Z))+g^{-1}(\theta,p)g^{-1}(pR(X,Z),p))
\end{aligned}$$

$v)$

$$\begin{aligned}
(L_{\omega^V}\, {}^{CG}g)(Y^H,\xi^V) &= L_{\omega^V}\, {}^{CG}g(Y^H,\xi^V) - {}^{CG}g(L_{\omega^V}Y^H,\xi^V) - {}^{CG}g(Y^H,L_{\omega^V}\xi^V) \\
&= {}^{CG}g((\nabla_Y\omega)^V,\xi^V) \\
&= \frac{1}{1+r^2}(g^{-1}((\nabla_Y\omega),\xi)+g^{-1}((\nabla_Y\omega),p)g^{-1}(\xi,p))
\end{aligned}$$

$vi)$

$$\begin{aligned}
(L_{X^H}\, {}^{CG}g)(Y^H,\xi^V) &= L_{X^H}\, {}^{CG}g(Y^H,\xi^V) - {}^{CG}g(L_{X^H}Y^H,\xi^V) - {}^{CG}g(Y^H,L_{X^H}\xi^V) \\
&= - {}^{CG}g([X,Y]^H+(pR(X,Y)^V,\xi^V) - {}^{CG}g(Y^H,(\nabla_X\xi)^V) \\
&= - {}^{CG}g((pR(X,Y))^V,\xi^V) \\
&= -\frac{1}{1+r^2}(g^{-1}(pR(X,Y),\xi)+g^{-1}(pR(X,Y),p)g^{-1}(\xi,p))
\end{aligned}$$

$vii)$

$$\begin{aligned}(L_{\omega^V}\ {}^{CG}g)(Y^H,Z^H) &= L_{\omega^V}\ {}^{CG}g(Y^H,Z^H) - {}^{CG}g(L_{\omega^V}Y^H,Z^H) - {}^{CG}g(Y^H,L_{\omega^V}Z^H)\\ &= L_{\omega^V}(g(Y,Z))^V + {}^{CG}g((\nabla_Y\omega)^V,Z^H) + {}^{CG}g(Y^H,(\nabla_Z\omega)^V)\\ &= \omega^V(g(Y,Z))^V\\ &= 0\end{aligned}$$

$viii)$

$$\begin{aligned}(L_{X^H}\ {}^{CG}g)(Y^H,Z^H) &= L_{X^H}\ {}^{CG}g(Y^H,Z^H) - {}^{CG}g(L_{X^H}Y^H,Z^H) - {}^{CG}g(Y^H,L_{X^H}Z^H)\\ &= L_{X^H}(g(Y,Z))^V - {}^{CG}g([X,Y]^H + (pR(X,Y)^V,Z^H)\\ &\quad - {}^{CG}g(Y^H,[X,Z]^H + (pR(X,Z))^V)\\ &= X^H(g(Y,Z))^V - {}^{CG}g([X,Y]^H,Z^H) - {}^{CG}g((pR(X,Y))^V,Z^H)\\ &\quad - {}^{CG}g(Y^H,[X,Z]^H) - {}^{CG}g(Y^H,(pR(X,Z))^V)\\ &= (L_Xg(Y,Z))^V - (g(L_XY,Z))^V - (g(Y,L_XZ))^V\\ &= ((L_Xg)(Y,Z))^V.\end{aligned}$$

□

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