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Some properties of a sequence whose limit is the generalized Ioachimescu's constant

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Abstract. In this short note we have shown some properties of a sequence whose limit is the generalized Ioachimescu's constant. More precise, we show its decreasing monotonicity, concavity, log-concavity, and starshapedness. Our results are simple to be proved but new and with interest. To achieve this we have used on line version of computational knowledge engine Wolfram Alpha.

In 1895 A. G. Ioachimescu [1] proposed a problem regarding to the sequence $(S_n)_{n=1}^{\infty}$ defined by

$$S_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n}, \quad n \in \{1, 2, \dots\},$$

who asked to be shown that it is convergent and its limit lies between -2 and -1.

In [2], the author considered the following modification of above sequence, denoted by $(I_n)_{n=1}^{\infty}$,

$$I_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - 2(\sqrt{n} - 1), \quad n \in \{1, 2, \dots\}.$$

She has denoted its limit by $\mathcal{J}$ and called it Ioachimescu's constant. Moreover, the same author [3], proved that

$$\frac{1}{2\sqrt{n + \frac{1}{5}}} < I_n - \mathcal{J} < \frac{1}{2\sqrt{n + \frac{1}{6}}}, \quad n \in \{1, 2, \dots\}.$$

Using these inequalities she got that $\mathcal{J} = 0.53964549119....$ In addtion, let us also mention here that the following inequalities pertaining to the sequences $(S_n)_{n=1}^{\infty}$ and $(I_n)_{n=1}^{\infty}$ are given in [5]:

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$$2\left(\sqrt{n+1}-\sqrt{n-1}\right) \leq S_n \leq -1, \quad \frac{1}{n}-2 \leq S_n \leq -1,$$

and

$$\frac{1}{\sqrt{n}} < I_n \leq 1, \quad 2\left(\sqrt{n+1}-\sqrt{n}\right) < I_n \leq 1.$$

To our best knowledge, we have not seen so far to be shown the properties of the sequence $(I_n)_{n=1}^{\infty}$, such as decreasing monotonicity, concavity, log-concavity, and starshapedness. Now we recall some well-known concepts needed in this paper:

1° A sequence $(a_n)_{n=1}^{\infty}$ is said to be monotone increasing (decreasing) if $a_n \leq a_{n+1}$ ($a_n \geq a_{n+1}$) for all $n \in \{1, 2, \dots\}$.

2° A sequence $(a_n)_{n=1}^{\infty}$ is said to be convex (concave) if $a_n - 2a_{n+1} + a_{n+2} \geq 0$ ($a_n - 2a_{n+1} + a_{n+2} \leq 0$) for all $n \in \{1, 2, \dots\}$.

3° A sequence $(a_n)_{n=1}^{\infty}$ is said to be log-convex (log-concave) if $a_n \cdot a_{n+2} \geq a_{n+1}^2$ ($a_n \cdot a_{n+2} \leq a_{n+1}^2$) for all $n \in \{1, 2, \dots\}$.

4° A sequence $(a_n)_{n=0}^{\infty}$ is said to be starshaped ([4]) if $\frac{a_n - a_0}{n} \leq \frac{a_{n+1}-a_0}{n+1}$ for all $n \in \{1, 2, \dots\}$.

Showing the decreasing monotonicity, concavity, log-concavity, and starshapedness properties of the sequence $(I_n)_{n=1}^{\infty}$ we accomplish the main aim of this paper.

1. Main Results

Theorem 1.1. *The sequence $(I_n)_{n=1}^{\infty}$ is a strictly monotone decreasing sequence.*

Proof. Namely, we have

$$I_{n+1} - I_n = \frac{1}{\sqrt{n+1}} + 2(\sqrt{n}-\sqrt{n+1}) = \frac{2\sqrt{n(n+1)}-2n-1}{\sqrt{n+1}} < 0$$

for all $n = 1, 2, \dots$, because of $4(n^2+n) < (2n+1)^2$.

The proof is completed. □

Theorem 1.2. *The sequence $(I_n)_{n=1}^{\infty}$ is a strictly concave sequence.*

Proof. For all $n = 1, 2, \dots$, we have

$$\begin{aligned} &I_{n+2} - 2I_{n+1} + I_n \\ &= \frac{2\sqrt{(n+1)(n+2)}-2n-3}{\sqrt{n+2}} - \frac{2\sqrt{n(n+1)}-2n-1}{\sqrt{n+1}} \\ &= \frac{(2n+3)\sqrt{n+1}-2(n+1)\sqrt{n+2}-(2n+1)\sqrt{n+2}+2\sqrt{n(n+1)(n+2)}}{\sqrt{(n+1)(n+2)}}. \end{aligned}$$

Plotting the function, $x \geq 1$,

$$f(x) = (2x+3)\sqrt{x+1} - 2(x+1)\sqrt{x+2} - (2x+1)\sqrt{x+2} + 2\sqrt{x(x+1)(x+2)},$$

using on line version of computational knowledge engine Wolfram Alpha we obtain the graphic in Figure 1, from which we conclude that $f(x) < 0$ for all $x \geq 1$.

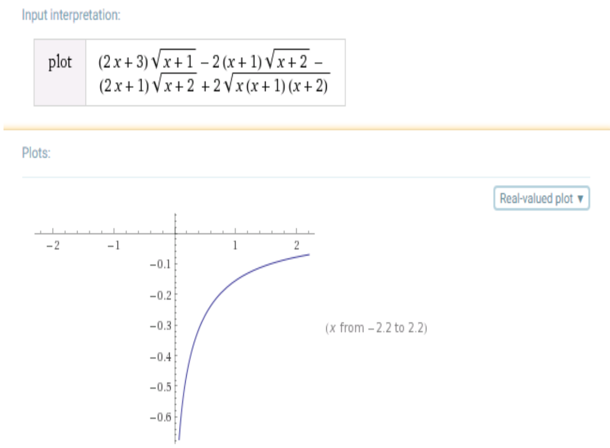

Figure 1

This means that $I_{n+2} - 2I_{n+1} + I_n < 0$ for all $n = 1, 2, \dots$, i.e. the sequence $(I_n)_{n=1}^{\infty}$ is a strictly concave sequence.

The proof is completed. $\square$

Theorem 1.3. *The sequence $(I_n)_{n=1}^{\infty}$ is a strictly log-concave sequence.*

Proof. It is easy to check that

$$I_n = I_{n+1} + 2(\sqrt{n+1} - \sqrt{n}) - \frac{1}{\sqrt{n+1}}$$

and

$$I_{n+2} = I_{n+1} - 2(\sqrt{n+2} - \sqrt{n+1}) + \frac{1}{\sqrt{n+2}}$$

for all $n \in \{1, 2, \dots\}$.

Hence,

$$\begin{aligned} I_{n+1}^2 - I_n I_{n+2} = I_{n+1}^2 - &\left(I_{n+1} + 2(\sqrt{n+1} - \sqrt{n}) - \frac{1}{\sqrt{n+1}}\right) \times \\ &\left(I_{n+1} - 2(\sqrt{n+2} - \sqrt{n+1}) + \frac{1}{\sqrt{n+2}}\right) = I_{n+1} \times \\ &\frac{2(n+2)\sqrt{n+1} - 4(n+1)\sqrt{n+2} + 2\sqrt{n(n+1)(n+2)} + \sqrt{n+2} - \sqrt{n+1}}{\sqrt{(n+1)(n+2)}}. \end{aligned}$$

For $x \geq 1$, we consider the functions

$$g_1(x) = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{x}},$$

$$g_2(x) = 2(\sqrt{x} - 1),$$

and

$$g_3(x) = 2(x+2)\sqrt{x+1} - 4(x+1)\sqrt{x+2} + 2\sqrt{x(x+1)(x+2)} + \sqrt{x+2} - \sqrt{x+1}.$$

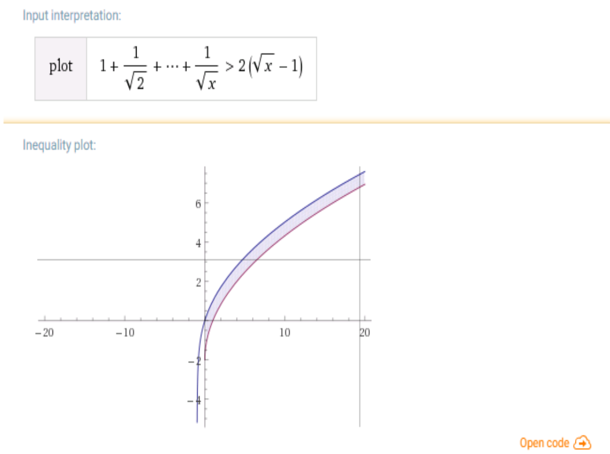

Figure 2

In one hand, plotting the inequality, $g_1(x) > g_2(x)$, using on line version of computational knowledge engine Wolfram Alpha we obtain the graphic in Figure 2, from which we conclude that $g_1(x) > g_2(x)$ holds true for all $x \geq 1$.

In the other hand, plotting the function $g_3(x)$, we obtain the graphic in Figure 3, from which we derive that $g_3(x) < 0$ for all $x \geq 1$.

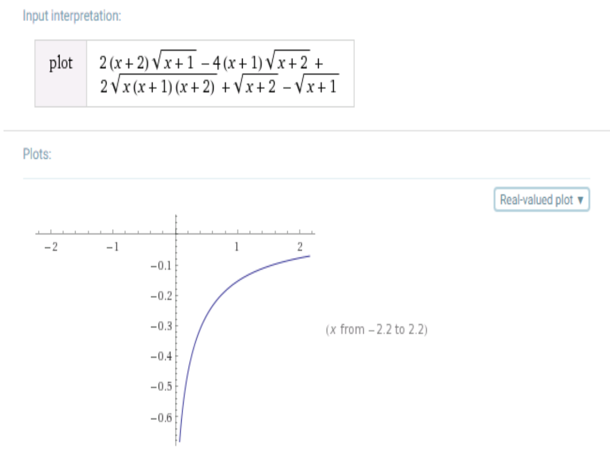

Figure 3

So, for $x \geq 1$ we have $[g_1(x) - g_2(x)] \cdot g_3(x) < 0$. Based on what we said so far we have shown that

$$I_{n+1}^2 - I_n I_{n+2} < 0,$$

i.e. the sequence $(I_n)_{n=1}^{\infty}$ is a strictly log-concave sequence.

The proof is completed.

$\square$

Theorem 1.4. *The sequence $(I_n)_{n=1}^{\infty}$ is a star-shaped sequence.*

Proof. After some simply calculations we have

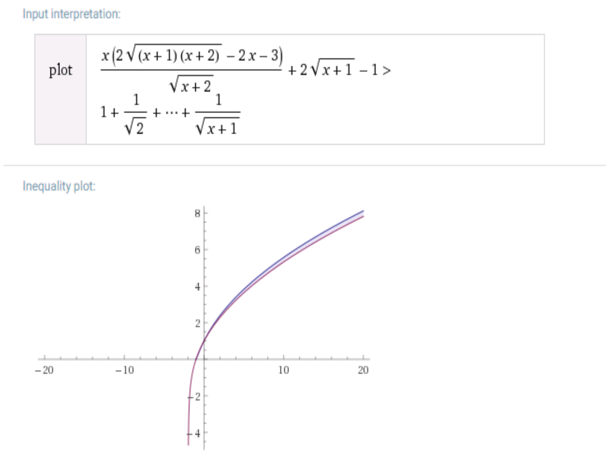

Figure 4

$$\frac{I_{n+2}-I_1}{n+1}-\frac{I_{n+1}-I_1}{n}=\frac{n[2\sqrt{(n+1)(n+2)}-2n-3]-(I_{n+1}-1)\sqrt{n+2}}{n(n+1)\sqrt{n+2}}.$$

Plotting the inequality

$$\frac{x[2\sqrt{(x+1)(x+2)}-2x-3]}{\sqrt{x+2}}>1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{x+1}}-2\sqrt{x+1}+1,$$

for $x \geq 1$ we obtain the graphics in Figure 4, which obviously imply that

$$\frac{I_{n+2}-I_1}{n+1}-\frac{I_{n+1}-I_1}{n}>0$$

for all $n \in \{1,2,\dots\}$.

The proof is completed. □

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