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On some problems regarding growth analysis of composite entire and meromorphic function

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Abstract. In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using relative L^*-order, relative L^*- type and relative L^*-weak type.

1. Introduction, Definitions and Notations.

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to f is defined on $|z| = r$ as follows:

$$M_f(r) = \max_{|z|=r} |f(z)| .$$

When f is meromorphic, $M_f(r)$ cannot be defined as f is not analytic throughout the complex plane. In this situation, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of f, playing the same role as maximum modulus function in the following manner:

$$T_f(r) = N_f(r) + m_f(r),$$

where

$$N_f(r) = \int_0^r \frac{n_f(t) - n_f(0)}{t} dt + n_f(0) \log r$$

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is the pole-counting contribution, where $n_f(r)$ is the number of poles of f, including multiplicities, for $|z| \leq r$. On the other hand, the function $m_f(r)$ known as the proximity function is defined as

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta,$$

$$\text{where } \log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0 .$$

If f is an entire function, then the Nevanlinna's Characteristic $T_f(r)$ of f reduces to $m_f(r)$. For a non-constant entire function f, $T_f(r)$ is strictly increasing and continuous and its inverse $T_f^{-1} : \left(T_f(0), \infty\right) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} T_f^{-1}(s) = \infty$. In this connection we just recall the following definition which is relevant:

Definition 1.1. {[1]} A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $\left[M_f(r)\right]^2 \leq M_f(r^\sigma)$ holds. For examples of functions with or without the Property (A), one may see [1].

However, we denote the order and lower order of growth of meromorphic f by ρ_f and λ_f respectively and they are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r} .$$

When f is entire, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

Somasundaram and Thamizharasi [9] introduced the notions of L-order and L-type for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. The more generalized concept for L-order and L-type for entire and meromorphic functions are L^*-order and L^*-type. Their definitions are as follows:

Definition 1.2. [9] The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]}.$$

When f is meromorphic, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]} .$$

Definition 1.3. [9] The L^*-type $\sigma_f^{L^*}$ of an entire function f is defined as

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}\ 0 < \rho_f^{L^*} < \infty .$$

For meromorphic f,

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}} \quad 0 < \rho_f^{L^*} < \infty .$$

Similarly, L^*-lower type $\overline{\sigma}_f^{L^*}$ of an entire function f is defined as

$$\overline{\sigma}_f^{L^*} = \liminf_{r\to\infty} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}} \quad 0 < \rho_f^{L^*} < \infty .$$

When f is meromorphic,

$$\overline{\sigma}_f^{L^*} = \liminf_{r\to\infty} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}} \quad 0 < \rho_f^{L^*} < \infty .$$

Analogously in order to determine the relative growth of two entire or meromorphic functions having same non zero finite L^*-lower order one may introduce the definition of L^*-weak type of entire and meromorphic functions having finite positive L^*-lower order in the following way:

Definition 1.4. The L^*-weak type denoted by $\tau_f^{L^*}$ of an entire function f is defined as follows:

$$\tau_f^{L^*} = \liminf_{r\to\infty} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}} , \quad 0 < \lambda_f^{L^*} < \infty .$$

Also one may define the growth indicator $\overline{\tau}_f^{L^*}$ of an entire function f in the following manner :

$$\overline{\tau}_f^{L^*} = \limsup_{r\to\infty} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}} , \quad 0 < \lambda_f^{L^*} < \infty,$$

For meromorphic f,

$$\overline{\tau}_f^{L^*} = \limsup_{r\to\infty} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}} \text{ and } \tau_f^{L^*} = \liminf_{r\to\infty} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}}, \quad 0 < \lambda_f^{L^*} < \infty.$$

Lahiri and Banerjee [7] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

Definition 1.5. [7] Let f be meromorphic and g be entire. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as

$$\begin{aligned} \rho_g(f) &= \inf\left\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\right\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} . \end{aligned}$$

The definition coincides with the classical one [7] if $g(z) = \exp z$.

Similarly one can define the relative lower order of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

In the line of Somasundaram and Thamizharasi [9] and Lahiri and Banerjee [7], one may define the relative L^*-order and relative L^*-lower order of a meromorphic function f with respect to an entire function g in the following manner:

Definition 1.6. The relative L^*-order $\rho_g^{L^*}(f)$ and the relative L^*-lower order $\lambda_g^{L^*}(f)$ of a meromorphic function f with respect to an entire function g are defined by

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log\left[re^{L(r)}\right]}.$$

Further to compare the relative growth of two meromorphic functions having same non zero finite relative L^*-order with respect to another entire function, one may introduce the definitions of *relative L^*-type* and *relative L^*-lower type* in the following manner:

Definition 1.7. The relative L^*-type and relative L^*-lower type denoted respectively by $\sigma_g^{L^*}(f)$ and $\overline{\sigma}_g^{L^*}(f)$ of a meromorphic function f with respect to an entire function g are respectively defined as follows:

$$\sigma_g^{L^*}(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}} \text{ and } \overline{\sigma}_g^{L^*}(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}},$$

$$0 < \rho_g^{L^*}(f) < \infty.$$

Analogously to determine the relative growth of two meromorphic functions having same non zero finite *relative L^*-lower order* with respect to an entire function one may introduce the definition of *relative L^*-weak type* in the following way,

Definition 1.8. The relative L^*-weak type denoted by $\tau_g^{L^*}(f)$ of a meromorphic function f with respect to an entire function g is defined as follows:

$$\tau_g^{L^*}(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{\left[re^{L(r)}\right]^{\lambda_g^{L^*}(f)}},\ 0 < \lambda_g^{L^*}(f) < \infty.$$

Also one may define the growth indicator $\overline{\tau}_g^{L^*}(f)$ of an entire function f with respect to another entire function g in the following manner :

$$\overline{\tau}_g^{L^*}(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{\left[re^{L(r)}\right]^{\lambda_g^{L^*}(f)}},\ 0 < \lambda_g^{L^*}(f) < \infty.$$

In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using L^*-order, L^*-type and L^*-weak type. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [6] and [10].

2. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [2] *If f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f) .$$

Lemma 2.2. [3] *Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T(r, f \circ g) \geq T(\exp(r^{\mu}), f) .$$

Lemma 2.3. [8] *Let f be meromorphic and g be entire such that $0 < \rho_g < \infty$ and $0 < \lambda_f$. Then for a sequence of values of r tending to infinity,*

$$T_{f \circ g}(r) > T_g(\exp(r^{\mu})),$$

where $0 < \mu < \rho_g$.

Lemma 2.4. [4] *Let f be a meromorphic function and g be an entire function such that $\lambda_g < \mu < \infty$ and $0 < \lambda_f \leq \rho_f < \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f \circ g}(r) < T_f(\exp(r^{\mu})) .$$

Lemma 2.5. [4] *Let f be a meromorphic function of finite order and g be an entire function such that $0 < \lambda_g < \mu < \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f \circ g}(r) < T_g(\exp(r^{\mu})) .$$

Lemma 2.6. [5] *Let f be an entire function which satisfies the Property (A), $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then*

$$\beta T_f(r) < T_f\left(\alpha r^{\delta}\right).$$

3. Theorems.

In this section we present the main results of the paper.

Theorem 3.1. *Let f be a meromorphic function and g and h be any two entire functions such that $0 < \lambda_h^{L^*}(f) < \infty$ and $\sigma_g^{L^*} < \infty$. Also let h satisfy the Property (A). If $L\left(M_g(r)\right) = o([r \exp L(r)]^{\alpha})$ as $r \to \infty$ and for some positive $\alpha < \rho_g^{L^*}$, then for any $\delta > 1$*

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f\left(\exp[r \exp L(r)]^{\rho_g^{L^*}}\right)} \leq \delta \cdot \sigma_g^{L^*} .$$

Proof. Let us consider that $\beta > 2$ and $\delta > 1$. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.1, Lemma 2.6 and the inequality $T_g(r) \leq \log M_g(r)$ {cf. [6] } for a sequence of values of r tending to infinity that

$$\begin{aligned}
T_h^{-1} T_{f \circ g}(r) &\leqslant T_h^{-1}\left[\{1 + o(1)\} T_f\left(M_g(r)\right)\right] \\
\text{i.e., } T_h^{-1} T_{f \circ g}(r) &\leqslant \beta\left[T_h^{-1} T_f\left(M_g(r)\right)\right]^{\delta} \\
\text{i.e., } \log T_h^{-1} T_{f \circ g}(r) &\leqslant \delta \log T_h^{-1} T_f\left(M_g(r)\right) + O(1)
\end{aligned}$$

$$\begin{aligned} i.e.,\ \log T_h^{-1} T_{f\circ g}(r) &\leqslant \delta\left(\lambda_h^{L^*}(f)+\varepsilon\right)\left(\log M_g(r)+L\left(M_g(r)\right)\right) \\ &\quad +O(1)\,. \end{aligned}$$

$$\begin{aligned} &i.e.,\ \log T_h^{-1} T_{f\circ g}(r) \leqslant \delta\left(\lambda_h^{L^*}(f)+\varepsilon\right)\cdot \\ &\left[\left(\sigma_g^{L^*}+\varepsilon\right)[r\exp L(r)]^{\rho_g^{L^*}}+L\left(M_g(r)\right)\right]+O(1)\,. \end{aligned} \tag{3.1}$$

Further, we obtain for all sufficiently large values of r that

$$\begin{aligned} &\log T_h^{-1} T_f\left(\exp[r\exp L(r)]^{\rho_g^{L^*}}\right) \geq \\ &\qquad \left(\lambda_h^{L^*}(f)-\varepsilon\right)\left[[r\exp L(r)]^{\rho_g^{L^*}}+\left[L\left(\exp[r\exp L(r)]^{\rho_g^{L^*}}\right)\right]\right] \end{aligned}$$

$$i.e.,\ \log T_h^{-1} T_f\left(\exp[r\exp L(r)]^{\rho_g^{L^*}}\right) \geq \left(\lambda_h^{L^*}(f)-\varepsilon\right)\cdot[r\exp L(r)]^{\rho_g^{L^*}}\,.$$

Now from (3.1) and above, it follows for a sequence of values of r tending to infinity that

$$\begin{aligned} &\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(\exp[r\exp L(r)]^{\rho_g^{L^*}}\right)} \\ &\qquad \leq \frac{O(1)+\delta\left(\lambda_h^{L^*}(f)+\varepsilon\right)\cdot\left[\left(\sigma_g^{L^*}+\varepsilon\right)[r\exp L(r)]^{\rho_g^{L^*}}+L\left(M_g(r)\right)\right]}{\left(\lambda_h^{L^*}(f)-\varepsilon\right)\cdot[r\exp L(r)]^{\rho_g^{L^*}}} \end{aligned}$$

$$\begin{aligned} i.e.,\ \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(\exp[r\exp L(r)]^{\rho_g^{L^*}}\right)} &\leq \frac{O(1)}{\left(\lambda_h^{L^*}(f)-\varepsilon\right)\cdot[r\exp L(r)]^{\rho_g^{L^*}}}+ \\ &\quad \frac{\delta\left(\lambda_h^{L^*}(f)+\varepsilon\right)\cdot\left[\left(\sigma_g^{L^*}+\varepsilon\right)+\frac{L\left(M_g(r)\right)}{[r\exp L(r)]^{\rho_g^{L^*}}}\right]}{\left(\lambda_h^{L^*}(f)-\varepsilon\right)}\,. \end{aligned} \tag{3.2}$$

As $\alpha < \rho_g^{L^*}$ and $L\left(M_g(r)\right) = o\left([r\exp L(r)]^{\alpha}\right)$ as $r\to\infty$, we obtain that

$$\lim_{r\to\infty}\frac{L\left(M_g(r)\right)}{[r\exp L(r)]^{\rho_g^{L^*}}} = 0\,. \tag{3.3}$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (3.2) and (3.3) that

$$\liminf_{r\to\infty}\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(\exp[r\exp L(r)]^{\rho_g^{L^*}}\right)} \leq \delta\cdot\sigma_g^{L^*}\,.$$

Thus the theorem is established. □

In the line of Theorem 3.1, the following two theorems can be carried out and therefore their proofs are omitted:

Theorem 3.2. *Let f be a meromorphic function and g and h be any two entire functions such that $0 < \rho_h^{L^*}(f) < \infty$ and $\sigma_g^{L^*} < \infty$. Also let h satisfy the Property (A). If $L\left(M_g(r)\right) = o([r \exp L(r)]^{\alpha})$ as $r \to \infty$ and for some positive $\alpha < \rho_g^{L^*}$, then for any $\delta > 1$*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(\exp[r \exp L(r)]^{\rho_g^{L^*}}\right)} \leq \delta \cdot \sigma_g^{L^*}.$$

Theorem 3.3. *Let f be a meromorphic function and g and h be any two entire functions such that $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ and $\sigma_g^{L^*} < \infty$. Also let h satisfy the Property (A). If $L\left(M_g(r)\right) = o([r \exp L(r)]^{\alpha})$ as $r \to \infty$ and for some positive $\alpha < \rho_g^{L^*}$, then for any $\delta > 1$*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(\exp[r \exp L(r)]^{\rho_g^{L^*}}\right)} \leq \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \sigma_g^{L^*}}{\lambda_h^{L^*}(f)}.$$

Using the notion of L^*-lower type, we may state the following theorem without its proof because it can be proved in the line of Theorem 3.3:

Theorem 3.4. *Let f be a meromorphic function and g and h be any two entire functions such that $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ and $\overline{\sigma}_g^{L^*} < \infty$. Also let h satisfy the Property (A). If $L\left(M_g(r)\right) = o([r \exp L(r)]^{\alpha})$ as $r \to \infty$ and for some positive $\alpha < \rho_g^{L^*}$, then for any $\delta > 1$*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(\exp[r \exp L(r)]^{\rho_g^{L^*}}\right)} \leq \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \overline{\sigma}_g^{L^*}}{\lambda_h^{L^*}(f)}.$$

Now we state the following three theorems without their proofs as those can be carried out in the line of Theorem 3.1, Theorem 3.2 and Theorem 3.3 respectively.

Theorem 3.5. *Let f be a meromorphic function and g, h and k be any three entire functions such that $\lambda_h^{L^*}(f) < \infty$, $\lambda_k^{L^*}(g) > 0$ and $\sigma_g^{L^*} < \infty$. Also let h satisfy the Property (A). If $L\left(M_g(r)\right) = o([r \exp L(r)]^{\alpha})$ as $r \to \infty$ and for some positive $\alpha < \rho_g^{L^*}$, then for any $\delta > 1$*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g\left(\exp[r \exp L(r)]^{\rho_g^{L^*}}\right)} \leq \delta \cdot \frac{\lambda_h^{L^*}(f) \cdot \sigma_g^{L^*}}{\lambda_k^{L^*}(g)}.$$

Theorem 3.6. *Let f be a meromorphic function and g, h and k be any three entire functions such that $\rho_h^{L^*}(f) < \infty$, $\rho_k^{L^*}(g) > 0$ and $\sigma_g^{L^*} < \infty$. Also let h satisfy the Property (A). If $L\left(M_g(r)\right) = o([r \exp L(r)]^{\alpha})$ as $r \to \infty$ and for some positive $\alpha < \rho_g^{L^*}$, then for any $\delta > 1$*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g\left(\exp[r \exp L(r)]^{\rho_g^{L^*}}\right)} \leq \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \sigma_g^{L^*}}{\rho_k^{L^*}(g)}.$$

Theorem 3.7. *Let f be a meromorphic function and g, h and k be any three entire functions such that $\rho_h^{L^*}(f) < \infty$, $\lambda_k^{L^*}(g) > 0$ and $\sigma_g^{L^*} < \infty$. Also let h satisfy the Property (A). If*

$L\left(M_g(r)\right) = o([r \exp L(r)]^{\alpha})$ *as* $r \to \infty$ *and for some positive* $\alpha < \rho_g^{L^*}$, *then for any* $\delta > 1$

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g\left(\exp[r \exp L(r)]^{\rho_g^{L^*}}\right)} \leq \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \sigma_g^{L^*}}{\lambda_k^{L^*}(g)} .$$

Theorem 3.8. *Let* f *be a meromorphic function and* g, h *and* k *be any three entire functions such that* $\rho_h^{L^*}(f) < \infty$, $\lambda_k^{L^*}(g) > 0$ *and* $\overline{\sigma}_g^{L^*} < \infty$. *Also let* h *satisfy the Property (A). If* $L\left(M_g(r)\right) = o([r \exp L(r)]^{\alpha})$ *as* $r \to \infty$ *and for some positive* $\alpha < \rho_g^{L^*}$, *then for any* $\delta > 1$

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g\left(\exp[r \exp L(r)]^{\rho_g^{L^*}}\right)} \leq \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \overline{\sigma}_g^{L^*}}{\lambda_k^{L^*}(g)} .$$

We omit the proof of Theorem 3.8 as it can easily be established in the line of Theorem 3.4.

Further we state the following two theorems which are based on L^* -weak type:

Theorem 3.9. *Let* f *be a meromorphic function and* g *and* h *be any two entire functions such that* $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ *and* $\tau_g^{L^*} < \infty$. *Also let* h *satisfy the Property (A). If* $L\left(M_g(r)\right) = o([r \exp L(r)]^{\alpha})$ *as* $r \to \infty$ *and for some positive* $\alpha < \lambda_g^{L^*}$, *then for any* $\delta > 1$

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(\exp[r \exp L(r)]^{\lambda_g^{L^*}}\right)} \leq \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \tau_g^{L^*}}{\lambda_h^{L^*}(f)} .$$

Theorem 3.10. *Let* f *be a meromorphic function and* g, h *and* k *be any three entire functions such that* $\rho_h^{L^*}(f) < \infty$, $\lambda_k^{L^*}(g) > 0$ *and* $\tau_g^{L^*} < \infty$. *Also let* h *satisfy the Property (A). If* $L\left(M_g(r)\right) = o([r \exp L(r)]^{\alpha})$ *as* $r \to \infty$ *and for some positive* $\alpha < \lambda_g^{L^*}$, *then for any* $\delta > 1$

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g\left(\exp[r \exp L(r)]^{\lambda_g^{L^*}}\right)} \leq \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \tau_g^{L^*}}{\lambda_k^{L^*}(g)} .$$

The proofs of the above two theorems can be carried out in the line of Theorem 3.4 and Theorem 3.8 respectively and therefore their proofs are omitted.

Using the concept of the growth indicator $\overline{\tau}_g^{L^*}$ of an entire function g, we may state the subsequent six theorems without their proofs since those can be carried out in the line of Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.5, Theorem 3.6 and Theorem 3.7 respectively.

Theorem 3.11. *Let* f *be a meromorphic function and* g *and* h *be any two entire functions such that* $0 < \lambda_h^{L^*}(f) < \infty$ *and* $\overline{\tau}_g^{L^*} < \infty$. *Also let* h *satisfy the Property (A). If* $L\left(M_g(r)\right) = o([r \exp L(r)]^{\alpha})$ *as* $r \to \infty$ *and for some positive* $\alpha < \lambda_g^{L^*}$, *then for any* $\delta > 1$

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(\exp[r \exp L(r)]^{\lambda_g^{L^*}}\right)} \leq \delta \cdot \overline{\tau}_g^{L^*} .$$

Theorem 3.12. *Let f be a meromorphic function and g and h be any two entire functions such that $0 < \rho_h^{L^*}(f) < \infty$ and $\overline{\tau}_g^{L^*} < \infty$. Also let h satisfy the Property (A). If $L\left(M_g(r)\right) = o([r\exp L(r)]^{\alpha})$ as $r \to \infty$ and for some positive $\alpha < \lambda_g^{L^*}$, then for any $\delta > 1$*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(\exp[r\exp L(r)]^{\lambda_g^{L^*}}\right)} \leq \delta \cdot \overline{\tau}_g^{L^*} .$$

Theorem 3.13. *Let f be a meromorphic function and g and h be any two entire functions such that $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ and $\overline{\tau}_g^{L^*} < \infty$. Also let h satisfy the Property (A). If $L\left(M_g(r)\right) = o([r\exp L(r)]^{\alpha})$ as $r \to \infty$ and for some positive $\alpha < \lambda_g^{L^*}$, then for any $\delta > 1$*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(\exp[r\exp L(r)]^{\lambda_g^{L^*}}\right)} \leq \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \overline{\tau}_g^{L^*}}{\lambda_h^{L^*}(f)} .$$

Theorem 3.14. *Let f be a meromorphic function and g, h and k be any three entire functions such that $\lambda_h^{L^*}(f) < \infty$, $\lambda_k^{L^*}(g) > 0$ and $\overline{\tau}_g^{L^*} < \infty$. Also let h satisfy the Property (A). If $L\left(M_g(r)\right) = o([r\exp L(r)]^{\alpha})$ as $r \to \infty$ and for some positive $\alpha < \lambda_g^{L^*}$, then for any $\delta > 1$*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g\left(\exp[r\exp L(r)]^{\lambda_g^{L^*}}\right)} \leq \delta \cdot \frac{\lambda_h^{L^*}(f) \cdot \overline{\tau}_g^{L^*}}{\lambda_k^{L^*}(g)} .$$

Theorem 3.15. *Let f be a meromorphic function and g, h and k be any three entire functions such that $\rho_h^{L^*}(f) < \infty$, $\rho_k^{L^*}(g) > 0$ and $\overline{\tau}_g^{L^*} < \infty$. Also let h satisfy the Property (A). If $L\left(M_g(r)\right) = o([r\exp L(r)]^{\alpha})$ as $r \to \infty$ and for some positive $\alpha < \lambda_g^{L^*}$, then for any $\delta > 1$*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g\left(\exp[r\exp L(r)]^{\lambda_g^{L^*}}\right)} \leq \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \overline{\tau}_g^{L^*}}{\rho_k^{L^*}(g)} .$$

Theorem 3.16. *Let f be a meromorphic function and g, h and k be any three entire functions such that $\rho_h^{L^*}(f) < \infty$, $\lambda_k^{L^*}(g) > 0$ and $\overline{\tau}_g^{L^*} < \infty$. Also let h satisfy the Property (A). If $L\left(M_g(r)\right) = o([r\exp L(r)]^{\alpha})$ as $r \to \infty$ and for some positive $\alpha < \lambda_g^{L^*}$, then for any $\delta > 1$*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g\left(\exp[r\exp L(r)]^{\lambda_g^{L^*}}\right)} \leq \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \overline{\tau}_g^{L^*}}{\lambda_k^{L^*}(g)} .$$

Theorem 3.17. *Let f be a meromorphic function and g and h be any two entire functions such that $0 < \rho_h^{L^*}(f) < \rho_g$ and $\sigma_h^{L^*}(f) > 0$. If $L\left(\exp\left(re^{L(r)}\right)^{\alpha}\right) = o\left(\left[re^{L(r)}\right]^{\alpha}\right)$ $(r \to \infty)$ for any $\alpha > 0$, then*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}\left(re^{L(r)}\right)}{T_h^{-1} T_f(r)} \geq \frac{\lambda_h^{L^*}(f)}{\sigma_h^{L^*}(f)} .$$

Proof. From the definition of relative L^*- type of meromorphic function, we obtain for all sufficiently large values of r that

$$T_h^{-1}T_f(r) \le \left(\sigma_h^{L^*}(f)+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_h^{L^*}(f)} . \tag{3.4}$$

As $0<\rho_h^{L^*}(f)<\rho_g$, we obtain in view of Lemma 2.2 for a sequence of values of r tending to infinity that

$$\log T_h^{-1}T_{f\circ g}\left(re^{L(r)}\right) \ge \log T_h^{-1}T_f\left(\exp\left(re^{L(r)}\right)^{\rho_h^{L^*}(f)}\right)$$

$$i.e.,\ \log T_h^{-1}T_{f\circ g}\left(re^{L(r)}\right) \ge \left(\lambda_h^{L^*}(f)-\varepsilon\right)\left[\left[re^{L(r)}\right]^{\rho_h^{L^*}(f)}+L\left(\exp\left(re^{L(r)}\right)^{\rho_h^{L^*}(f)}\right)\right].$$

Therefore from (3.4) and above, it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1}T_{f\circ g}\left(re^{L(r)}\right)}{T_h^{-1}T_f(r)} \ge \frac{\left(\lambda_h^{L^*}(f)-\varepsilon\right)\left[\left[re^{L(r)}\right]^{\rho_h^{L^*}(f)}+L\left(\exp\left(re^{L(r)}\right)^{\rho_h^{L^*}(f)}\right)\right]}{\left(\sigma_h^{L^*}(f)+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_h^{L^*}(f)}}$$

Since $\lim\limits_{r\to\infty}\frac{L\left(\exp\left(re^{L(r)}\right)^{\rho_h^{L^*}(f)}\right)}{\left[re^{L(r)}\right]^{\rho_h^{L^*}(f)}}=0$ as $L\left(\exp\left(re^{L(r)}\right)^{\alpha}\right)=o\left(\left[re^{L(r)}\right]^{\alpha}\right)$ $(r\to\infty)$ for any $\alpha>0$, we obtain from above that

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}\left(re^{L(r)}\right)}{T_h^{-1}T_f(r)} \ge \frac{\lambda_h^{L^*}(f)}{\sigma_h^{L^*}(f)} .$$

Thus the theorem follows. □

Now using the concept of the growth indicator $\overline{\tau}_h^{L^*}(f)$ of a meromorphic function f, we may state the following theorem without its proof since it can be carried out in the line of Theorem 3.17.

Theorem 3.18. *Let f be a meromorphic function and g and h be any two entire functions such that $0<\rho_h^{L^*}(f)<\rho_g$ and $\overline{\tau}_h^{L^*}(f)>0$. If $L\left(\exp\left(re^{L(r)}\right)^{\alpha}\right)=o\left(\left[re^{L(r)}\right]^{\alpha}\right)$ $(r\to\infty)$ for any $\alpha>0$, then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}\left(re^{L(r)}\right)}{T_h^{-1}T_f(r)} \ge \frac{\lambda_h^{L^*}(f)}{\overline{\tau}_h^{L^*}(f)} .$$

Theorem 3.19. *Let f be a meromorphic function and g and h be any two entire functions such that (i) $\rho_h^{L^*}(f)=\rho_g^{L^*}$, (ii) $\sigma_g^{L^*}<\infty$ and (iii) $\overline{\sigma}_h^{L^*}(f)>0$. Also let h satisfy the Property*

(A). Then for any $\delta > 1$
(a) if $L\left(M_g(r)\right) = o\left\{T_h^{-1} T_f(r)\right\}$ then

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r) + L\left(M_g(r)\right)} \leq \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \sigma_g^{L^*}}{\overline{\sigma}_h^{L^*}(f)}$$

and (b) if $T_h^{-1} T_f(r) = o\left\{L\left(M_g(r)\right)\right\}$ then

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r) + L\left(M_g(r)\right)} \leq \delta \cdot \rho_h^{L^*}(f) .$$

Proof. Let us consider that $\beta > 2$ and $\delta > 1$. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.1, Lemma 2.6 and the inequality $T_g(r) \leq \log M_g(r)$ {cf. [6] } for all sufficiently large values of r that

$$\begin{aligned} T_h^{-1} T_{f\circ g}(r) &\leqslant T_h^{-1}\left[\{1+o(1)\} T_f\left(M_g(r)\right)\right] \\ i.e.,\ T_h^{-1} T_{f\circ g}(r) &\leqslant \beta\left[T_h^{-1} T_f\left(M_g(r)\right)\right]^{\delta} \\ i.e.,\ \log T_h^{-1} T_{f\circ g}(r) &\leqslant \delta \log T_h^{-1} T_f\left(M_g(r)\right) + O(1) \\ i.e.,\ \log T_h^{-1} T_{f\circ g}(r) &\leqslant \delta\left(\rho_h^{L^*}(f)+\varepsilon\right)\left(\log M_g(r) + L\left(M_g(r)\right)\right) \\ &\quad +O(1) \end{aligned}$$

$$\begin{aligned} i.e.,\ \log T_h^{-1} T_{f\circ g}(r) \leqslant{}& \delta\left(\rho_h^{L^*}(f)+\varepsilon\right)\left(\sigma_g^{L^*}+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_g^{L^*}} \\ &+\delta\left(\rho_h^{L^*}(f)+\varepsilon\right) L\left(M_g(r)\right) + O(1). \end{aligned}$$

In view of condition (ii), we obtain from above for all sufficiently large values of r that

$$\begin{aligned} \log T_h^{-1} T_{f\circ g}(r) \leqslant{}& \delta\left(\rho_h^{L^*}(f)+\varepsilon\right)\left(\sigma_g^{L^*}+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_h^{L^*}(f)} \\ &+ \delta\left(\rho_h^{L^*}(f)+\varepsilon\right) L\left(M_g(r)\right) + O(1) . \end{aligned} \tag{3.5}$$

Again from the definition of relative L^*-lower type, we get for all sufficiently large values of r that

$$\begin{aligned} T_h^{-1} T_f(r) &\geq \left(\overline{\sigma}_h^{L^*}(f)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_h^{L^*}(f)} \\ i.e.,\ \left[re^{L(r)}\right]^{\rho_h^{L^*}(f)} &\leq \frac{T_h^{-1} T_f(r)}{\left(\overline{\sigma}_h^{L^*}(f)-\varepsilon\right)} . \end{aligned} \tag{3.6}$$

Now from (3.5) and (3.6), it follows for all sufficiently large values of r that

$$\begin{aligned} \log T_h^{-1} T_{f\circ g}(r) \leqslant{}& \delta\left(\rho_h^{L^*}(f)+\varepsilon\right)\left(\sigma_g^{L^*}+\varepsilon\right)\frac{T_h^{-1} T_f(r)}{\left(\overline{\sigma}_h^{L^*}(f)-\varepsilon\right)} \\ &+\delta\left(\rho_h^{L^*}(f)+\varepsilon\right) L\left(M_g(r)\right) + O(1) \end{aligned}$$

$$ie.,\ \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r) + L\left(M_g(r)\right)} \leq \frac{O(1)}{T_h^{-1} T_f(r) + L\left(M_g(r)\right)}$$

$$+\frac{\frac{\delta\left(\rho_h^{L^*}(f)+\varepsilon\right)\left(\sigma_g^{L^*}+\varepsilon\right)}{\left(\overline{\sigma}_h^{L^*}(f)-\varepsilon\right)}}{1+\frac{L\left(M_g(r)\right)}{T_h^{-1}T_f(r)}}+\frac{\delta\left(\rho_h^{L^*}(f)+\varepsilon\right)}{1+\frac{T_h^{-1}T_f(r)}{L\left(M_g(r)\right)}}\,. \tag{3.7}$$

If $L\left(M_g(r)\right)=o\left\{T_h^{-1}T_f(r)\right\}$ then from (3.7), we get that

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r)+L\left(M_g(r)\right)}\le\frac{\delta\left(\rho_h^{L^*}(f)+\varepsilon\right)\left(\sigma_g^{L^*}+\varepsilon\right)}{\left(\overline{\sigma}_h^{L^*}(f)-\varepsilon\right)}\,.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r)+L\left(M_g(r)\right)}\le\delta\cdot\frac{\rho_h^{L^*}(f)\cdot\sigma_g^{L^*}}{\overline{\sigma}_h^{L^*}(f)}\,.$$

Thus the first part of the theorem follows.

Again if $T_h^{-1}T_f(r)=o\left\{L\left(M_g(r)\right)\right\}$ then from (3.7) it follows that

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r)+L\left(M_g(r)\right)}\le\delta\left(\rho_h^{L^*}(f)+\varepsilon\right)\,.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from above that

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r)+L\left(M_g(r)\right)}\le\ \delta\cdot\rho_h^{L^*}(f)\,.$$

Thus the second part of the theorem is established. □

Theorem 3.20. *Let f be a meromorphic function and g and h be any two entire functions such that (i) $\lambda_h^{L^*}(f)<\infty$, (ii) $\rho_h^{L^*}(f)=\rho_g^{L^*}$, (iii) $\sigma_g^{L^*}<\infty$ and (iv) $\overline{\sigma}_h^{L^*}(f)>0$. Also let h satisfy the Property (A). Then for any $\delta>1$,*
(a) if $L\left(M_g(r)\right)=o\left\{T_h^{-1}T_f(r)\right\}$ then

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r)+L\left(M_g(r)\right)}\le\delta\cdot\frac{\lambda_h^{L^*}(f)\cdot\sigma_g^{L^*}}{\overline{\sigma}_h^{L^*}(f)}$$

and (b) if $T_h^{-1}T_f(r)=o\left\{L\left(M_g(r)\right)\right\}$ then

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r)+L\left(M_g(r)\right)}\le\ \delta\cdot\lambda_h^{L^*}(f)\,.$$

Theorem 3.21. *Let f be a meromorphic function and g and h be any two entire functions such that (i) $\rho_h^{L^*}(f)=\rho_g^{L^*}$, (ii) $\sigma_g^{L^*}<\infty$ and (iii) $\sigma_h^{L^*}(f)>0$. Also let h satisfy the Property (A). Then for any $\delta>1$,*
(a) if $L\left(M_g(r)\right)=o\left\{T_h^{-1}T_f(r)\right\}$ then

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r)+L\left(M_g(r)\right)}\le\delta\cdot\frac{\rho_h^{L^*}(f)\cdot\sigma_g^{L^*}}{\sigma_h^{L^*}(f)}$$

and (b) if $T_h^{-1}T_f(r) = o\left\{L\left(M_g(r)\right)\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r) + L\left(M_g(r)\right)} \le \delta \cdot \rho_h^{L^*}(f) .$$

Theorem 3.22. *Let* f *be a meromorphic function and* g *and* h *be any two entire functions such that (i)* $\rho_h^{L^*}(f) = \rho_g^{L^*}$*, (ii)* $\overline{\sigma}_g^{L^*} < \infty$ *and (iii)* $\overline{\sigma}_h^{L^*}(f) > 0$*. Also let* h *satisfy the Property (A). Then for any* $\delta > 1$*,*
(a) if $L\left(M_g(r)\right) = o\left\{T_h^{-1}T_f(r)\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r) + L\left(M_g(r)\right)} \le \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \overline{\sigma}_g^{L^*}}{\overline{\sigma}_h^{L^*}(f)}$$

and (b) if $T_h^{-1}T_f(r) = o\left\{L\left(M_g(r)\right)\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r) + L\left(M_g(r)\right)} \le \delta \cdot \rho_h^{L^*}(f) .$$

We omit the proof of the above three theorems as those can be carried out in the line of Theorem 3.19.

Similarly using the concept of the growth indicator $\tau_h^{L^*}(f)$ and $\overline{\tau}_g^{L^*}$, we may state the subsequent four theorems without their proofs since those can be carried out in the line of Theorem 3.19, Theorem 3.20, Theorem 3.21 and Theorem 3.22 respectively.

Theorem 3.23. *Let* f *be a meromorphic function and* g *and* h *be any two entire functions such that (i)* $\rho_h^{L^*}(f) < \infty$*, (ii)* $\lambda_h^{L^*}(f) = \lambda_g^{L^*}$*, (iii)* $\overline{\tau}_g^{L^*} < \infty$ *and (iv)* $\tau_h^{L^*}(f) > 0$*. Also let* h *satisfy the Property (A). Then for any* $\delta > 1$*,*
(a) if $L\left(M_g(r)\right) = o\left\{T_h^{-1}T_f(r)\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r) + L\left(M_g(r)\right)} \le \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \overline{\tau}_g^{L^*}}{\tau_h^{L^*}(f)}$$

and (b) if $T_h^{-1}T_f(r) = o\left\{L\left(M_g(r)\right)\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r) + L\left(M_g(r)\right)} \le \delta \cdot \rho_h^{L^*}(f) .$$

Theorem 3.24. *Let* f *be a meromorphic function and* g *and* h *be any two entire functions such that (i)* $\lambda_h^{L^*}(f) = \lambda_g^{L^*}$*, (ii)* $\overline{\tau}_g^{L^*} < \infty$ *and (iii)* $\tau_h^{L^*}(f) > 0$*. Also let* h *satisfy the Property (A). Then for any* $\delta > 1$*,*
(a) if $L\left(M_g(r)\right) = o\left\{T_h^{-1}T_f(r)\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r) + L\left(M_g(r)\right)} \le \delta \cdot \frac{\lambda_h^{L^*}(f) \cdot \overline{\tau}_g^{L^*}}{\tau_h^{L^*}(f)}$$

and (*b*) *if* $T_h^{-1} T_f(r) = o\left\{L\left(M_g(r)\right)\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r) + L\left(M_g(r)\right)} \leq \delta \cdot \lambda_h^{L^*}(f) \ .$$

Theorem 3.25. *Let* f *be a meromorphic function and* g *and* h *be any two entire functions such that* (*i*) $\rho_h^{L^*}(f) < \infty$, (*ii*) $\lambda_h^{L^*}(f) = \lambda_g^{L^*}$, (*iii*) $\overline{\tau}_g^{L^*} < \infty$ *and* (*iv*) $\overline{\tau}_h^{L^*}(f) > 0$. *Also let* h *satisfy the Property (A). Then for any* $\delta > 1$,
(*a*) *if* $L\left(M_g(r)\right) = o\left\{T_h^{-1} T_f(r)\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r) + L\left(M_g(r)\right)} \leq \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \overline{\tau}_g^{L^*}}{\overline{\tau}_h^{L^*}(f)}$$

and (*b*) *if* $T_h^{-1} T_f(r) = o\left\{L\left(M_g(r)\right)\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r) + L\left(M_g(r)\right)} \leq \delta \cdot \rho_h^{L^*}(f) \ .$$

Theorem 3.26. *Let* f *be a meromorphic function and* g *and* h *be any two entire functions such that* (*i*) $\rho_h^{L^*}(f) < \infty$, (*ii*) $\lambda_h^{L^*}(f) = \lambda_g^{L^*}$, (*iii*) $\tau_g^{L^*} < \infty$ *and* (*iv*) $\tau_h^{L^*}(f) > 0$. *Also let* h *satisfy the Property (A). Then for any* $\delta > 1$,
(*a*) *if* $L\left(M_g(r)\right) = o\left\{T_h^{-1} T_f(r)\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r) + L\left(M_g(r)\right)} \leq \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \tau_g^{L^*}}{\tau_h^{L^*}(f)}$$

and (*b*) *if* $T_h^{-1} T_f(r) = o\left\{L\left(M_g(r)\right)\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r) + L\left(M_g(r)\right)} \leq \delta \cdot \rho_h^{L^*}(f) \ .$$

Analogously we may state the following four theorems under some different conditions which can also be carried out using the same technique of Theorem 3.19 and therefore their proofs are omitted.

Theorem 3.27. *Let* f *be a meromorphic function and* g *and* h *be any two entire functions such that* (*i*) $\rho_h^{L^*}(f) < \infty$, (*ii*) $\lambda_h^{L^*}(f) = \rho_g^{L^*}$, (*iii*) $\sigma_g^{L^*} < \infty$ *and* (*iv*) $\tau_h^{L^*}(f) > 0$. *Also let* h *satisfy the Property (A). Then for any* $\delta > 1$,
(*a*) *if* $L\left(M_g(r)\right) = o\left\{T_h^{-1} T_f(r)\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r) + L\left(M_g(r)\right)} \leq \delta \cdot \frac{\rho_h^{L^*}(f) \cdot \sigma_g^{L^*}}{\tau_h^{L^*}(f)}$$

and (*b*) *if* $T_h^{-1} T_f(r) = o\left\{L\left(M_g(r)\right)\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r) + L\left(M_g(r)\right)} \leq \delta \cdot \rho_h^{L^*}(f) \ .$$

Theorem 3.28. *Let f be a meromorphic function and g and h be any two entire functions such that (i) $\lambda_h^{L^*}(f) = \rho_g^{L^*}$, (ii) $\sigma_g^{L^*} < \infty$ and (iii) $\tau_h^{L^*}(f) > 0$. Also let h satisfy the Property (A). Then for any $\delta > 1$,*
(a) if $L\left(M_g(r)\right) = o\left\{T_h^{-1}T_f(r)\right\}$ then

$$\liminf_{r\to\infty} \frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r) + L\left(M_g(r)\right)} \leq \delta \cdot \frac{\lambda_h^{L^*}(f)\cdot\sigma_g^{L^*}}{\tau_h^{L^*}(f)}$$

and (b) if $T_h^{-1}T_f(r) = o\left\{L\left(M_g(r)\right)\right\}$ then

$$\liminf_{r\to\infty} \frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r) + L\left(M_g(r)\right)} \leq \delta\cdot\lambda_h^{L^*}(f) .$$

Theorem 3.29. *Let f be a meromorphic function and g and h be any two entire functions such that (i) $\rho_h^{L^*}(f) = \lambda_g^{L^*}$, (ii) $\overline{\tau}_g^{L^*} < \infty$ and (iii) $\overline{\sigma}_h^{L^*}(f) > 0$. Also let h satisfy the Property (A). Then for any $\delta > 1$,*
(a) if $L\left(M_g(r)\right) = o\left\{T_h^{-1}T_f(r)\right\}$ then

$$\limsup_{r\to\infty} \frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r) + L\left(M_g(r)\right)} \leq \delta \cdot \frac{\rho_h^{L^*}(f)\cdot\overline{\tau}_g^{L^*}}{\overline{\sigma}_h^{L^*}(f)}$$

and (b) if $T_h^{-1}T_f(r) = o\left\{L\left(M_g(r)\right)\right\}$ then

$$\limsup_{r\to\infty} \frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r) + L\left(M_g(r)\right)} \leq \delta\cdot\rho_h^{L^*}(f) .$$

Theorem 3.30. *Let f be a meromorphic function and g and h be any two entire functions such that (i) $\rho_h^{L^*}(f) = \lambda_g^{L^*}$, (ii) $\overline{\tau}_g^{L^*} < \infty$ and (iii) $\overline{\sigma}_h^{L^*}(f) > 0$. Also let h satisfy the Property (A). Then for any $\delta > 1$,*
(a) if $L\left(M_g(r)\right) = o\left\{T_h^{-1}T_f(r)\right\}$ then

$$\liminf_{r\to\infty} \frac{\log T_h^{-1}T_{f\circ g}(r)}{nf\,T_h^{-1}T_f(r) + L\left(M_g(r)\right)} \leq \delta \cdot \frac{\lambda_h^{L^*}(f)\cdot\overline{\tau}_g^{L^*}}{\overline{\sigma}_h^{L^*}(f)}$$

and (b) if $T_h^{-1}T_f(r) = o\left\{L\left(M_g(r)\right)\right\}$ then

$$\liminf_{r\to\infty} \frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r) + L\left(M_g(r)\right)} \leq \delta\cdot\lambda_h^{L^*}(f) .$$

Theorem 3.31. *Let f be a meromorphic function and h be an entire function such that $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$. Also let g be an entire function. Then*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(\exp r^{\mu})} \geq \frac{\lambda_h^{L^*}(f)}{\rho_h^{L^*}(f)}$$

where $0 < \mu < \rho_g \leq \infty$.

Proof. In view of Lemma 2.2, we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f \circ g}(r) &\geq \log T_h^{-1} T_f(\exp r^\mu) \\ i.e., \log T_h^{-1} T_{f \circ g}(r) &\geq \left(\lambda_h^{L^*}(f) - \varepsilon\right)[r^\mu + L(\exp r^\mu)] . \end{aligned} \quad (3.8)$$

Also for any arbitrary $\varepsilon (> 0)$, it follows for all sufficiently large values of r that

$$\log T_h^{-1} T_f(\exp r^\mu) \leq \left(\rho_h^{L^*}(f) + \varepsilon\right)[r^\mu + L(\exp r^\mu)] . \quad (3.9)$$

Now from (3.8) and (3.9), we get for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(\exp r^\mu)} \geq \frac{\left(\lambda_h^{L^*}(f) - \varepsilon\right)[r^\mu + L(\exp r^\mu)]}{\left(\rho_h^{L^*}(f) + \varepsilon\right)[r^\mu + L(\exp r^\mu)]} .$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(\exp r^\mu)} \geq \frac{\lambda_h^{L^*}(f)}{\rho_h^{L^*}(f)} .$$

Thus the theorem follows. □

Theorem 3.32. *Let f be a meromorphic function, g be an entire function with non zero finite order and h be any entire function such that $0 < \lambda_f$ and $0 < \lambda_h^{L^*}(g) \leq \rho_h^{L^*}(g) < \infty$. Then*

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_g(\exp r^\mu)} \geq \frac{\lambda_h^{L^*}(g)}{\rho_h^{L^*}(g)}$$

where $0 < \mu < \rho_g$.

We omit the proof of the above theorem as it can be carried out in the line of Theorem 3.31 and with the help of Lemma 2.3.

Theorem 3.33. *Let f be a meromorphic function of non zero finite order and lower order and h be an entire function such that $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$. Also let g be an entire function. Then*

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(\exp r^\mu)} \leq \frac{\rho_h^{L^*}(f)}{\lambda_h^{L^*}(f)}$$

where $\lambda_g < \mu < \infty$.

Proof. In view of Lemma 2.4, we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f \circ g}(r) &< \log T_h^{-1} T_f(\exp r^\mu) \\ i.e., \log T_h^{-1} T_{f \circ g}(r) &< \left(\rho_h^{L^*}(f) + \varepsilon\right)[r^\mu + L(\exp r^\mu)] . \end{aligned} \quad (3.10)$$

Also for any arbitrary $\varepsilon (> 0)$, it follows for all sufficiently large values of r that

$$\log T_h^{-1} T_f(\exp r^\mu) \geq \left(\lambda_h^{L^*}(f) - \varepsilon\right)[r^\mu + L(\exp r^\mu)] . \quad (3.11)$$

Now from (3.10) and (3.11), we get for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(\exp r^\mu)} < \frac{\left(\rho_h^{L^*}(f)+\varepsilon\right)[r^\mu + L(\exp r^\mu)]}{\left(\lambda_h^{L^*}(f)-\varepsilon\right)[r^\mu + L(\exp r^\mu)]}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(\exp r^\mu)} \geq \frac{\rho_h^{L^*}(f)}{\lambda_h^{L^*}(f)}.$$

Thus the theorem follows. □

Now we state the following theorem without its proof as it can be carried out in the line of the above theorem and with the help of Lemma 2.5:

Theorem 3.34. *Let f be a meromorphic function of finite order and g, h be any two entire functions such that $0 < \lambda_h^{L^*}(g) \leq \rho_h^{L^*}(g) < \infty$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_g(\exp r^\mu)} \leq \frac{\rho_h^{L^*}(g)}{\lambda_h^{L^*}(g)}$$

where $0 < \lambda_g < \mu < \infty$..

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