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Markov moment problems in concrete spaces

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Abstract. In the first part of this work a necessary and sufficient condition and respectively a sufficient condition for the existence of a solution of a concrete Markov moment problem are given. The latter result also yields the uniqueness of the solution. Both these two results are based on previous theorems on the abstract moment problem, applied to concrete spaces. Namely, the space which the solution is defined on is a space of analytic functions of several variables, while the target space is an order complete Banach lattice of selfadjoint operators (the bicommutant). The purpose of the second part is to continue the previous study of the author on inverse problems related to the multidimensional Markov moment problem.

1. Introduction

The classical moment problem is an interpolation problem with a constraint on the positivity of the (linear) solution. Usually, the representation of the solution by means of a positive scalar or vector measure is pointed out. The Markov moment problem involves an additional upper constraint. This constraint controls the norm of the solution. In case of representation of the solution as an element h of an L^{∞} - space, the two constraints mentioned above lead to the conclusion $0 \leq h \leq b$ a.e., where b is a positive real constant which appears naturally from a computation. All the aspects on the classical moment problem are strongly related to the form of positive polynomials on the set under discussion [1], [2], [5], [23], [24], [29], [31], [33]. For the classical moment problem (respectively Markov moment problem) see [1] (respectively [11]). Meanwhile, connections of the moment problem with some other fields have been done [4]. In the latter work, elements of fixed point theory related to the moment problem are applied. For operator valued moment problems see [16] - [19], [33]. Some other connections to operator theory are revealed in [29], [33]. The latter work contains also solutions of moment problems on unbounded subsets. Three main aspects are studied in the moment problem: existence, uniqueness and construction of the solution. The first part of the present work focuses on the existence problem. One gives necessary

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and sufficient or only sufficient conditions for the existence of a solution. In some cases, the uniqueness of the solution follows too. Most of these results are based on Hahn - Banach type arguments [1], [2], [5], [6], [11] - [15], [19], [21], [23] - [29], [31], [33]. In the first part of the present work generalizations of Hahn - Banach theorem are applied (see [21] - [25]). The two following sections of the present work solve Markov moment problems, by quite different methods. However, the Hahn - Banach theorem and its generalizations are applied in Section 2 (theorems 2.1, 2.3). The whole work is based on order vector structures on concrete spaces and positive linear operators between such spaces. Some of our solutions are also unique by their continuity, using the density of the subspace of polynomials in the concerned function spaces. The first purpose of this work is to apply earlier results [23] on the abstract Markov moment problem to some concrete moment problems in spaces of analytic functions. The second purpose of this work is to improve and complete previous results on inverse problems [28]. For the uniqueness of the solution, see [3], [9], [10], [32]. For the construction of a solution see [19]. The background of the paper is contained in some chapters from [7], [8], [20], [30]. The rest of this work is organized as follows. Section 2 contains two new applications (Theorems 2.2, 2.4) of some earlier results to the Markov moment problem on concrete spaces. Section 3 is devoted to an inverse problem related to a multidimensional moment problem. One continues the study from [28] on this subject.

2. Moment problems in spaces of analytic functions

Theorem 2.1 (Theorem 4 [23]). . *Let X be an ordered vector space, Y an order complete vector lattice, $\{x_j\}_{j\in J} \subset X$, $\{y_j\}_{j\in J} \subset Y$ given families and $F_1, F_2 \in L(X,Y)$ two linear operators. The following statements are equivalent*

(a) there exists a linear operator $F \in L(X,Y)$ such that

$$F_1(x) \le F(x) \le F_2(x), \quad \forall x \in X_+,\ F(x_j) = y_j, \quad \forall j \in J;$$

(b) for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j\in J_0} \subset \mathbb{R}$, we have:

$$\left(\sum_{j\in J_0} \lambda_j x_j = \psi_2 - \psi_1,\ \psi_1, \psi_2 \in X_+\right) \Rightarrow \sum_{j\in J_0} \lambda_j y_j \le F_2(\psi_2) - F_1(\psi_1).$$

Let the space X be formed by all power series having real coefficients, absolutely convergent in the closed polydisc

$$\bar{D}_r = \{z = (z_1, \ldots, z_n) : |z_p| \le r_p,\ p \in \{1, \ldots, n\}\}, \quad r := (r_1, \ldots, r_n).$$

The order relation on the space X is defined by the convex cone of those power series with all their coefficients nonnegative numbers. The norm on X is defined by

$$\|\varphi\|_\infty = \sup\{|\varphi(z)| : z \in \bar{D}_r\}.$$

Denote

$$h_k(z) = z_1^{k_1} \ldots z_n^{k_n}, \quad k = (k_1, \ldots, k_n) \in \mathbb{N}^n, \quad z \in \bar{D}_r.$$

On the other hand, let H be a complex Hilbert space, $\mathcal{A}$ the real vector space of all self adjoint operators acting on $H, U_0 \in \mathcal{A}$. Define

$$Y_1 := \{A \in \mathcal{A} : AU_0 = U_0 A\}, \quad Y := \{U \in Y_1 : UV = VU, \quad \forall\, V \in Y_1\},$$
$$Y_+ := \{U \in Y : \langle Uh, h\rangle \ge 0, \quad \forall\, h \in H\}.$$

Endowed with the order relation defined by this cone, and with the operatorial norm, Y is an order complete Banach lattice [8] (which is also a commutative algebra of self - adjoint operators). Notice also that the norm on Y is solid: $|U| \leq |V| \Rightarrow \|U\| \leq \|V\|$, $U, V \in Y$. Here $|U| = \sup\{U, -U\} = \sqrt{U^2}$, $\|U\| = \sup_{\|h\|=1} \|U(h)\| = \sup_{\|h\|=1} |\langle U(h), h\rangle|$, for any self adjoint operator U in Y. Let $(B_k)_{k\in N^n}$ be a multi - indexed sequence of operators in Y. Assume that $A_p \in Y$, $0 \leq A_p \leq r_p I$, $p \in \{1, \ldots, n\}$.

Theorem 2.2. *Let $a > 0$. The following statements are equivalent*
(a) there exists a linear operator F applying X into Y, such that

$$F(h_k) = B_k, \quad \forall\, k \in \mathbb{N}^{\mathbf{n}}, \quad |F(\psi)| \leq a \cdot \psi(A_1, \ldots, A_n), \quad \forall\, \psi \in X_+;$$

(b) for any finite subset $J_0 \subset \mathbb{N}^{\mathbf{n}}$, and any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, we have

$$\sum_{j\in J_0} \lambda_j B_j \leq a \cdot \left(\sum_{j\in J_0} |\lambda_j| A_1^{j_1} \ldots A_n^{j_n} \right).$$

Proof. The implication (a) $\Rightarrow$ (b) is almost obvious:

$$\sum_{j\in J_0} \lambda_j B_j = F\left(\sum_{j\in J_0} \lambda_j h_j \right) \leq \sum_{j\in J_0} |\lambda_j| |F(h_j)| \leq a \cdot \left(\sum_{j\in J_0} |\lambda_j| h_j(A_1, \ldots, A_n) \right)$$

$$= a \cdot \left(\sum_{j\in J_0} |\lambda_j| A_1^{j_1} \ldots A_n^{j_n} \right), \quad j = (j_1, \ldots, j_n) \in \mathbb{N}^n.$$

To prove the converse, we apply Theorem 2.1, (b) $\Rightarrow$ (a). Let us verify the implication stated at point (b), Theorem 2.1, where h_j stands for x_j, B_j stands for y_j, $\forall\, j \in \mathbb{N}^n =: J$. If

$$\sum_{j\in J_0} \lambda_j h_j = \psi_2 - \psi_1 = \sum_{k\in\mathbb{N}^n} \alpha_k h_k - \sum_{k\in\mathbb{N}^n} \beta_k h_k, \quad \alpha_k \geq 0,\ \beta_k \geq 0,\ \forall\, k \in \mathbb{N}^n,$$

then

$$\lambda_j = \alpha_j - \beta_j,\ j \in J_0 \Rightarrow |\lambda_j| \leq \alpha_j + \beta_j,\ j \in J_0.$$

The hypothesis (b) of the present theorem yields

$$\sum_{j\in J_0} \lambda_j B_j \leq a \left(\sum_{j\in J_0} |\lambda_j| A_1^{j_1} \ldots A_n^{j_n} \right)$$

$$\leq a \left(\sum_{j\in J_0} \alpha_j A_1^{j_1} \ldots A_n^{j_n} \right) + a \left(\sum_{j\in J_0} \beta_j A_1^{j_1} \ldots A_n^{j_n} \right) \leq$$

$$a \left(\sum_{k\in\mathbb{N}^n} \alpha_k A_1^{k_1} \ldots A_n^{k_n} \right) + a \left(\sum_{k\in\mathbb{N}^n} \beta_k A_1^{k_1} \ldots A_n^{k_n} \right) = F_2(\psi_2) - F_1(\psi_1),$$

$$F_2(\psi) = F_2 \left(\sum_{k\in\mathbb{N}^n} \alpha_k h_k \right) := a \sum_{k\in\mathbb{N}^n} \alpha_k A_1^{k_1} \ldots A_n^{k_n},\ k = (k_1, \ldots, k_n),\ F_1 := -F_2.$$

Hence the conditions mentioned at point (b) Theorem 2.1 are verified, so that there exists a linear operator F from X into Y such that the moment conditions $F(h_k) = B_k$,

$\forall\, k \in \mathbb{N}^n$ are verified and F is between F_1 and F_2 on the positive cone of X. The latter condition can be written as

$$-a\psi(A_1,\ldots,A_n) = -a\sum_{k\in\mathbb{N}^n}\alpha_k A_1^{k_1}\ldots A_n^{k_n} = F_1(\psi) \le F(\psi) \le F_2(\psi) =$$

$$a\sum_{k\in\mathbb{N}^n}\alpha_k A_1^{k_1}\ldots A_n^{k_n} = a\cdot\psi(A_1,\ldots,A_n), \quad \psi \in X_+\,,$$

that is

$$|F(\psi)| \le a\cdot\psi(A_1,\ldots,A_n), \quad \forall\,\psi \in X_+\,.$$

This concludes the proof. □

We go on with an application of another variant of the abstract moment problem.

Theorem 2.3 (Theorem 1 [23])**.** *. Let X, Y be as in Theorem 2.1, $\{x_j\}_{j\in J} \subset X$, $\{y_j\}_{j\in J} \subset Y$ be given families, $T : X \to Y$ a convex operator. The following assertions are equivalent:*
(a) there exists a linear positive operator $F : X \to Y$ such that

$$F(x_j) = y_j\ \forall j \in J,\ F(x) \le T(x) \quad \forall\, x \in X;$$

(b) for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j\in J_0} \subset \mathbb{R}$, we have:

$$\sum_{j\in J_0}\lambda_j x_j \le x \Rightarrow \sum_{j\in J_0}\lambda_j y_j \le T(x).$$

Here is a new application of Theorem 2.3. The space Y is as mentioned above. The space X consists in all absolutely convergent power series in the closed unit polydisc

$$\bar{D}_1 = \{z = (z_1,\ldots,z_n);\ |z_p| \le 1, \quad p \in 1,\ldots,n\},$$

having real coefficients. The positive cone of X consists in all such power series having all nonnegative coefficients. The norm on X is defined by

$$\|h\|_\infty = \sup\{|h(z)|;\ z \in \bar{D}_1\}, \quad h \in X.$$

As we have mentioned above, one denotes by h_k the basic polynomials

$$h_k(z) = z_1^{k_1}\ldots z_n^{k_n}, \quad k = (k_1,\ldots,k_n) \in \mathbb{N}^n,\ z \in \bar{D}_1.$$

Let $A_1,\ldots,A_n$ be positive operators in Y, such that $\|A_p\| < 1$, $\forall\, p \in \{1,\ldots,n\}$, $a > 0$ a real number and $\{B_k\}_{k\in\mathbb{N}^n}$ a multi - indexed sequence in Y.

Theorem 2.4. *Assume that $0 \le B_k \le aA_1^{k_1}\ldots A_n^{k_n}$ for all $k = (k_1,\ldots,k_n) \in \mathbb{N}^n$. Then there exists a unique linear bounded positive operator $F \in B_+(X,Y)$ such that*

$$F(h_k) = B_k,\ k \in \mathbb{N}^n,\ F(\varphi) \le a\|\varphi\|_\infty\prod_{p=1}^{n}(I - A_p)^{-1},$$

$$\varphi \in X,\ \|F\| \le a\prod_{p=1}^{n}(1 - \|A_p\|)^{-1}.$$

Proof. . We apply Theorem 2.3 (b) $\Rightarrow$ (a), to $x_j = h_j$, $y_j = B_j$, $j \in J := \mathbb{N}^n$. To this end, we verify the implication mentioned at point (b) of the latter theorem. Cauchy's inequalities lead to

$$\sum_{j \in J_0} \lambda_j h_j \leq \varphi = \sum_{k \in \mathbb{N}^n} \beta_k h_k \Rightarrow \lambda_j \leq \beta_j \leq |\beta_j| \leq \frac{\|\varphi\|_\infty}{(1-\varepsilon)^{j_1+\ldots+j_n}}$$

for all positive $\varepsilon < \min\{1 - \|A_p\|;\ p \in \{1,\ldots,n\}\}$ and all $j \in J_0$. Making $\varepsilon \downarrow 0$, we get $\lambda_j \leq \|\varphi\|_\infty$, $\forall\, j \in J_0$. Now application of the relations verified by the operators B_j, $j \in J_0$ yields

$$\lambda_j B_j \leq \|\varphi\|_\infty B_j \leq a\|\varphi\|_\infty A_1^{j_1} \ldots A_n^{j_n},\ j \in J_0 \Rightarrow$$

$$\sum_{j \in J_0} \lambda_j B_j \leq a\|\varphi\|_\infty \sum_{j \in J_0} A_1^{j_1} \ldots A_n^{j_n} \leq a\|\varphi\|_\infty \left(\sum_{k_1 \in \mathbb{N}} A_1^{k_1}\right) \ldots \left(\sum_{k_n \in \mathbb{N}} A_n^{k_n}\right) =$$

$$a\|\varphi\|_\infty \prod_{p=1}^{n} (I - A_p)^{-1} =: T(\varphi).$$

Thus, the implication (b) from Theorem 2.3 is verified. Application of the latter theorem, leads to the existence of a linear positive operator $F : X \to Y$, such that the interpolation conditions

$$F(h_k) = B_k, \quad k \in \mathbb{N}^n$$

are verified and $F(\varphi) \leq T(\varphi) = a\|\varphi\|_\infty \prod_{p=1}^{n} (I - A_p)^{-1}$ for all φ in X. In particular, the last relations yield

$$\|F\| \leq a \prod_{p=1}^{n} \left\| \sum_{m=0}^{\infty} A_p^m \right\| \leq a \prod_{p=1}^{n} \left(\sum_{m=0}^{\infty} \|A_p\|^m \right) = a \prod_{p=1}^{n} (1 - \|A_p\|)^{-1}.$$

The uniqueness of the solution follows from its continuity, also using the density of polynomials in the space X. This concludes the proof. □

3. An inverse problem related to a Markov moment problem

In this section we continue the study from [28] on the inverse problems related to the multidimensional Markov moment problem. The next result furnishes the solution on whose moments the inverse problem will be based on. Consider the following measure on $\mathbb{R}_+^n$

$$d\nu = \exp\left(-\sum_{j=1}^{n} t_j\right) dt, \quad dt = dt_1 \ldots dt_n,\ n \in \mathbb{N},\ n \geq 2.$$

Let

$$\varphi_j(t) = t_1^{j_1} \ldots t_n^{j_n},\ j = (j_1,\ldots,j_n) \in \mathbb{N}^n,\ t_j \geq 0,\ \{m_j\}_{j \in \mathbb{N}^n} \subset \mathbb{R}.$$

Starting from the given numbers m_j, $j \in \mathbb{N}^n$, assume that there exists $h \in L_\nu^\infty(\mathbb{R}_+^n)$, $0 \leq h \leq 1$ such that m_j are the classical moments of the function h: $m_j = \int_{\mathbb{R}_+^n} h\varphi_j d\nu$, $j \in \mathbb{N}^n$. One can solve the following inverse problem (see Theorem 3.1. from below).

Step 1. Find an approximation of the solution h in terms of the moments m_j, $j \in \mathbb{N}^n$. To this end, since $h \in L^\infty_\nu(\mathbb{R}^n_+) \subset L^2_\nu(\mathbb{R}^n_+)$, h has a Fourier expansion with respect to the Hilbert base $(\psi_j)_{j_k \geq 0}$ associated following Gramm-Schmidt procedure to the complete system of linearly independent polynomials $(\varphi_j)_{j_k \geq 0}$. The Fourier coefficients $\langle h, \psi_j \rangle$ are given by:

$$\langle h, \psi_j \rangle = \sum_{\substack{l_k \leq j_k, \\ k=0,\dots,n}} \alpha_l \langle h, \varphi_l \rangle = \sum_{\substack{l_k \leq j_k, \\ k=0,\dots,n}} \alpha_l m_l,$$

where α_l are given by the Gram-Schmidt procedure, so that we know h in terms of the moments. Recall that there exists a subsequence of the sequence of Fourier partial sums, which converges pointwise to h [30]. One can write: $h \approx \tilde{h}$, where $\tilde{h}$ is a partial sum of the Fourier series of h. Note that all these partial sums are polynomials, so that they are continuous.

Step 2. Let $\tilde{h}$ be a partial sum of the Fourier series of h with respect to the orthogonal polynomials $(\psi_j)_{j_k \geq 1}$. Using Schwarz inequality, and approximation of continuous functions by simple functions, one deduces

$$\begin{aligned} m_j &\approx \int_A \varphi_j \tilde{h}\, d\nu \approx \int_A \varphi_j \left(\sum_{p,q} c_{p,q} \cdot \chi_{D_{p,q}}(t_1, \dots, t_n) \right) d\nu = \\ &= \sum_{p,q} c_{p,q} \left(\sum_{m \in \mathbb{N}} \int_A \varphi_j \cdot \chi_{[x_{1,m,p,q}, y_{1,m,p,q})}(t_1) \dots \chi_{[x_{n,m,p,q}, y_{n,m,p,q})}(t_n) d\nu \right) = \\ &= \sum_{p,q} c_{p,q} \left(\sum_{m \in \mathbb{N}} \int_{x_{1,m,p,q}}^{y_{1,m,p,q}} t_1^{j_1} \exp(-t_1) dt_1 \dots \int_{x_{n,m,p,q}}^{y_{n,m,p,q}} t_n^{j_n} \exp(-t_n) dt_n \right), \\ & j \in \mathbb{N}^n. \end{aligned} \tag{3.1}$$

$$\tilde{h}(t_1, \dots, t_n) \approx \sum_{p,q \leq M} c_{p,q} \chi_{D_{p,q}(t_1, \dots, t_n)},$$

Consider the system of equations with the unknowns $x_{1,m,p,q}, y_{1,m,p,q}, \dots, x_{n,m,p,q}, y_{n,m,p,q}$:

$$\begin{aligned} m_j = \sum_{p,q} c_{p,q} \Bigg(\sum_{m \in N} & \int_{x_{1,m,p,q}}^{y_{1,m,p,q}} t_1^{j_1} \exp(-t_1) dt_1 \dots \\ & \dots \int_{x_{n,m,p,q}}^{y_{n,m,p,q}} t_n^{j_n} \exp(-t_n) dt_n \Bigg), \ j \in \mathbb{N}^n. \end{aligned} \tag{3.2}$$

The above computation (3.1) involves terms appearing from the approximation of the polynomial $\tilde{h}$ by simple functions [30] (the last relation (3.1) from above), where p is large and m_q is suitable chosen for approximating $\tilde{h}$. For a one - dimensional example see [24]. The conclusion is that we can determinate (approximating) the "unknowns" $y_{k,m,p,q}, x_{k,m,p,q}$, $k = 1, \dots, n$ by means of the cell decomposition of the open subsets $D_{p,q}$ associated to the known polynomial $\tilde{h}$ (cf. [30, section 2.19]). In the above computation, we have applied Fubini's theorem on each cell from the cell decomposition of a measure - approximating open subset $D_{p,q}$,

$$D_{p,q} = \bigcup_{m \in N} [x_{1,m,p,q}, y_{1,m,p,q}) \times \ldots \times [x_{n,m,p,q}, y_{n,m,p,q}) \supset$$

$$\supset \left\{ (t_1, \ldots, t_n); \frac{m_q}{2^p} \leq \tilde{h}(t_1, \ldots, t_n) < \frac{m_q + 1}{2^p} \right\},$$

$$\nu(D_{p,q}) \approx \nu\left(\left\{ (t_1, \ldots, t_n); \frac{m_q}{2^p} \leq \tilde{h}(t_1, \ldots, t_n) < \frac{m_q + 1}{2^p} \right\} \right),$$

where $p, m_q, c_{p,q}$ (from (3.2)) are suitable chosen in order to approximate $\tilde{h}$ by simple functions. Such open subsets $D_{p,q}$ do exist since the measure ν is outer regular [30]. We have proved the following result.

Theorem 3.1. *If one considers the system (3.2) with the unknowns $x_{k,m,p,q}$, $y_{k,m,p,q}$, $k \in \{1, \ldots, n\}$, $p, q \in \mathbb{N}$ suitable chosen, then the unknowns are the coordinates of the vertices of the cells from the cell-decomposition of the subsets $D_{p,q}$. The numbers $c_{p,q}$ are the values of the known polynomial $\tilde{h}$ from Step 1 at some points in the "known" subsets*

$$\left\{ (t_1, \ldots, t_n); \frac{m_q}{2^p} \leq \tilde{h}(t_1, \ldots, t_n) < \frac{m_q + 1}{2^p} \right\}.$$

Clearly, the solution is not unique (a big "semi - closed" cell can be decomposed as a joint of smaller such disjoint sub - cells).

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