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On hyponormal operators and related classes of operators

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Abstract. There are many classes of operators defined based on operator inequalities. The class of hyponormal operators is perhaps the first such class of operators. An operator T defined on a Hilbert space H is called hyponormal if $T^*T \geq TT^*$, where T^* is the adjoint of the operator T. Since then, many other classes of operators have been introduced, mainly by weakening or modifying the inequality $T^*T \geq TT^*$ in various ways. Among these classes of operators are semi-hyponormal operators, p-hyponormal operators (for $0 < p < 1/2$), log-hyponormal operators , and w-hyponormal operators. Hyponormal operators possess many interesting properties. And the other classes of operators possess the same properties or somewhat weaker properties. In this paper, we will give a review of these classes of operators and their properties.

1. Introduction

Let H be a separable Hilbert Space and let $L(H)$ denote the algebra of bounded linear operators on H. The adjoint operator T^* of an operator T on H is defined by $\langle T^*x, y\rangle = \langle x, Ty\rangle$ for all x and y in H. An operator $T \in L(H)$ is called self-adjoint if $T^* = T$ or equivalently if $\langle Tx, x\rangle$ is real for all $x \in H$. A self-adjoint operator $T \in L(H)$ is called positive semi-definite, denoted $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for all $x \in H$. An operator $T \in L(H)$ is called normal if $T^*T = TT^*$ (or equivalently $||T^*x|| = ||Tx||$ for all $x \in H$) and hyponormal if $T^*T \geq TT^*$ (or equivalently $||T^*x|| \leq ||Tx||$ for all $x \in H$). An operator $U \in L(H)$ is called unitary if $U^*U = I = UU^*$, where I is the identity operator on H, an isometry if $U^*U = I$ or equivalently, $||Ux|| = ||x||$ for all $x \in H$, and a partial isometry if U^*U is projection from H to a subspace of H. The polar decomposition of an operator $T \in L(H)$ is denoted by $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry from $\overline{R(|T|)}$ to $\overline{R(T)}$ such that $N(U) = N(|T|)$ with $\overline{N(\,)}$ and $\overline{R(\,)}$ being the closures of null space and the range, respectively, of an operator. The spectrum of an operator T is defined by $\sigma(T) = \{z \in \mathbb{C} : (T - zI)^{-1}\text{does not exist}\}$ and the spectral radius defined by $r_{sp} = \sup\{|z| : z \in \sigma(T)\}$.

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Example 1.1 (A Normal Operator [15])**.** Let H be the Hilbert space of all two-way (bilateral) square-summable sequences. The elements of H can be written in the form $\xi = \langle ..., \xi_{-2}, \xi_{-1}, (\xi_0), \xi_1 \xi_2, ...\rangle$. Define the forward bilateral shift V on H as $V\langle ..., \xi_{-2}, \xi_{-1}, (\xi_0), \xi_1, \xi_2, ...\rangle = \langle ..., \xi_{-2}, (\xi_{-1}), \xi_0, \xi_1, \xi_2, ...\rangle$. Then the adjoint V^* is the backward bilateral shift is given by $V^*\langle ..., \xi_{-2}, \xi_{-1}, (\xi_0), \xi_1, \xi_2, ...\rangle = \langle ..., \xi_{-2}, \xi_{-1}, \xi_0, (\xi_1), \xi_2, ...\rangle$. Then for any $\xi \in H$, we have,

$$\begin{aligned} V^*V\xi &= V^*(V\langle ..., \xi_{-2}, \xi_{-1}, (\xi_0), \xi_1, \xi_2, ...\rangle) \\ &= V^*\langle ..., \xi_{-2}, (\xi_{-1}), \xi_0, \xi_1, \xi_2, ...\rangle \\ &= \langle ..., \xi_{-2}, \xi_{-1}, (\xi_0), \xi_1, \xi_2, ...\rangle = \xi. \end{aligned}$$

Similarly, we have

$$\begin{aligned} VV^*\xi &= V(V^*\langle ..., \xi_{-2}, \xi_{-1}, (\xi_0), \xi_1, \xi_2, ...\rangle) \\ &= V\langle ..., \xi_{-2}, \xi_{-1}, \xi_0, (\xi_1), \xi_2, ...\rangle \\ &= \langle ..., \xi_{-2}, \xi_{-1}, (\xi_0), \xi_1, \xi_2, ...\rangle = \xi. \end{aligned}$$

Therefore, $V^*V = I = VV^*$ and V is normal. In fact, in this example, the bilateral shift V is a unitary operator.

Normal operators have been studied by many authors including P. Halmos [15], C. R. Putnum [22], and D. Xia [25].

Example 1.2 (A Hyponormal Operator [15])**.** Let H be the Hilbert space of one-way (unilateral) square-summable sequences. The elements of H can be written in the form $\xi = \langle \xi_0, \xi_1, \xi_2, ...\rangle$. Define the forward unilateral shift U on H as $U\langle \xi_0, \xi_1, \xi_2, ...\rangle = \langle 0, \xi_0, \xi_1, \xi_2, ...\rangle$. Then the adjoint U^* is the backward unilateral shift is given by $U^*\langle \xi_0, \xi_1, \xi_2, \xi_3, ...\rangle = \langle \xi_1, \xi_2, \xi_3, ...\rangle$. For any $\xi \in H$, it can easily seen that

$$\begin{aligned} U^*U\xi &= U^*(U\langle \xi_0, \xi_1, \xi_2, \xi_3, ...\rangle) \\ &= U^*\langle 0, \xi_0, \xi_1, \xi_2, ...\rangle \\ &= \langle \xi_0, \xi_1, \xi_2, ...\rangle = \xi. \end{aligned}$$

Similarly, we have

$$\begin{aligned} UU^*\xi &= U(U^*\langle \xi_0, \xi_1, \xi_2, \xi_3, ...\rangle) \\ &= U\langle \xi_1, \xi_2, \xi_3 ...\rangle \\ &= \langle 0, \xi_1, \xi_2, ...\rangle. \end{aligned}$$

It can be shown that $\langle U^*U\xi, \xi\rangle = ||\xi||^2$ and $\langle UU^*\xi, \xi\rangle = ||\xi||^2 - |\xi_0|^2$. Therefore, $\langle U^*U\xi, \xi\rangle \geq \langle UU^*\xi, \xi\rangle$ for every $\xi \in H$ and hence $U^*U \geq UU^*$. Therefore, the unilateral shift U is hyponormal.

Remark 1.3. : It is easily seen that every normal operator is hyponormal. But Example 1.2 shows that the unilateral shift is hyponormal, but not normal.

Hyponormal operators have been studied by many authors including P. Halmos [15], J. G. Stampfli [23], D. Xia [25], and T. Furuta [14].

1.1. Some Important Operator Inequalities. Since the classes of operators discussed in this work are defined based on operator inequalities, the following two operator inequalities are very important.

Lemma 1.4 (Löwner-Heinz inequality [20]). *For two operators A and B on a Hilbert Space H, if $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ for $0 \leq \alpha \leq 1$.*

Remark 1.5. : But the above result does not necessarily hold for $\alpha > 1$. For example, $A \geq B \geq 0$ does not always imply $A^2 \geq B^2$ in general. For example, let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $A \geq B \geq 0$, but $A^2 \ngeq B^2$.

Lemma 1.6 (Furuta Inequality [12]). *Let A and B be positive semi-definite operators on a Hilbert space H such that $A \geq B \geq 0$. Then for each $r \geq 0$,*

$$(B^r A^p B^r)^{\frac{1}{q}} \geq B^{\frac{p+2r}{q}} \quad \text{and} \quad A^{\frac{p+2r}{q}} \geq (A^r B^p A^r)^{\frac{1}{q}}$$

hold for all $p \geq 0$, $q \geq 1$ such that $(1+2r)q \geq p + 2r$.

Remark 1.7. : When $r = 0$, Furuta inequality becomes the Löwner-Heinz inequality.

Lemma 1.8 (Hansen's Inequality [16]). *If A is a positive semi-definite operator and E is contraction on a separable Hilbert space H, then for any $0 < \alpha \leq 1$, $E^* A^\alpha E \leq (E^* A E)^\alpha$.*

2. p-Hyponormal Operators

An operator $T \in L(H)$ is called p-hyponormal if $(T^*T)^p \geq (TT^*)^p$, where $p > 0$. Let $T = U|T|$ be the polar decomposition of T. Then if T is p-hyponormal, we have

$$U^*|T|^{2p}U \geq |T|^{2p} \geq U|T|^{2p}U^*$$

If $p = 1$, then T is hyponormal. If $p = 1/2$, then T is called semi-hyponormal. If $T = U|T|$ is semi-hyponormal, then $U^*|T|U \geq |T| \geq U|T|U^*$. By Löwner-Heinz inequality, if T is p-hyponormal, then it is q-hyponormal for $q \leq p$. Clearly, the class of p-hyponormal operators has been introduced by weakening the definition of hyponormal operators. So any hyponormal operator is semi-hyponormal. It can be shown that if T is invertible and p-hyponormal, then so is its inverse T^{-1}. Study of semi-hyponormal operators was initiated by D. Xia [25]. Study of p-hyponormal operators for $0 < p < 1/2$ was initiated by this author [1]. It is shown in [1] and [2] that p-hyponormal operators for $1/2 < p < 1$ possess many of the properties the hyponormal operators possess and p-hyponormal operators for $0 < p < 1/2$ possess many of the properties the semi-hyponormal operators possess.

2.1. The Operator Transform $\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. The operator transform

$$\widetilde{T} = \Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$$

was introduced by this author [1] in the study of p-hyponormal operators. By applying this operator transform, we are able to extend or generalize many properties of hyponormal and semi-hyponormal operators to p-hyponormal operators for $0 < p < 1/2$. The operator $\Delta(T)$ has many important properties: The operator T is invertible if and only if $\Delta(T)$ is invertible. Both T and $\Delta(T)$ have the same spectrum. Other important properties of $\Delta(T)$ will be introduced later in this survey.

2.2. Polar Decomposition of the Operator Transform $\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. We apply the following lemma to obtain a polar decomposition of $\Delta(T)$.

Lemma 2.1 ([21])**.** *Let E and F be positive semi-definite operators, and assume that E^{-1} exists. Then a necessary and sufficient condition for the existence of a positive operator S such that $SES = F$ is that $(E^{\frac{1}{2}}FE^{\frac{1}{2}})^{\frac{1}{2}} \leq aE$ for some $a > 0$. If this condition is satisfied, then the operator S is uniquely given by $S = E^{-\frac{1}{2}}(E^{\frac{1}{2}}FE^{\frac{1}{2}})^{\frac{1}{2}}E^{-\frac{1}{2}}$.*

Theorem 2.2. *Let $T = U|T|$ be a p-hyponormal operator and assume that $|T|^{-1}$ exists. If $\Delta(T) = \widetilde{U}|\Delta(T)|$ is a polar decomposition of $\Delta(T)$, then*

$$|\Delta(T)| = |T|^{\frac{1}{2}}S^{-1}|T|^{\frac{1}{2}} \quad \text{and} \quad \widetilde{U} = |T|^{\frac{1}{2}}US|T|^{-\frac{1}{2}},$$

where S is the unique positive operator such that $SU^*|T|US = |T|$.

Proof. If $\Delta(T) = \widetilde{U}|\Delta(T)|$ is the polar decomposition of $\Delta(T)$, then

$$|\Delta(T)| = (\Delta(T)^*\Delta(T))^{\frac{1}{2}} = (|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}}.$$

Let $E = U^*|T|U$ and $F = |T|$. Then p-hyponormality of T implies $(E^{\frac{1}{2}}FE^{\frac{1}{2}})^{\frac{1}{2}} \leq E$ by Lemma 1.6. So by the previous lemma, there is a unique positive operator S such that

$$SES = F \quad \text{or equivalently} \quad SU^*|T|US = |T|.$$

Since both E and F are invertible, $S = E^{-\frac{1}{2}}(E^{\frac{1}{2}}FE^{\frac{1}{2}})^{\frac{1}{2}}E^{-\frac{1}{2}}$ is also invertible. Therefore, $SU^*|T|US = |T|$ implies $U^*|T|U = S^{-1}|T|S^{-1}$. Hence

$$|\Delta(T)| = \left(|T|^{\frac{1}{2}}S^{-1}|T|S^{-1}|T|^{\frac{1}{2}}\right)^{\frac{1}{2}} = |T|^{\frac{1}{2}}S^{-1}|T|^{\frac{1}{2}}.$$

Finally, $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = \Delta(T) = \widetilde{U}|T|^{\frac{1}{2}}S^{-1}|T|^{\frac{1}{2}}$ implies $\widetilde{U} = |T|^{\frac{1}{2}}US|T|^{-\frac{1}{2}}$. □

Though the above polar decomposition of $\Delta(T)$ is useful, Ito, Yamazaki and Yanagida [18] proved the following polar decomposition of $\Delta(T)$ which has even more applications.

Theorem 2.3 ([18])**.** *Let $T = U|T|$ be the polar decomposition of T and $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = V||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}|$ be the polar decomposition of $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}$. Then $\Delta(T) = VU|\Delta(T)|$ is the polar decomposition of $\Delta(T)$.*

Proof. First we notice that $|T^*| = (TT^*)^{\frac{1}{2}} = U|T|U^*$ and hence $|T^*|^{\frac{1}{2}} = U|T|^{\frac{1}{2}}U^*$ and $|T^*|^{\frac{1}{2}}U = U|T|^{\frac{1}{2}}$.

(i) Proof of $\Delta(T) = VU|\Delta(T)|$.

$$\begin{aligned} VU|\Delta(T)| &= VU(|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^{\frac{1}{2}}U^*U \\ &= V(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}}U \\ &= V||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}|U = |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U \\ &= |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = \Delta(T). \end{aligned}$$

□

(ii) Proof of the kernel condition $N(\Delta(T)) = N(VU)$.

$$\begin{aligned} VUx = 0 &\Leftrightarrow |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}Ux = 0 \quad \text{by} \quad N(V) = N(|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}) \\ &\Leftrightarrow |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = 0 \Leftrightarrow \Delta(T)x = 0. \end{aligned}$$

Therefore, $N(\Delta(T)) = N(VU)$.
(iii) The proof that VU is a partial isometry. By (ii), we have $N(VU)^{\perp} = N(|\Delta(T)|)^{\perp} = \overline{R(|\Delta(T)|)}$. Then for any $x \in N(VU)^{\perp}$, there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset H$ such that $x = \lim_{n\to\infty} |\Delta(T)|y_n$. From that we obtain,

$$\begin{aligned} ||VUx|| &= ||VU \lim_{n\to\infty} |\Delta(T)|y_n|| = || \lim_{n\to\infty} VU|\Delta(T)|y_n|| = || \lim_{n\to\infty} \Delta(T)y_n|| \quad \text{by (i)} \\ &= \lim_{n\to\infty} ||\Delta(T)y_n|| = \lim_{n\to\infty} ||\,|\Delta(T)|y_n|| = || \lim_{n\to\infty} |\Delta(T)|y_n|| = ||x||. \end{aligned}$$

That is, VU is a partial isometry. Therefore, the proof is complete.

Theorem 2.4 ([1])**.** *Let T be a p-hyponormal operator on a separable Hilbert space H. Then the operator $\Delta(T)$ is hyponormal if $1/2 \le p < 1$ and $(p+1/2)$-hyponormal if $0 < p < 1/2$.*

Proof. If T is a $p-$hyponormal operator for $1/2 \le p \le 1$, then T is semi-hyponormal. Therefore, we have

$$(T^*T)^{\frac{1}{2}} \ge (TT^*)^{\frac{1}{2}} \quad \text{or} \quad \text{equivalently} \quad U^*|T|U \ge |T| \ge U|T|U^*.$$

Thus,

$$\begin{aligned} \Delta(T)^*\Delta(T) - \Delta(T)\Delta(T)^* &= |T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}} - |T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}} \\ &= |T|^{\frac{1}{2}}(U^*|T|U - U|T|U^*)|T|^{\frac{1}{2}} \ge 0 \end{aligned}$$

and hence $\Delta(T)$ is hyponormal.

For the case $0 < p < 1/2$, we apply Furuta inequality. If $T = U|T|$ be a $p-$hyponormal operator for $0 < p < 1/2$, then, we have

$$(T^*T)^p \ge (TT^*)^p \quad \text{or} \quad \text{equivalently} \quad U^*|T|^{2p}U \ge |T|^{2p} \ge U|T|^{2p}U^*.$$

Let

$$A = U^*|T|^{2p}U, \quad B = |T|^{2p}, \quad \text{and} \quad C = U|T|^{2p}U^*.$$

Then, since $(1 + \frac{2}{4p})\frac{2}{2p+1} = \frac{1}{p} = \frac{1}{2p} + \frac{2}{4p}$, we have by Furuta inequality,

$$\begin{aligned} \big(\Delta(T)^*\Delta(T)\big)^{(p+\frac{1}{2})} &= (|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^{(p+\frac{1}{2})} = (B^{\frac{1}{4p}}A^{\frac{1}{2p}}B^{\frac{1}{4p}})^{(p+\frac{1}{2})} \\ &\ge B^{(\frac{1}{2p}+\frac{2}{4p})(p+\frac{1}{2})} = B^{(1+\frac{1}{2p})} = |T|^{2p+1}. \end{aligned}$$

Similarly,

$$\begin{aligned} \big(\Delta(T)\Delta(T)^*\big)^{(p+\frac{1}{2})} &= (|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}})^{(p+\frac{1}{2})} = (B^{\frac{1}{4p}}C^{\frac{1}{2p}}B^{\frac{1}{4p}})^{(p+\frac{1}{2})} \\ &\le B^{(\frac{1}{2p}+\frac{2}{4p})(p+\frac{1}{2})} = B^{(1+\frac{1}{2p})} = |T|^{2p+1}. \end{aligned}$$

Thus, $\big(\Delta(T)^*\Delta(T)\big)^{(p+\frac{1}{2})} \ge |T|^{2p+1} \ge \big(\Delta(T)\Delta(T)^*\big)^{(p+\frac{1}{2})}$ and hence $\Delta(T)$ is $(p+\frac{1}{2})$-hyponormal. □

Remark 2.5. : One of the main differences between hyponormal operators and p-hyponormal operators for $0 < p < 1$ is that hyponormal operators have the property that if T is hyponormal, then, for any $\lambda \in \mathbb{C}$, $T - \lambda I$ is also hyponormal. But p-hyponormal operators with $0 < p < 1/2$ do not have this property.

For any eigenvalue z of T, the subspace, $M_z = \{x \in H : Tx = zx\}$ of H is called the eigenspace of T corresponding to the eigenvalue z. A subset M of H reduces an operator T if $TM \subseteq M$ and $T^*M \subseteq M$. The following results are already shown for semi-hyponormal operators in Xia [25].

Theorem 2.6 ([1]). *Let $T = U|T|$ be p-hyponormal for $0 < p < 1/2$ and U be unitary. Then the eigenspaces of U reduce T.*

Proof. It suffices to prove the case for $p = \frac{1}{2^n}$ for some $n \in \mathbb{N}$. Then we have $Q_n = (T^*T)^{\frac{1}{2^n}} - (TT^*)^{\frac{1}{2^n}} = |T|^{\frac{1}{2^{n-1}}} - U|T|^{\frac{1}{2^{n-1}}}U^* \geq 0$. Let $M_z = \{x \in H : Ux = zx,\ |z| = 1\}$ be the eigenspace of U corresponding to the eigenvalue z. Since U is unitary, we have $U^*x = \bar{z}x$ for any $z \in M_z$ and hence $\langle Q_n x, x\rangle = \langle |T|^{\frac{1}{2^{n-1}}}x, x\rangle - \langle U|T|^{\frac{1}{2^{n-1}}}U^*x, x\rangle = \langle |T|^{\frac{1}{2^{n-1}}}x, x\rangle - \langle |T|^{\frac{1}{2^{n-1}}}\bar{z}x, \bar{z}x\rangle = 0$. Since $Q_n \geq 0$, we have $Q_n x = 0$. Therefore,

$$U|T|^{\frac{1}{2^{n-1}}}U^*x = |T|^{\frac{1}{2^{n-1}}}x, \quad \text{or equivalently} \quad U|T|^{\frac{1}{2^{n-1}}}x = z|T|^{\frac{1}{2^{n-1}}}x,$$

and hence $|T|^{\frac{1}{2^{n-1}}}x \in M_z$. Therefore, M_z is invarient with respect to $|T|^{\frac{1}{2^{n-1}}}$ and hence with respect to $|T|$. Thus, M_z reduces $|T|$ and finally it reduces T. □

We quote the following lemma from [1] to prove our next theorem.

Lemma 2.7. *Let X be self-adjoint, $Y > 0$ and $T = X + iY$. If $T_0 = X + iY^{\frac{1}{2^{n-1}}}$ is hyponormal, then the eigenspaces of Y reduce X.*

Theorem 2.8 ([1]). *Let $T = U|T|$ be p-hyponormal for $0 < p < 1/2$ and U be unitary. If $\sigma(U) \neq \{z : |z| = 1\}$, then the eigenspaces of $|T|$ reduce T.*

Proof. It suffices to prove the case for $p = \frac{1}{2^n}$ for some $n \in \mathbb{N}$. Since $\sigma(U) \neq \{z : |z| = 1\}$, without loss of generality, we may assume that $1 \notin \sigma(U)$. Let $B = i(U + I)(U - I)^{-1}$ be the inverse Caley transform of U (see Xia [25]). Then $i(B|T|^{\frac{1}{2^{n-1}}} - |T|^{\frac{1}{2^{n-1}}}B) = 2(U - I)^{-1}Q_n(U^* - I)^{-1} \geq 0$, and hence the operator $T_0 = B + i|T|^{\frac{1}{2^{n-1}}}$ is hyponormal, where $Q_n = (T^*T)^{\frac{1}{2^n}} - (TT^*)^{\frac{1}{2^n}} = |T|^{\frac{1}{2^{n-1}}} - U|T|^{\frac{1}{2^{n-1}}}U^*$. So by the previous lemma, the eigenspaces of $|T|$ reduce B. Hence they reduce U. So the eigenspaces of $|T|$ reduce T. □

The point spectrum of T is defined by $\sigma_p(T) = \{z \in \mathbb{C} : Tx = \lambda x \text{ for some } x \neq 0\}$ while the joint point spectrum is defined by $\sigma_{jp}(T) = \{z = re^{i\theta} \in \mathbb{C} : |T|x = rx \text{ and } Ux = e^{i\theta}x \text{ for some } x \neq 0\}$. Equivalently, $\sigma_{jp}(T) = \{z \in \mathbb{C} : Tx = zx \text{ and } T^*x = \bar{z}x \text{ for some } x \neq 0\}$. We have the following results from [1] involving the point spectrum and the joint point spectrum of a p-hyponormal operator for $0 < p < 1/2$.

Theorem 2.9 ([1]). *Let $T = U|T|$ be an invertible p-hyponormal operator,$0 < p < 1/2$. Then*

$$\sigma_{jp}(T) = \sigma_p(T).$$

Remark 2.10. : Xia [25] proved the above theorem for semi-hyponormal operators.

Proof. It suffices to show that $\sigma_p(T) \subset \sigma_{jp}(T)$, since it is obvious that $\sigma_{jp}(T) \subset \sigma_p(T)$. Let $re^{i\theta} \in \sigma_p(T), r \geq 0$. Then, there exists a vector $x \in H, x \neq 0$, such that

$$U|T|x = re^{i\theta}x.$$

Hence

$$|T|^{\frac{1}{2}}U|T|x = re^{i\theta}|T|^{\frac{1}{2}}x \quad \text{or equivalently} \quad \Delta(T)\big(|T|^{\frac{1}{2}}x\big) = re^{i\theta}\big(|T|^{\frac{1}{2}}x\big).$$

So $|T|^{\frac{1}{2}}x$ is an eigenvector of $\Delta(T)$ corresponding to the eigenvalue $re^{i\theta}$ of T. Now since $\Delta(T)$ is semi-hyponormal and the property holds for semi-hyponormal operators, we have

$$|\Delta(T)|\big(|T|^{\frac{1}{2}}x\big) = r\big(|T|^{\frac{1}{2}}x\big) \quad \text{and} \quad \widetilde{U}\big(|T|^{\frac{1}{2}}x\big) = e^{i\theta}\big(|T|^{\frac{1}{2}}x\big),$$

where $\Delta(T) = \widetilde{U}|\Delta(T)|$ is the polar decomposition of $\Delta(T)$. The operators $|\Delta(T)|$ and $\widetilde{U}$ have the form $|\Delta(T)| = |T|^{\frac{1}{2}}S^{-1}|T|^{\frac{1}{2}}$ and $\widetilde{U} = |T|^{\frac{1}{2}}US|T|^{-\frac{1}{2}}$, where S is a contraction satisfying the condition $SU^*|T|US = |T|$ by Theorem 2.2. Therefore, the above equations are equivalent to

$$USx = e^{i\theta}x \quad \text{and} \quad S^{-1}|T|x = rx,$$

respectively. Combining above two equations, we get

$$S^{-1}|T|S^{-1}U^*x = re^{-i\theta}x \quad \text{and hence} \quad U^*|T|x = re^{-i\theta}x.$$

Combining

$$U|T|x = re^{i\theta}x \quad \text{and} \quad U^*|T|x = re^{-i\theta}x$$

we get $U^2x = e^{2i\theta}x$. Equality $U^2x = e^{2i\theta}x$ implies that $(U + e^{i\theta}I)x$ and $(U - e^{i\theta}I)x$ are eigenvectors of U corresponding to eigenvalues $e^{i\theta}$ and $-e^{i\theta}$, respectively. Since eigenspaces of U reduce $|T|$ by Theorem 2.6, we have

$$U\big[|T|(U + e^{i\theta}I)x\big] = e^{i\theta}\big[|T|(U + e^{i\theta}I)x\big]$$

and

$$U\big[|T|(U - e^{i\theta}I)x\big] = -e^{i\theta}\big[|T|(U - e^{i\theta}I)x\big]$$

which imply $|T|Ux = U|T|x = re^{i\theta}x$. Equations $U^*|T|x = re^{-i\theta}x$ and $|T|Ux = U|T|x = re^{i\theta}x$ imply

$$|T|^2x = r^2x \quad \text{and hence} \quad |T|x = rx.$$

Finally we get $Ux = e^{i\theta}x$. Therefore

$$\sigma_p(T) \subset \sigma_{jp}(T)$$

as desired. □

Theorem 2.11 ([23]). *Let T be a hyponormal operator and $z \notin \sigma(T)$. Then*

$$||(T - zI)^{-1}|| = \frac{1}{dist(z, \sigma(T))}.$$

Theorem 2.12 ([1]). *Let $T = U|T|$ be an invertible p-hyponormal operator, $0 < p < 1$, and $z \notin \sigma(T)$. Then the following properties hold.*

(1) $||\, |T|^{\frac{1}{2}}(T - zI)^{-1}|T|^{-\frac{1}{2}}\, || = \frac{1}{\mathrm{dist}(z,\sigma(T))}$ *if* $1/2 \leq p < 1$.

(2) $||\, |\Delta(T)|^{\frac{1}{2}}\, |T|^{\frac{1}{2}}(T - zI)^{-1}|T|^{-\frac{1}{2}}\, |\Delta(T)|^{-\frac{1}{2}}\, || = \frac{1}{\mathrm{dist}(z,\sigma(T))}$ *if* $0 < p < 1/2$

(3) $||T^{-1}|| = \frac{1}{min\{|z|,\ z \in \sigma(T)\}}$.

Proof. (1) If $1/2 \leq p < 1$, then $\Delta(T)$ is hyponormal and by applying Theorem 2.11 to $\Delta(T)$, we get

$$||(\Delta(T) - zI)^{-1}|| = \frac{1}{dist(z, \sigma(\Delta(T))}.$$

But since $\sigma(\Delta(T)) = \sigma(T)$ and $(\Delta(T) - zI)^{-1} = |T|^{\frac{1}{2}}(T - zI)^{-1}|T|^{-\frac{1}{2}}$, we get the desired result

$$||\, |T|^{\frac{1}{2}}(T - zI)^{-1}|T|^{-\frac{1}{2}}\, || = \frac{1}{\mathrm{dist}(z, \sigma(T))}.$$

(2) For the case $0 < p < 1/2$, apply the above result to $\Delta^2(T) = \Delta(\Delta(T))$ since $\Delta^2(T)$ is hyponormal.

(3) Since T^{-1} is p-hyponormal and since $0 \notin \sigma(T)$, we have

$$||T^{-1}|| = r_{sp}(T^{-1}) = \max\{|z|, z \in \sigma(T^{-1})\} = \frac{1}{\min\{|z|\ z \in \sigma(T)\}}.$$

□

2.3. Powers of p-Hyponormal Operators. An important question is whether when a p-hyponormal operator operator is raised to a positive integer power, does the new operator retain some degree of hyponormality. The following theorem answers this question.

Theorem 2.13 ([3]). *If T is p-hyponormal, $0 < p \leq 1$, then for any positive integer n, T^n is $\frac{p}{n}$-hyponormal.*

Proof. Let $T = U|T|$ be the polar decomposition of T. For each positive integer n, let $A_n = \left(T^{n*}T^n\right)^{p/n}$, $B_n = \left(T^nT^{n*}\right)^{p/n}$ and $C_n = (U^*{A_n}^{n/p}U)^{p/n}$. We apply Hansen's inequality and use induction to prove the inequalities

$$A_n \geq A_1 \geq B_1 \geq B_n. \tag{2.1}$$

The inequalities (1) clearly hold for $n = 1$. So assume (1) holds for some positive integer $n = k$. Since T is p-hyponormal, using the inductive hypothesis, we get

$$U^*A_kU \geq U^*A_1U \geq A_1. \tag{2.2}$$

Hansen's inequality implies $C_k \geq U^*A_kU \geq A_1$. Thus,

$$\begin{aligned}
A_{k+1} &= \left({T^*}^{k+1}T^{k+1}\right)^{p/(k+1)} \\
&= \left(T^*({T^*}^kT^k)T\right)^{p/(k+1)} \\
&= \left(|T|U^*{A_k}^{k/p}U|T|\right)^{p/(k+1)} \\
&= \left({A_1}^{1/2p}{C_k}^{k/p}{A_1}^{1/2p}\right)^{p/(k+1)} \\
&\geq A_1
\end{aligned}$$

by Lemma 1.6. On the other hand, the inductive hypothesis implies $B_k \leq B_1 \leq A_1$ and hence

$$\begin{aligned} B_{k+1} &= \left(T^{k+1}T^{*k+1}\right)^{p/(k+1)} \\ &= \left(T(T^kT^{*k})T^*\right)^{p/(k+1)} \\ &= \left(U|T|{B_k}^{k/p}|T|U^*\right)^{p/(k+1)} \\ &= U\left(|T|{B_k}^{k/p}|T|\right)^{p/(k+1)}U^* \\ &= U\left({A_1}^{1/2p}{B_k}^{k/p}{A_1}^{1/2p}\right)^{p/(k+1)}U^* \\ &\leq UA_1U^* \\ &= B_1. \end{aligned}$$

by Lemma 1.6. Thus, the inequalities (1) hold by induction. Therefore, $(T^{n*}T^n)^{p/n} \geq (T^nT^{n*})^{p/n}$ and hence T^n is $\frac{p}{n}$-hyponormal. $\square$

As mentioned before, any hyponormal operator is p-hyponormal for $0 < p < 1$. But the converse is not true. Below we give an example of a semi-hyponormal operator which is not hyponormal.

Example 2.14 (A semi-hyponormal operator which is not hyponormal)**.** Let $H = l^2$, the Hilbert space of one-way (unilateral) square-summable sequences and U be the unilateral shift and let $T = U^* + 2U$. Then, $T^*T - TT^* = 3(1 - UU^*) \geq 0$, and hence T is hyponormal. Let $f = e_0 - 2e_2$, where $\{e_n\}$ is an orthonormal basis for l^2. Then it can be shown that $||T^2f|| = 80$ and $||T^{*2}f|| = 89$. Therefore, $||T^{*2}f|| \nleq ||T^2f||$ and T^2 is not hyponormal. But the operator T^2 is semi-hyponormal by Theorem 2.13. So the operator T^2 in this example is semi-hyponormal, but not hyponormal.

3. Log-Hyponormal Operators

Log-hyponormal operators were introduced by Tanahashi [24]. An invertible operator T is called log-hyponormal if $\log(T^*T) \geq \log(TT^*)$ or equivalently $\log(|T|) \geq \log(U|T|U^*)$. If T is invertible, since $(T^*T)^p \geq (TT^*)^p$ implies $\log(T^*T)^p \geq \log(TT^*)^p$ which implies $\log(T^*T) \geq \log(TT^*)$, every invertible p-hyponormal operator is log-hyponormal. The converse of the above conclusion is not true. But log-hyponormal operators possess many properties of p-hyponormal operators. We next give an example of a log-hyponormal operator which is not p-hyponormal for any $p > 0$.

Example 3.1 (A log hyponormal operator which is not p-hyponormal [24])**.** Let $H = \bigoplus \sum_{n=-\infty}^{\infty} \mathbf{C}^2$ and $\bar{x} = (..., x_{-1}, \widehat{x_0}, x_1, ...) \in H$ with $||\bar{x}||^2 = \sum_{n=-\infty}^{\infty} |x_i|^2 < \infty$.

Let A and B be positive matrices such that

$$\log(A) = \begin{bmatrix} 1 & \sqrt{3/2} \\ \sqrt{3/2} & 1/2 \end{bmatrix} \qquad \text{and} \qquad \log(B) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

Clearly, $\log(B) \leq \log(A)$ and it can be shown that

$$A^p = \frac{1}{5}\begin{bmatrix} 3e^{2p} + 2e^{-\frac{p}{2}} & \sqrt{6}(e^{2p} - e^{-\frac{p}{2}}) \\ \sqrt{6}(e^{2p} - e^{-\frac{p}{2}}) & 3e^{2p} + 2e^{-\frac{p}{2}} \end{bmatrix} \qquad \text{and} \qquad B^p = \begin{bmatrix} 1 & 0 \\ 0 & e^{-p} \end{bmatrix}.$$

So there exists no $p>0$ such that $A^p \geq B^p$.

Define $P \in L(H)$ as

$$(P\bar{x})_n = P_n x_n, \quad \text{where} \quad P_n = \begin{cases} B & \text{for} \quad n \leq 0 \\ A & \text{for} \quad n > 0 \end{cases}.$$

Let U be the bilateral shift defined by $(U\bar{x})_n = x_{n-1}$ and $T = UP$. Then

$$\Big(\big((T^*T)^p - (TT^*)^p\big)\bar{x}\Big)_n = \begin{cases} 0 & \text{if} \quad n \neq 1 \\ \big(A^{2p} - B^{2p}\big)x_1 & \text{if} \quad n = 1 \end{cases}$$

and

$$\Big(\big(\log(T^*T) - \log(TT^*)\big)\bar{x}\Big)_n = \begin{cases} 0 & \text{if} \quad n \neq 1 \\ (2\log A - 2\log B)x_1 & \text{if} \quad n = 1 \end{cases}.$$

Therefore, T is log-hyponormal, but not p-hyponormal for any $p > 0$.

3.1. Some Key Properties of Log-Hyponormal Operators. Assume T is an invertible log-hyponormal operator. The inverse T^{-1} is also log-hyponormal. If T is log-hyponormal and $\Delta(T)$ is normal, then T is normal. If T is a log-hyponormal operator, then $|\Delta(T)| \geq |T| \geq |\Delta(T)^*|$, [5]. If T is log-hyponormal, then for any positive integer n,

$$\log\big(T^{n*}T^n\big)^{1/n} \geq \log(T^*T) \geq \log(TT^*) \geq \log\big(T^nT^{n*}\big)^{1/n}$$

and hence T^n is log-hyponormal, [26]. If T log-hyponormal, then T is normaloid; that is, $r_{sp}(T) = ||T||$.

For invertible operators, A, $B > 0$, $\log(A) \geq \log(B)$ does not necessarily imply that $A \geq B$. Ando [7] proved that $\log(A) \geq \log(B)$ if and only if $A^r \geq \big(A^{r/2}B^rA^{r/2}\big)^{r/2}$ for all $r \geq 0$. Let $r = 1$. Then $\log(A) \geq \log(B)$ implies $A \geq \big(A^{1/2}BA^{1/2}\big)^{1/2}$. Let T be log-hyponormal. Then $\log(|T|) \geq \log(U|T|U^*)$. By letting $A = |T|$ and $B = U|T|U^*$ above we get if T is log-hyponormal, then $|T| \geq |\Delta(T)^*|$. Aluthge and Wang [5] proved that $|T| \geq |\Delta(T)^*|$ and $|\Delta(T)| \geq |T|$ are equivalent.

Theorem 3.2. *If T is an invertible log-hyponormal operator, then*

$$|\Delta(T)| \geq |T| \geq |\Delta(T)^*|.$$

4. w-Hyponormal Operators

Recall that if T is p-hyponormal for $0 < p < 1/2$, then $\Delta(T)$ is $\big(p + \frac{1}{2}\big)$-hyponormal and hence semi-hyponormal. More precisely,

$$|\Delta(T)| \geq |T| \geq |\Delta(T)^*|.$$

It was also shown earlier that if T is log-hyponormal, then T satisfies the above inequality. Which leads us to the following question: Are there operators that are not necessarily p-hyponormal or log-hyponormal, but satisfies the above inequality? So we define the class of w-hyponormal operators as follows [5], [6].

Definition 4.1. : An operator T is called w-hyponormal if it satisfies

$$|\Delta(T)| \geq |T| \geq |\Delta(T)^*|.$$

Clearly, any p-hyponormal or log-hyponormal operator is w-hyponormal.

Example 4.2 (A w-hyponormal operator which is neither log-hyponormal nor p-hyponormal)**.** Let T be the operator in Example 3.1. Consider the operator $S = T \oplus 0$ defined on $K = H \oplus H$. Then S is w-hyponormal, but is neither log-hyponormal nor p-hyponormal.
Note that $(S^*S)^p = (T^*T)^p \oplus 0$ and $(SS^*)^p = (TT^*)^p \oplus 0$. So since T is not p-hyponormal, S is not p-hyponormal. Since S is not invertible, it is not log-hyponormal. Note that $\Delta(S) = \Delta(T) \oplus 0$ and $\Delta(S)^* = \Delta(T)^* \oplus 0$ and hence $|\Delta(S)| = |\Delta(T)| \oplus 0$ and $|\Delta(S)^*| = |\Delta(T)^*| \oplus 0$. Since T is log-hyponormal, we have $|\Delta(T)| \geq |T| \geq |\Delta(T)^*|$. Also, $|S| = |T| \oplus 0$. Therefore, $|\Delta(S)| \geq |S| \geq |\Delta(S)^*|$ and hence S is w-hyponormal. In general, any operator $T \oplus R$, where R is a non-invertible p-hyponormal operator, is w-hyponormal, but is neither log-hyponormal nor p-hyponormal.

Theorem 4.3 ([5])**.** *If T is a w-hyponormal operator, then $|T^2| \geq |T|^2$ and $|T^*|^2 \geq |T^{*2}|$.*

Proof. Let $\Delta(T) = V|\Delta(T)|$ be the polar decomposition of $\Delta(T)$. Then $V^*|\Delta(T)^*|V = |\Delta(T)| \geq |T|$. Thus,

$$\begin{aligned}
|T^2| &= (T^{2^*}T^2)^{1/2} = (T^*|T|^2T)^{1/2} \\
&= (|T|U^*|T|^2U|T|)^{1/2} \\
&= (|T|^{1/2}\Delta(T)^*|T|\Delta(T)|T|^{1/2})^{1/2} \\
&\geq (|T|^{1/2}\Delta(T)^*|\Delta(T)^*|\Delta(T)|T|^{1/2})^{1/2} \\
&= (|T|^{1/2}|\Delta(T)|V^*|\Delta(T)^*|V|\Delta(T)||T|^{1/2})^{1/2} \\
&\geq (|T|^{1/2}|\Delta(T)||T||\Delta(T)||T|^{1/2})^{1/2} \\
&= |T|^{1/2}|\Delta(T)||T|^{1/2} \geq |T|^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|T^{*2}| &= (T^2T^{*2})^{1/2} \\
&= (T|T^*|^2T)^{1/2} \\
&= (U|T|U|T|^2U^*|T|U^*)^{1/2} \\
&= U(|T|^{1/2}\Delta(T)|T|\Delta(T)^*|T|^{1/2})^{1/2}U^* \\
&\leq U(|T|^{1/2}\Delta(T)V^*|\Delta(T)^*|V\Delta(T)^*|T|^{1/2})^{1/2}U^* \\
&\leq U|T|^{1/2}|\Delta(T)^*||T|^{1/2}U^* \\
&\leq U|T|^2U^* \\
&= |T^*|^2.
\end{aligned}$$

□

5. Paranormal Operators

An operator T is called paranormal if $||T^2x|| \geq ||Tx||^2$ for all unit vectors $x \in H$.

Theorem 5.1 ([7]). *An operator T is paranormal if and only if the inequality*

$$|T^2|^2 - 2\lambda|T|^2 + \lambda^2 I \geq 0$$

for all $\lambda \geq 0$.

Lemma 5.2. *(Schwartz Inequality) If $A \geq 0$, then*

$$|\langle Ax, y\rangle| \leq \langle Ax, x\rangle^{1/2}\langle Ay, y\rangle^{1/2}$$

holds for for all $x, y \in H$.

Lemma 5.3. *(Hölder-McCarthy Inequality) If $A \geq 0$ and $x \in H$ is nonzero, then $||Ax|| \geq ||A^{1/2}x||^2||x||^{-1}$.*

Theorem 5.4 ([4]). *If T is w-hyponormal, then T is paranormal.*

Proof. Since T is w-hyponormal, $|\Delta(T)| \geq |T| \geq |\Delta(T)^*|$. Let $x \in H$ be a unit vector. Need to show that $||T^2x|| \geq ||Tx||^2$. If $Tx = 0$, the result holds. So let $Tx \neq 0$. Then,

$$\begin{aligned}
||T^2x|| &= |||T|Tx|| \\
&\geq ||\,|T|^{1/2}Tx||^2||Tx||^{-1} \text{(by Lemma 5.3)} \\
&= ||\Delta(T)|T|^{1/2}x||^2||Tx||^{-1} \\
&= ||\,|\Delta(T)||T|^{1/2}x||^2||Tx||^{-1} \\
&\geq ||\,|\Delta(T)|^{1/2}|T|^{1/2}x||^4||\,|T|^{1/2}x||^{-2}||Tx||^{-1} \text{(by Lemma 5.3)} \\
&\geq ||\,|T|x||^4||\,|T|^{1/2}x||^{-2}||Tx||^{-1} \text{(by } w\text{-hyponormality of } T) \\
&\geq ||\,|T|x||^4||\,|T|x||^{-1}||Tx||^{-1} = ||Tx||^2 \text{(by Lemma 5.3)}
\end{aligned}$$

□

Definition 5.5. : An operator T is called quassinormal if T commutes with T^*T or equivalently, $U|T| = |T|U$.

Theorem 5.6. *We have $\Delta(T) = T$ if and only if T is quassinormal.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Then,

$$\begin{aligned}
T = \Delta(T) &\Leftrightarrow U|T| = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \\
&\Leftrightarrow U|T|^{\frac{1}{2}}|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \\
&\Leftrightarrow U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}U \quad \text{on} \quad H = N(|T|^{\frac{1}{2}}) \oplus N(|T|^{\frac{1}{2}})^{\perp} \\
&\Leftrightarrow U|T| = |T|U.
\end{aligned}$$

□

6. Generalizations of $\Delta(T)$

Many generalizations the operator trnsform $\Delta(T)$ have been given including the generalization $T_\epsilon = \Delta_\epsilon(T) = |T|^\epsilon U|T|^{1-\epsilon}$, where $0 < \epsilon < 1$, by this aurthor. Furuta [13] introduced the generalization $\widehat{T} = |T|^p U|T|^q$ while Huruya [17] introduced the generalization $T(r,t) = |T|^t U |T|^{r-t},\ r \geq t > 0$. Emphasis on these generalizations have been to preserv some properties such as as having the same norm and the same spectrum, and increasing the degree of hyponormality.

Theorem 6.1 ([2]). *Let $T = U|T|$ be p-hyponormal and $0 < p < \frac{1}{2}$. Then the operator $\Delta_\epsilon(T) = |T|^\epsilon U |T|^{1-\epsilon}$, $0 < \epsilon \leq \frac{1}{2}$, is $(p+\epsilon)$-hyponormal.*

Proof. Let

$$A = U^*|T|^{2p}U, \quad B = |T|^{2p} \quad \text{and} \quad C = U|T|^{2p}U^*.$$

Then, $A \geq B \geq C$ by p-hyponormality of T, and

$$\begin{aligned}
\big(\Delta_\epsilon(T)\Delta_\epsilon(T)^*\big)^{(p+\epsilon)} &= \big(|T|^\epsilon U |T|^{2(1-\epsilon)} U^* |T|^\epsilon\big)^{(p+\epsilon)} \\
&= \big(B^{\epsilon/2p} C^{(1-\epsilon)/p} B^{\epsilon/2p}\big)^{(p+\epsilon)} \\
&= \big(B^r C^s B^r\big)^{1/q} \leq B^{(s+2r)/q} = |T|^{2(p+\epsilon)},
\end{aligned}$$

by Lemma 1.6 since $(1+2r)q \geq s+2r$ holds with $r = \epsilon/2p$, $s = (1-\epsilon)/p$, and $q = 1/(p+\epsilon)$. Similarly,

$$\begin{aligned}
\big(\Delta_\epsilon(T)^*\Delta_\epsilon(T)\big)^{(p+\epsilon)} &= \big(|T|^{1-\epsilon} U^* |T|^{2\epsilon} U |T|^{1-\epsilon}\big)^{(p+\epsilon)} \\
&= \big(B^{(1-\epsilon)/2p} A^{\epsilon/p} B^{(1-\epsilon)/2p}\big)^{(p+\epsilon)} \\
&= \big(B^r A^s B^r\big)^{1/q} \geq B^{(s+2r)/q} = |T|^{2(p+\epsilon)},
\end{aligned}$$

again by Furuta inequality since $(1+2r)q \geq s+2r$ holds with $r = (1-\epsilon)/2p$, $s = \epsilon/p$, and $q = 1/(p+\epsilon)$.

Thus, $\big(\Delta_\epsilon(T)^*\Delta_\epsilon(T)\big)^{(p+\epsilon)} \geq |T|^{2(p+\epsilon)} \geq \big(\Delta_\epsilon(T)\Delta_\epsilon(T)^*\big)^{(p+\epsilon)}$ and $\Delta_\epsilon(T)$ is $(p+\epsilon)$-hyponormal. □

Remark 6.2. : This operator T_ϵ has the same spectrum as T and if T is p-hyponormal for any $p > 0$, then they also have the same norm.

Theorem 6.3 ([13]). *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator for $0 < p \leq 1$ with $N(T) = N(T^*)$. Then $\widehat{T} = |T|^p U |T|^q$ is $\frac{1}{2}\big(1+\frac{p}{q}\big)$-hyponormal for any q such that $q \geq p$.*

Proof.

$$\begin{aligned}
\big(\widehat{T}^*\widehat{T}\big)^{(p+q)/2q} &= \big(|T|^q U^* |T|^{2p} U |T|^q\big)^{(p+q)/2q} \\
&= \big(B^{q/2p} A^{2q/2p} B^{q/2p}\big)^{(p+q)/2q} \\
&\geq B^{(2q/2p+2q/2p)(p+q)/2q} = B^{(1+q/p} = |T|^{2p+q},
\end{aligned}$$

since $\big(1+\frac{2q}{2p}\big)\frac{2q}{p+q} = \frac{2q}{2p}+\frac{2q}{2p}$ and $\frac{2q}{p+q} \geq 1$, where A and B are as defined in the proof of Theorem 6.1.

$$\begin{aligned}\big(\widehat{T}\widehat{T}^*\big)^{(p+q)/2q} &= \big(|T|^pU|T|^{2q}U^*|T|^p\big)^{(p+q)/2q}\\ &= \big(B^{q/2p}C^{2q/2p}B^{q/2p}\big)^{(p+q)/2q}\\ &\le B^{(2q/2p+2q/2p)(p+q)/2q} = B^{(1+q)/p} = |T|^{2p+q},\end{aligned}$$

since $\big(1+\frac{2q}{2p}\big)\frac{2q}{p+q} = \frac{2q}{2p}+\frac{2q}{2p}$ and $\frac{2q}{p+q} \ge 1$, where C is as defined in the proof of Theorem 6.1. Therefore, $\big(\widehat{T}^*\widehat{T}\big)^{(p+q)/2q} \ge \big(\widehat{T}\widehat{T}^*\big)^{(p+q)/2q}$ and hence $\widehat{T}$ is $\frac{1}{2}\big(1+\frac{p}{q}\big)$-hyponormal. □

Even if T is not p-hyponormal, we have $\sigma(\Delta_\epsilon(T)) = \sigma(T)$. If T is p-hyponormal for any $p > 0$, then $||\Delta_\epsilon(T)|| = ||T||$.

Theorem 6.4 ([17])**.** *Let $T = U|T|$ be p-hyponormal for $p > 0$. Then for any $r > 0$ and $r \ge t \ge 0$, let $q = min\{\frac{p+t}{r}, \frac{p+(r-t)}{r}, 1\}$ and $T(r,t) = |T|^tU|T|^{r-t}$. Then $T(r,t)$ satisfies $\big((T(r,t)^*T(r,t)\big)^q \ge |T|^{2rq} \ge \big(T(r,t)T(r,t)^*\big)^q$ and hence is q-hyponormal.*

Remark 6.5. If $r \ne 1$, $\widehat{T}$ and T do not have the same spectrum or the norm.

7. Applications of $\Delta(T)$

In additions to the applications to p-hyponormal operators, there are many more applications of the operator transform $\Delta(T)$. These include applications invloving numerical range the spectral radius of an operator. There are applications to other classes of operators also. Below we list a few results without proof by Yamazaki [27], [28] on spectral radius and by Ando [8] on the numerical range.

Theorem 7.1 ([27])**.** *For each n, define the n^{th} transform as*

$$\Delta^n(T) = \Delta(\Delta^{n-1}(T)), \qquad \Delta^0(T) = T.$$

Then for any $T \in L(H)$, we have

$$r(T) = \lim_{n\to\infty} ||\Delta^n(T)||,$$

where $r(T)$ is the spectral radius of T.

Remark 7.2. : The above result is similar to the already known result that for any $T \in L(H)$, the spectral radius $r(T)$ is given by

$$r(T) = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}}.$$

Definition 7.3. The numerical range an operator T is defined by $W(T) = \{\langle Tx, x\rangle, ||x|| = 1, x \in H\}$. The convex hull of $\sigma(T)$ is the intersection of all convex sets that contains the spectrum $\sigma(T)$.

Remark 7.4. It is known that conv$\sigma(T) =$ conv$\sigma(\Delta(T)) \subset W(\Delta(T)) \subset W(T)$, See [9].

Theorem 7.5 ([8])**.** *Let $T = U|T|$ be an $m \times m$ complex matrix, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is unitary. Then*

$$\mathrm{conv}(\sigma(T)) = \bigcap_{n=1}^{\infty} W(\Delta^n(T)).$$

Below we give an example to illustrate Ando's result for a 5×5 matrix.

Example 7.6. Let $T = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 4 & -1 \\ 2 & 0 & 3 & -2 & 0 \\ 2 & 0 & 3 & -2 & 0 \\ -5 & 3 & 6 & 2 & -3 \end{bmatrix} = U|T|$. Then,

$$\Delta(T) = \begin{bmatrix} -0.54040 & 1.89407 & 2.34884 & 2.51379 & 4.45816 \\ 1.45465 & 0.133716 & 1.59104 & 1.09987 & -1.55918 \\ 3.39366 & -0.28571 & 3.81554 & -1.24208 & -0.76634 \\ 0.59826 & 0.11018 & 2.64224 & 4.30297 & 0.57032 \\ -4.02159 & 2.65391 & 4.70995 & 1.92198 & -1.71183 \end{bmatrix},$$

$$\Delta^2(T) = \begin{bmatrix} -0.58209 & 1.65441 & 1.60478 & 1.21738 & 4.66880 \\ 1.30153 & 0.08781 & 1.35699 & 0.16681 & -2.17684 \\ 4.19113 & -0.50885 & 4.36806 & -0.80805 & -0.70566 \\ 1.15547 & -0.72437 & 2.54050 & 3.84160 & -1.23690 \\ -3.56948 & 2.28149 & 3.60285 & 1.46415 & -1.71539 \end{bmatrix}, \text{and}$$

$$\Delta^3(T) = \begin{bmatrix} -0.23903 & 1.53807 & 1.55063 & 0.61403 & 4.95372 \\ 1.06787 & 0.24486 & 0.98382 & -0.37814 & -2.42279 \\ 4.37618 & -0.51658 & 4.50346 & -0.63326 & -0.19030 \\ 1.38195 & -1.20418 & 2.02767 & 3.35244 & -1.20896 \\ -3.67017 & 2.26129 & 3.11002 & 1.02570 & -1.86171 \end{bmatrix}.$$

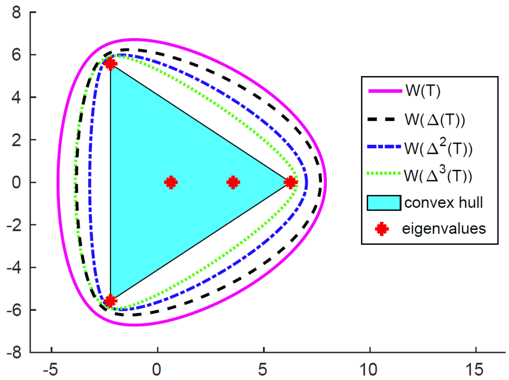

FIGURE 1. The figure shows the convex hull of the spectrum (the triangle) and numerical ranges of $T, \Delta(T), \Delta^2(T)$, and $\Delta^3(T)$.The outermost graph is the boundary of the numerical range of T.

7.1. Convergence of the operator Transform $\Delta(T)$. Every time the when the operator tansform $\Delta(T)$ is applied on an operator T, it seems to get closer to becoming a normal operator. Ando and Yamazaki Proved that for a 2×2 matrix T, the sequence $\{\Delta^n(T)\}$ converges as $n \to \infty$. Antezana, Pujals, and Stojanoff proved that the above result for any $n \times n$ square matrix. But the result does not seem to hold for infinite dimensional operators. Cho, Jung, and Lee [19] gave an example based on a weighted shift where the sequence $\Delta^n(T)$ does not converge even with respect to the week operator topology. Also, see [10] and [11].

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