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An application of Wilson system in numerical solution of Fredholm integral equations

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Abstract. A numerical method based on the Wilson wavelets for solving Fredholm integral equations and Fredholm integro-differential (FID) equations is provided. Using Wilson wavelets approximation method Fredholm integral equations is reduced to algebraic equations. Furthermore the operational matrix of integration for Wilson wavelets is obtained. Finally, a numerical comparison between results obtained by Wilson wavelets and CAS wavelets is carried out. The applicability and accuracy of the proposed method are also checked using numerical examples.

1. Introduction

Integral equations are used as mathematical models in many physical situations. There exist different methods for solving integral equations and recently methods using wavelets have been applied in engineering and sciences[16]. Haar wavelets [3, 13], CAS wavelets [10, 17], Harmonic wavelets [6], Chebyshev wavelets [1, 2, 14], Legendre wavelets [11, 15, 18] for solving integral and differential equations has produced significant results. In this paper we propose solving integral and differential equations using Wilson wavelets. These wavelets are a special case of orthonormal wavelets with a simple structure and methods based on them are expected to give better results. In order to compute an approximate solution for the second Fredholm integral equations, we first analyze the convergence of Wilson wavelets and then obtain a new operational matrix of integration for the basis functions to eliminate the integral operation and reduce the problem into solving a system of algebraic equations. We also provide illustrative examples to demonstrate the validity and applicability of the method.
The paper is organized as follows: Section 2, contains introducing Wilson wavelets and their properties. In section 3, we present a method to solve the second Fredholm integral equations. In section 4, we present a method for solving FID equations using

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the operational matrix and we provide illustrative test problems in section 5. Section 6 provides a conclusion.

2. The Wilson wavelets and their properties

In this section, we briefly introduce the Wilson wavelets and some of its properties which are used in the sequel of the paper.

2.1. The Wilson wavelets. Wilson introduced a system of basis functions as follows

$$\psi_{nm}(t) = \begin{cases} \epsilon_n \cos(2n\pi t)\omega(t - \frac{m}{2}), & m \text{ is even}, \\ \sqrt{2}\sin(2(n+1)\pi t)\omega(t - \frac{m+1}{2}), & m \text{ is odd}, \end{cases} \tag{2.1}$$

where

$$\epsilon_n = \begin{cases} 1, & n = 0, \\ \sqrt{2}, & n \in \mathbb{N}, \end{cases}$$

with a smooth well-localized window function ω [4, 5, 7, 9, 12]. The elements of this system are localized around positive and negative frequency. Based on this system of basis functions, Daubechies constructed an orthonormal system and called it as "Wilson bases"[8]. Now we consider $\omega = \chi_{[r,r+1)}$, where $r \in \mathbb{Z}$ in Eq. (2.1), i.e.

$$\psi_{nm}(t) = \begin{cases} \epsilon_n \cos(2n\pi t)\chi_{[r,r+1)}(t - \frac{m}{2}), & m \text{ is even}, \\ \sqrt{2}\sin(2(n+1)\pi t)\chi_{[r,r+1)}(t - \frac{m+1}{2}), & m \text{ is odd}. \end{cases} \tag{2.2}$$

where

$$\epsilon_n = \begin{cases} 1, & n = 0, \\ \sqrt{2}, & n \in \mathbb{N}. \end{cases}$$

The set $\{\psi_{nm}(t) | m \in \mathbb{Z},\ n \in \mathbb{N} \cup \{0\}\}$ is a tight frame for $L^2(\mathbb{R})$ with bound 1[12]. Now, we can show that for $r \in \mathbb{Z}$, the set $\{\psi_{nm}(t) | m \in \{-2r-1, -2r\}, n \in \mathbb{N} \cup \{0\}\}$ in Eq. (2.2) is an orthonormal basis for $L^2[0,1)$, which we called the Wilson wavelets.

2.2. Functions approximation. Any square integrable function $f(t)$ defined on $[0,1)$ can be expanded in terms of the Wilson wavelets as

$$f(t) = \sum_{n=0}^{\infty} c_{n,-2r}\psi_{n,-2r}(t) + \sum_{n=0}^{\infty} c_{n,-2r-1}\psi_{n,-2r-1}(t), \tag{2.3}$$

where $c_{nm} = < f(t), \psi_{nm}(t) >$, $m \in \{-2r-1, -2r\}$ and $< .,. >$ denotes the inner product on $L^2[0,1)$. If the infinite series in Eq. (2.3) is truncated, then it can be written as

$$f(t) \simeq \sum_{n=0}^{2^k-1} \sum_{m=-2r-1}^{-2r} c_{nm}\psi_{nm}(t) \triangleq C^T \Psi(t), \tag{2.4}$$

where C and $\Psi(t)$ are $\hat{m} = 2^{k+1}$ column vectors given by

$$C \triangleq [c_{0,-2r-1}, c_{0,-2r}, c_{1,-2r-1}, c_{1,-2r}, \ldots, c_{2^k-1,-2r-1}, c_{2^k-1,-2r}]^T, \tag{2.5}$$

$$\Psi(t) \triangleq [\psi_{0,-2r-1}(t), \psi_{0,-2r}(t), \psi_{1,-2r-1}(t), \ldots, \psi_{2^k-1,-2r-1}(t), \psi_{2^k-1,-2r}(t)]^T. \tag{2.6}$$

For simplicity, Eq. (2.4) can be written as

$$f(t) \simeq \mathcal{P}_{\hat{m}} f(t) = \sum_{i=1}^{\hat{m}} c_i \psi_i(t) \triangleq C^T \Psi(t), \tag{2.7}$$

where $c_i = c_{nm}$, $\psi_i(t) = \psi_{nm}(t)$, and the index i is determined by the relation $i = 2n + m + 2 + 2r$.
Thus we have

$$C \triangleq [c_1, c_2, \ldots, c_{\hat{m}}]^T,$$

and

$$\Psi(t) \triangleq [\psi_1(t), \psi_2(t), \ldots, \psi_{\hat{m}}(t)]^T. \tag{2.8}$$

Similarly, an arbitrary function of two variables $K(s,t)$ defined over $L^2([0,1) \times [0,1))$, may be expanded using Wilson wavelets as follows:

$$K(s,t) \simeq \sum_{i=1}^{\hat{m}} \sum_{j=1}^{\hat{m}} k_{ij} \psi_i(x) \psi_j(t) \triangleq \Psi^T(s) K \Psi(t), \tag{2.9}$$

where $K = [k_{ij}]$ and $k_{ij} = (\psi_i(s), (K(s,t), \psi_j(t)))$.

2.3. Convergence analysis. Here, we investigate the convergence analysis of the Wilson wavelets expansion.

Theorem 2.1. *If the series in Eq. (2.3) associated to the continuous function $f(t)$ is uniformly convergent, then it converges to the function $f(t)$ in $L^2[0,1)$.*

Proof. Let $g(t) = \sum_{n=0}^{\infty} \sum_{m=-2r-1}^{-2r} c_{nm} \psi_{nm}(t)$, where $c_{nm} = < f(t), \psi_{nm}(t) >$, and $m \in \{-2r-1, -2r\}$ for $r \in \mathbb{Z}$. Multiplying both sides of the previous equation by $\psi_{pq}(t)$, $q \in \{-2r-1, -2r\}$, $p \in \mathbb{N} \cup \{0\}$ and integrating on [0,1), we have

$$\begin{aligned}
\int_0^1 g(t)\psi_{pq}(t)dt &= \int_0^1 \sum_{n=0}^{\infty} \sum_{m=-2r-1}^{-2r} c_{nm}\psi_{nm}(t)\psi_{pq}(t)dt \\
&= \sum_{n=0}^{\infty} \sum_{m=-2r-1}^{-2r} c_{nm} \int_0^1 \psi_{nm}(t)\psi_{pq}(t)dt \\
&= c_{pq},
\end{aligned}$$

and so $< g(t), \psi_{nm}(t) >= c_{nm}$. Hence f and g have the same wavelet expansion with respect to the Wilson wavelet and therefore $f(t) = g(t)$. □

Theorem 2.2. *Let $f(t)$ be an arbitrary function defined on $[0,1)$ with bounded second derivative, i.e. $|f''(t)| \leq M$. Then it can be expanded as an infinite series of the Wilson wavelets and the series is uniformly convergent to $f(t)$, i.e.*

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=-2r-1}^{-2r} c_{nm} \psi_{nm}(t), \quad r \in \mathbb{Z}.$$

Moreover, if $\mathcal{P}_{\hat{m}} f(t)$ is the Wilson wavelets expansion of $f(t)$, then we have

$$\|\mathcal{P}_{\hat{m}} f(t) - f(t)\| \leq \frac{M}{2\pi^2} \sum_{n=2^k+1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} \right).$$

Proof. Let $r \in \mathbb{Z}$. By considering the Wilson wavelets for $n = 0$ and $m = -2r$, we have

$$|c_{0,-2r}| = |\int_0^1 f(t)dt| \leq \int_0^1 |f(t)|dt < \infty, \tag{2.10}$$

and for $n = 0$ and $m = -2r - 1$, we have

$$|c_{0,-2r-1}| = |\int_0^1 f(t)\psi_{0,-2r-1}(t)dt| = |\int_0^1 \sqrt{2}f(t)\sin(2\pi t)dt| < \infty. \tag{2.11}$$

Now for $n \in \mathbb{N}$ and $m = -2r$ by using integration by parts, we get

$$|c_{n,-2r}| = |\sqrt{2}\int_0^1 f(t)\cos(2n\pi t)dt| = \frac{1}{2\sqrt{2}n^2\pi^2}|\int_0^1 f''(t)\cos(2n\pi t)dt|. \tag{2.12}$$

Since $|f''(t)| \leq M$, we have

$$|c_{n,-2r}| \leq \frac{M}{2\sqrt{2}n^2\pi^2}, \tag{2.13}$$

and

$$\sum_{n\in\mathbb{N}} |c_{n,-2r}| \leq \sum_{n=1}^{\infty} \frac{M}{2\sqrt{2}n^2\pi^2} < \infty. \tag{2.14}$$

In the case $n \in \mathbb{N}$ and $m = -2r - 1$, we have

$$c_{n,-2r-1} = \int_0^1 f(t)\psi_{n,-2r-1}(t)dt = \int_0^1 \sqrt{2}f(t)\sin(2(n+1)\pi t)dt.$$

Now by applying a similar process as in Eqs. (2.12) and (2.13), we get

$$|c_{n,-2r-1}| \leq \frac{M}{2\sqrt{2}(n+1)^2\pi^2}, \tag{2.15}$$

and therefore

$$\sum_{n\in\mathbb{N}} |c_{n,-2r-1}| \leq \sum_{n=1}^{\infty} \frac{M}{2\sqrt{2}(n+1)^2\pi^2} < \infty. \tag{2.16}$$

Hence, by considering Eqs. (2.10), (2.11), (2.14) and (2.16), we have

$$\sum_{n=0}^{\infty} \sum_{m=-2r-1}^{-2r} |c_{nm}| < \infty,$$

which shows the series $\sum_{n=0}^{\infty} \sum_{m=-2r-1}^{-2r} c_{nm}$ is absolutely convergent.
Also we have

$$|\sum_{n=0}^{\infty} \sum_{m=-2r-1}^{-2r} c_{nm}\psi_{nm}(t)| \leq \sum_{n=0}^{\infty} \sum_{m=-2r-1}^{-2r} |c_{nm}||\psi_{nm}(t)| \leq \sqrt{2}\sum_{n=0}^{\infty} \sum_{m=-2r-1}^{-2r} |c_{nm}| < \infty.$$

By considering Theorem 2.1 the series $\sum_{n=0}^{\infty}\sum_{m=-2r-1}^{-2r} c_{nm}\psi_{nm}(t)$ is convergent to $f(t)$. Moreover, by using Eqs. (2.13) and (2.15) for $n > 2^k$, we have

$$\begin{aligned}
\|\mathcal{P}_{\hat{m}} f(t) - f(t)\| &= \max_{t\in[0,1)} |\sum_{n=2^k+1}^{\infty}\sum_{m=-2r-1}^{-2r} c_{nm}\psi_{nm}(t)| \\
&\le \sqrt{2}\sum_{n=2^k+1}^{\infty}\sum_{m=-2r-1}^{-2r} |c_{nm}| \\
&\le \frac{M}{2\pi^2}\sum_{n=2^k+1}^{\infty}(\frac{1}{n^2}+\frac{1}{(n+1)^2}),
\end{aligned}$$

which completes the proof.
For simplicity we consider $r = 0$ in the next sections. □

2.4. Operational Matrix of integration. In this section we shall obtain the operational matrix P, satisfying $\int_0^t \Psi(y)dy = P\Psi(t)$. We obtain P by using

$$\int_0^t \psi_{n,m}(y)dy = \sum_{i=0}^{\infty}\sum_{j=-1}^{0} c_{ij}\psi_{ij}(t) \simeq \sum_{i=0}^{2^k-1} c_{i,0}\psi_{i,0}(t) + \sum_{i=0}^{2^k-1} c_{i,-1}\psi_{i,-1}(t) \tag{2.17}$$

$$= \sum_{i=0}^{2^k-1} c_{i,0}\epsilon_i cos(2i\pi t) + \sum_{i=0}^{2^k-1}\sqrt{2}c_{i,-1}sin(2(i+1)\pi t).$$

For $n \neq 0$ and $m = 0$ we have:

$$\int_0^t \psi_{n,0}(y)dy = \int_0^t \epsilon_n cos(2n\pi y)dy = \frac{\epsilon_n}{2n\pi}sin(2n\pi t). \tag{2.18}$$

Comparing Eqs. (2.17) and (2.18), we have:

$$\frac{\epsilon_n}{2n\pi}sin(2n\pi t) \simeq \sum_{i=0}^{2^k-1} c_{i,0}\epsilon_i cos(2i\pi t) + \sum_{i=0}^{2^k-1}\sqrt{2}c_{i,-1}sin(2(i+1)\pi t),$$

which implies,

$$c_{i,0} = 0 \qquad \text{for all} \qquad 0 \le i \le 2^k - 1, \tag{2.19}$$

and

$$c_{i,-1} = \begin{cases} 0 & i \neq n-1, \\ \frac{1}{2n\pi} & i = n-1. \end{cases} \tag{2.20}$$

Using Eqs. (2.19) and (2.20), for $k = 1$ and $n = 1$ we have:

$$\int_0^t \psi_{1,0}(y)dy = [\frac{1}{2\pi}, 0, 0, 0]\Psi_4(t), \tag{2.21}$$

Also for $n = 0$ and $m = 0$, we have:

$$\int_0^t \psi_{0,0}(y)dy = \int_0^t dy = t. \tag{2.22}$$

Comparing Eqs. (2.17) and (2.22) we have:

$$t=\sum_{i=0}^{2^k-1} c_{i,0}\epsilon_i cos(2i\pi t)+\sum_{i=0}^{2^k-1}\sqrt{2}c_{i,-1}sin(2(i+1)\pi t),$$

which implies,

$$c_{i,-1}=-\frac{\sqrt{2}}{2(i+1)\pi} \qquad \text{for all} \qquad 0\leq i\leq 2^k-1, \tag{2.23}$$

and

$$c_{i,0}=\begin{cases} 0 & i\neq 0,\\ \frac{1}{2} & i=0,\end{cases} \tag{2.24}$$

Using Eqs. (2.23) and (2.24) for $k=1$ we have:

$$\int_0^t \psi_{0,0}(y)dy=[-\frac{\sqrt{2}}{2\pi},\frac{1}{2},-\frac{\sqrt{2}}{4\pi},0]\Psi_4(t). \tag{2.25}$$

For $n=0$ and $m=-1$, we have:

$$\int_0^t \psi_{0,-1}(y)dy=\int_0^t \sqrt{2}sin(2\pi y)dy=\frac{\sqrt{2}}{2\pi}(1-cos(2\pi t)). \tag{2.26}$$

Comparing Eqs. (2.17) and (2.26), we have :

$$\frac{\sqrt{2}}{2\pi}(1-cos(2\pi t))=\sum_{i=0}^{2^k-1} c_{i,0}\epsilon_i cos(2i\pi t)+\sum_{i=0}^{2^k-1}\sqrt{2}c_{i,-1}sin(2(i+1)\pi t).$$

Using the same argument that we applied in previous paragraphs, we obtain:

$$\int_0^t \psi_{0,-1}(y)dy=[0,\frac{\sqrt{2}}{2\pi},0,-\frac{1}{2\pi}]\Psi_4(t). \tag{2.27}$$

Moreover for $n\neq 0$ and $m=-1$, we get:

$$\int_0^t \psi_{n,-1}(y)dy=\int_0^t \sqrt{2}sin(2(n+1)\pi y)dy=\frac{\sqrt{2}}{2(n+1)\pi}(1-cos(2(n+1)\pi t)). \tag{2.28}$$

Comparing Eqs. (2.17) and (2.28), we have:

$$\frac{\sqrt{2}}{2(n+1)\pi}(1-cos(2(n+1)\pi t))=\sum_{i=0}^{2^k-1} c_{i,0}\epsilon_i cos(2i\pi t)+\sum_{i=0}^{2^k-1}\sqrt{2}c_{i,-1}sin(2(i+1)\pi t),$$

which implies,

$$c_{i,-1}=0 \qquad \text{for all} \qquad 0\leq i\leq 2^k-1, \tag{2.29}$$

and

$$c_{i,0}=\begin{cases} 0 & i\neq n+1,\\ \frac{\sqrt{2}}{4\pi} & i=0,\\ -\frac{1}{2(n+1)\pi} & i=n+1.\end{cases} \tag{2.30}$$

Using Eqs. (2.29) and (2.30) for $k=1$ and $n=1$ we obtain :

$$\int_0^t \psi_{1,-1}(y)dy=[0,\frac{\sqrt{2}}{4\pi},0,0]\Psi_4(t). \tag{2.31}$$

Now using Eqs. (2.21),(2.25),(2.27)and(2.31), we have:

$$\int_0^t \Psi_4(y)dy = P_{4\times 4}\Psi_4(t),$$

where $\Psi_4(t) = [\psi_{0,-1}, \psi_{0,0}, \psi_{1,-1}, \psi_{1,0}]^T$ and $P_{4\times 4} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2\pi} & 0 & -\frac{1}{2\pi} \\ -\frac{\sqrt{2}}{2\pi} & \frac{1}{2} & -\frac{\sqrt{2}}{4\pi} & 0 \\ 0 & \frac{\sqrt{2}}{4\pi} & 0 & 0 \\ \frac{1}{2\pi} & 0 & 0 & 0 \end{bmatrix}$.

$P_{4\times 4}$ can be written as $P_{4\times 4} = \begin{bmatrix} F_{2\times 2} & Q_{2\times 2} \\ -Q^t_{2\times 2} & 0_{2\times 2} \end{bmatrix}$, where $F = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2\pi} \\ -\frac{\sqrt{2}}{2\pi} & \frac{1}{2} \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & -\frac{1}{2\pi} \\ -\frac{\sqrt{2}}{4\pi} & 0 \end{bmatrix}$. In general, we have $\int_0^t \Psi(y)dy = P\Psi(t)$, where $\Psi(t)$ is given in Eq. (2.6) and for $k \geq 2$, P is a $2^{k+1} \times 2^{k+1}$ matrix given by

$$P_{2^{k+1}\times 2^{k+1}} = \begin{bmatrix} F_{2\times 2} & V_{2\times(2^{k+1}-2)} \\ -V^t_{(2^{k+1}-2)\times 2} & R_{(2^{k+1}-2)\times(2^{k+1}-2)} \end{bmatrix},$$

where

$$R_{(2^{k+1}-2)\times(2^{k+1}-2)} = \begin{bmatrix} 0_{2\times 2} & S_2^t & 0 & 0 & \dots & 0 \\ -S_2 & 0_{2\times 2} & S_3^t & 0 & \ddots & 0 \\ 0 & -S_3 & 0_{2\times 2} & S_4^t & \ddots & 0 \\ \vdots & 0 & -S_4 & \ddots & \ddots & 0 \\ 0 & \vdots & \vdots & \ddots & \ddots & S^t_{2^k-1} \\ 0 & 0 & 0 & \ddots & -S_{2^k-1} & 0_{2\times 2} \end{bmatrix},$$

and

$$V_{2\times(2^{k+1}-2)} = \begin{bmatrix} Q & T_{2^{k-1}+1} & T_{2^{k-1}+2} & \dots & T_{2^k} \end{bmatrix},$$

which

$$S_l = -\frac{1}{2\pi l}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, l = 2, 3, ...,$$

$$T_r = -\frac{\sqrt{2}}{2\pi r}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, r = 3, 4, ...,$$

$$F = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2\pi} \\ -\frac{\sqrt{2}}{2\pi} & \frac{1}{2} \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 0 & -\frac{1}{2\pi} \\ -\frac{\sqrt{2}}{4\pi} & 0 \end{bmatrix}.$$

3. A method to solve Fredholm integral equations

Consider the following Fredholm integral equations of the second kind

$$u(x) = f(x) + \int_0^1 K(x,t)u(t)dt \qquad 0 \le x, t < 1. \tag{3.1}$$

Using Wilson wavelets, we approximate $u(x)$, $f(x)$ and $K(x,t)$ as

$$u(x) = C^T \Psi(x), \tag{3.2}$$

$$f(x) = d^T \Psi(x), \tag{3.3}$$

and

$$K(x,t) = \Psi(x)^T K \Psi(t), \tag{3.4}$$

where C and $\Psi(x)$ are defined similarly to equations (2.5) and (2.6). Also K is a $2^{k+1} \times 2^{k+1}$ matrix, where the elements of K are as follows:

$$\int_0^1 \int_0^1 \psi_{ni}(x)\psi_{lj}(t)K(x,t)dtdx,\ n = 0, ..., 2^k - 1,\ i = -1, 0,\ l = 0, ..., 2^k - 1,\ j = -1, 0.$$

Moreover d is a $2^{k+1} \times 1$ matrix that is defined similarly to Eq. (2.5) and the elements of d can be obtained by $\int_0^1 f(x)\psi_{nm}(x)dx$ for $n = 0, ..., 2^k - 1, m = -1, 0$. Then by replacing Eqs. (3.2), (3.3) and (3.4) in Eq. (3.1) we have:

$$C^T \Psi(x) = d^T \Psi(x) + \int_0^1 \Psi(x)^T K \Psi(t)\Psi(t)^T C dt.$$

Orthonormality of Wilson wavelets implies that:

$$\Psi(x)^T C = \Psi(x)^T d + \Psi(x)^T K C. \tag{3.5}$$

Multiplying Eq. (3.5) by $\Psi(x)$ and integrating on $[0, 1)$ we obtain $C = (I - K)^{-1} d$, where I is the identity matrix.

4. A method to solve FID equations

Consider the following FID equation

$$u'(x) = f(x) + u(x) + \int_0^1 K(x,t)u(t)dt,\ 0 \le x, t < 1,\ u(0) = u_0, \tag{4.1}$$

where the function $f(t) \in L^2[0, 1)$ and the kernel $K(x,t) \in L^2([0, 1) \times [0, 1))$ are known and $u(x)$ is the unknown function to be determined. In order to use the Wilson wavelet, we approximate $u'(x)$, $u(0)$, $f(x)$ and $K(x,t)$ as

$$u'(x) = U'^T \Psi(x),\ u(0) = U_0^T \Psi(x),\ f(x) = X^T \Psi(x) \text{ and } K(x,t) = \Psi(x)^T K \Psi(t),$$

where U', U_0 and X are $2^{k+1} \times 1$ matrices whose coefficients are defined similarly to Eq. (2.5) and $\Psi(x)$ is the same as Eq. (2.6). Also K is a $2^{k+1} \times 2^{k+1}$ matrix and elements of K can be calculated as

$$\int_0^1 \int_0^1 \psi_{nm}(x)\psi_{n'm'}(t)K(x,t)dxdt,\ n, n' = 0, 1, ..., 2^k - 1,\ m, m' = -1, 0.$$

Moreover

$$\begin{aligned} u(x) = \int_0^x u'(t)dt + u(0) &= \int_0^x U'^T\Psi(t)dt + U_0^T\Psi(x) \\ &= U'^T P\Psi(x) + U_0^T\Psi(x) \\ &= (U'^T P + U_0^T)\Psi(x), \end{aligned}$$

where P is the operational matrix defined in section 2.
We consider $U^T = U'^T P + U_0^T$. Putting the above relations in Eq. (4.1), we have:

$$\Psi(x)^T U' = \Psi(x)^T X + \Psi(x)^T (P^T U' + U_0) + \int_0^1 \Psi(x)^T K\Psi(t)\Psi(t)^T (P^T U' + U_0)dt,$$

and by orthonormality of the Wilson wavelets we have:

$$\Psi(x)^T U' = \Psi(x)^T X + \Psi(x)^T (P^T U' + U_0) + \Psi(x)^T K(P^T U' + U_0),$$

thus we obtain:

$$(I - KP^T - P^T)U' = KU_0 + U_0 + X.$$

Now the last equation can be solved and using $U^T = U'^T P + U_0^T$ one can get $u(x)$.

5. Illustrative test problems

In this section, we consider some numerical examples to illustrate the efficiency and reliability of the proposed method. These examples are considered because their exact solutions are available. In the following examples, $\hat{m}_C$ and $\hat{m}_W$ denote the number of vectors for CAS wavelet and Wilson wavelet respectively. Because of structure of the Wilson wavelets defined in Eq. (2.2) which $m \in \{-2r-1, -2r\}$ by choosing every $r \in \mathbb{Z}$ we obtain the same results.

Consider the following Fredholm integral and FID equations

Example 5.1.

$$u(x) = cos(4\pi x) + \int_0^1 tu(t)dt \quad 0 \le x, t < 1, \text{ with exact solution } u(x) = cos(4\pi x).$$

Example 5.2.

$$u(x) = e^{2x+1/3} - \frac{1}{3}\int_0^1 e^{2x-\frac{5t}{3}} u(t)dt \quad 0 \le x, t < 1, \text{ with exact solution } u(x) = e^{2x}.$$

Example 5.3.

$$u'(x) = 1 - \frac{4}{3}x + u(x) + \int_0^1 xtu(t)dt,\ u(0) = 0, \text{with exact solution } u(x) = x.$$

Tables 1 - 3 show the results of a comparison between the L_2-errors for various $\hat{m}$'s when using the Wilson wavelet and CAS wavelet. Figures 1 - 3 show absolute error of Examples 5.1 - 5.3 by CAS wavelet and Wilson wavelet.

TABLE 1. L_2-errors in Example 5.1 for CAS and Wilson wavelets

$\hat{m}$		CAS $(M=1)$	Wilson
$\hat{m}_C$= 24	$\hat{m}_W = 16$	2.5222E-1	2.6587E-15
$\hat{m}_C$= 48	$\hat{m}_W = 32$	1.8072E-1	5.1391E-15
$\hat{m}_C$= 96	$\hat{m}_W = 64$	1.2804E-1	1.2384E-14

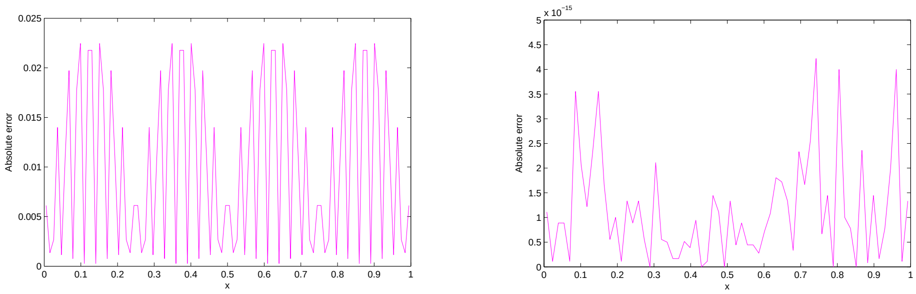

FIGURE 1. The graphs of the absolute errors in Example 5.1 for CAS wavelet (left) with $\hat{m}_C = 96$ and Wilson wavelet (right) with $\hat{m}_W$=64 .

TABLE 2. L_2-errors in Example 5.2 for CAS and Wilson wavelets

$\hat{m}$		CAS $(M=1)$	Wilson
$\hat{m}_C$= 384	$\hat{m}_W = 256$	2.9900E+1	5.8805E-1
$\hat{m}_C$= 768	$\hat{m}_W = 512$	7.2516E+1	5.8791E-1
$\hat{m}_C$= 1536	$\hat{m}_W = 1024$	1.2638E+2	5.8783E-1

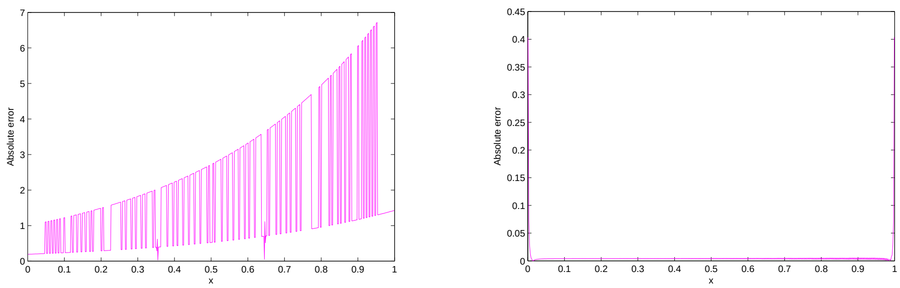

FIGURE 2. The graphs of the absolute errors for Example 5.2 by CAS wavelet (left) with $\hat{m}_C = 384$ and Wilson wavelet (right) with $\hat{m}_W$=256
.

6. Conclusion

A convergence analysis of the Wilson wavelets is investigated. A new operational matrix of integration for Wilson wavelets is derived and applied for solving of the FID

TABLE 3. L_2-errors in Example 5.3 for CAS and Wilson wavelets

$\hat{m}$		CAS ($M=1$)	Wilson
$\hat{m}_C$= 384	$\hat{m}_W = 256$	2.7717E0	9.1802E-2
$\hat{m}_C$= 768	$\hat{m}_W = 512$	9.2927E0	9.1891E-2
$\hat{m}_C$= 1536	$\hat{m}_W = 1024$	1.8226E+1	9.1940E-2

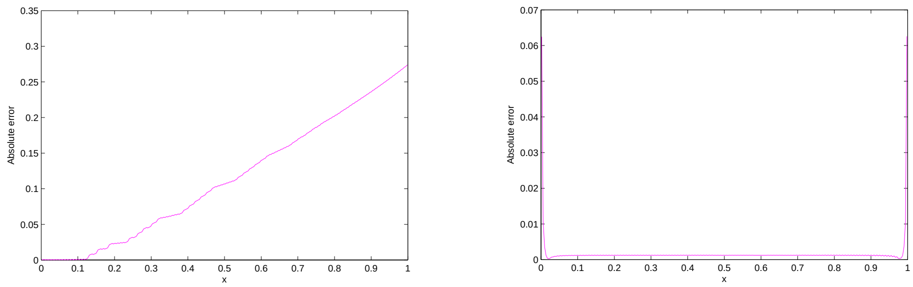

FIGURE 3. The graphs of the absolute errors for Example 5.3 by CAS wavelet (left) with $\hat{m}_C = 384$ and Wilson wavelet (right) with $\hat{m_W}$=256.

equations. Computational method based on these basis functions and their operational matrices of integration is proposed to solve the second Fredholm integral equations and the FID equations. Applicability and accuracy of the proposed method are checked on some numerical examples. Moreover, the results of the proposed method show that Wilson wavelets is effective.

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