Poincare Journal of Analysis & Applications
Vol. 2017 (2), 73-82

Analytic functions concerning with some subordinations

SHIGEYOSHI OWA†, JUNICHI NISHIWAKI, DANIEL BREAZ AND NICOLETA BREAZ

Date of Receiving	:	01.03.2017
Date of Revision	:	13.08.2017
Date of Acceptance	:	24.08.2017

Abstract. For analytic functions in the class $\mathcal{A}_n$ in the open unit disk $\mathbb{U}$, two subclasses $\mathcal{S}_n^*(\alpha)$ and $\mathcal{K}_n(\alpha)$ of starlike functions and convex functions are introduced. The object of the present paper is to discuss some interesting properties of functions in the classes $\mathcal{S}_n^*(\alpha)$ and $\mathcal{K}_n(\alpha)$ with some subordinations.

1. Introduction

Let $\mathcal{A}_n$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=n}^{\infty} a_k z^k \qquad (n \ = 2, 3, 4, \cdots) \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If we consider a function $f(z) \in \mathcal{A}_n$ which satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}) \tag{1.2}$$

then we say that $f(z)$ is starlike with respect to the origin in $\mathbb{U}$. We denote the subclass of $\mathcal{A}_n$ by $\mathcal{S}_n^*$ consisting of starlike functions in $\mathbb{U}$. Also, we say that $f(z)$ is convex in $\mathbb{U}$ if $f(z) \in \mathcal{A}_n$ satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathbb{U}). \tag{1.3}$$

This is equivalent to $zf'(z) \in \mathcal{S}_n^*$. We denote by $\mathcal{K}_n$ the subclass of $\mathcal{A}_n$ consisting of all convex functions in $\mathbb{U}$.

Let us consider a function $f(z)$ given by

$$f(z) = z + \frac{1}{n} z^n \qquad (n = 2, 3, 4, \cdots). \tag{1.4}$$

2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Analytic function, starlike function, convex function, subordination.
Communicated by: Shiv K. Kaushik
†Corresponding author.

This function $f(z)$ satisfies that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) = n - \mathrm{Re}\left(\frac{n(n-1)}{z^{n-1}+n}\right) = n - \frac{n(n-1)(n+r^{n-1}\cos(n-1)\theta)}{n^2+r^{2(n-1)}+2nr^{n-1}\cos(n-1)\theta} \tag{1.5}$$

for $z = re^{i\theta} \in \mathbb{U}$. Therefore, we see that

$$0 < \frac{n(1-r^{n-1})}{n-r^{n-1}} < \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \frac{n(1+r^{n-1})}{n+r^{n-1}} < \frac{2n}{n+1} \qquad (z \in \mathbb{U}). \tag{1.6}$$

Further, if we consider a function $f(z)$ given by

$$f(z) = z + \frac{1}{n^2}z^n \qquad (n = 2, 3, 4, \cdots), \tag{1.7}$$

then we have

$$0 < \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{2n}{n+1} \qquad (z \in \mathbb{U}). \tag{1.8}$$

Therefore, $f(z)$ given by (1.4) is in the class $\mathcal{S}_n^*$ and $f(z)$ given by (1.7) is in the class $\mathcal{K}_n$.

In view of the above, we say that $f(z) \in \mathcal{S}_n^*(\alpha)$ if $f(z) \in \mathcal{A}_n$ satisfies

$$0 < \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha \qquad (z \in \mathbb{U}) \tag{1.9}$$

for some real $\alpha > 1$, and that $f(z) \in \mathcal{K}_n(\alpha)$ if $f(z) \in \mathcal{A}_n$ satisfies

$$0 < \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \alpha \qquad (z \in \mathbb{U}) \tag{1.10}$$

for some real $\alpha > 1$. From the above definitions we say that $f(z) \in \mathcal{S}_n^*\left(\frac{2n}{n+1}\right)$ for $f(z)$ given by (1.4) and that $f(z) \in \mathcal{K}_n\left(\frac{2n}{n+1}\right)$ for $f(z)$ given by (1.7). Also, we know that $\mathcal{S}_n^*(\alpha) \subset \mathcal{S}_n^*(\beta)$ and $\mathcal{K}_n(\alpha) \subset \mathcal{K}_n(\beta)$ for $1 < \alpha < \beta$. It follows that if $f(z)$ is given by (1.4), then $f(z) \in \mathcal{S}_n^*(2)$ for any $n = 2, 3, 4, \cdots$, and that if $f(z)$ is given by (1.7), then $f(z) \in \mathcal{K}_n(2)$ for any $n = 2, 3, 4, \cdots$.

Let $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$. Then $p(z)$ is said to be subordinate to $q(z)$, written by $p(z) \prec q(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, and such that $p(z) = q(w(z))$. If $q(z)$ is univalent in $\mathbb{U}$, then the subordination $p(z) \prec q(z)$ is equivalent to $p(0) = q(0)$ and $p(\mathbb{U}) \subset q(\mathbb{U})$. We see many results of subordinations concerning with univalent, starlike and convex functions in $\mathbb{U}$ by Miller and Mocanu ([2],[3]), Obradović and Owa [7], Owa and Obradović [8], and Owa and Srivastava ([9],[10]).

2. Some Properties

We first consider

Theorem 2.1. *If $f(z) \in \mathcal{A}_n$ is given by*

$$f(z) = z + mz^n \qquad (n = 2, 3, 4, \cdots) \tag{2.1}$$

for some real $0 < m < 1$*, then*

$$\frac{1-mnr^{n-1}}{1-mr^{n-1}} \leqq \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \leqq \frac{1+mnr^{n-1}}{1+mr^{n-1}} \tag{2.2}$$

for $z = re^{i\theta} \in \mathbb{U}$ *and*

$$\frac{1-mn}{1-m} \leqq \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \leqq \frac{1+mn}{1+m} \qquad (z \in \mathbb{U}). \tag{2.3}$$

Proof. It follows from (2.1)

$$\frac{zf'(z)}{f(z)} = n - \frac{n-1}{1+mz^{n-1}}. \tag{2.4}$$

Letting $z = re^{i\theta} \in \mathbb{U}$, we have that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = n - \frac{(n-1)(1+mr^{n-1}\cos(n-1)\theta)}{1+m^2r^{2(n-1)}+2mr^{n-1}\cos(n-1)\theta}. \tag{2.5}$$

If we write

$$g(t) = \frac{1+mr^{n-1}t}{1+m^2r^{2(n-1)}+2mr^{n-1}t} \qquad (t = \cos(n-1)\theta), \tag{2.6}$$

then

$$g'(t) = \frac{mr^{n-1}(m^2r^{2(n-1)}-1)}{(1+m^2r^{2(n-1)}+2mr^{n-1}t)^2} < 0. \tag{2.7}$$

This shows us the inequality (2.2). Further, letting $r \to 1^-$ in (2.2), we obtain (2.3). $\square$

From Theorem 2.1, we easily say that

Theorem 2.2. *If* $f(z) \in \mathcal{A}_n$ *is given by*

$$f(z) = z + \frac{m}{n}z^n \qquad (n = 2, 3, 4, \cdots) \tag{2.8}$$

for some real m $(0 < m < 1)$*, then*

$$\frac{1-mnr^{n-1}}{1-mr^{n-1}} \leqq \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \leqq \frac{1+mnr^{n-1}}{1+mr^{n-1}} \tag{2.9}$$

for $z = re^{i\theta} \in \mathbb{U}$ *and*

$$\frac{1-mn}{1-m} \leqq \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \leqq \frac{1+mn}{1+m} \qquad (z \in \mathbb{U}). \tag{2.10}$$

Remark 2.3. Since $1 - mn \geqq 0$ for $0 < m \leqq \dfrac{1}{n}$, $f(z)$ given by (2.1) belongs to the class $\mathcal{S}_n^*\left(\dfrac{1+mn}{1+m}\right)$ for $0 < m \leqq \dfrac{1}{n}$ and $f(z)$ given by (2.8) belongs to the class $\mathcal{K}_n\left(\dfrac{1+mn}{1+m}\right)$ for $0 < m \leqq \dfrac{1}{n}$.

To discuss our next result, we have to recall here the following lemma due to Miller and Mocanu ([4],[5]) (also due to Jack [1]).

Lemma 2.4. *Let $w(z)$ be analytic in $\mathbb{U}$ with $w(0) = 0$. Then, if $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$, then we have*

$$z_0 w'(z_0) = m w(z_0) \tag{2.11}$$

and

$$\mathrm{Re}\left(1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right) \geqq m, \tag{2.12}$$

where $m \geqq 1$.

With the help of Lemma 2.1, we derive

Theorem 2.5. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha(1 + z^{n-1})}{\alpha + (2-\alpha)z^{n-1}} \qquad (z \in \mathbb{U}) \tag{2.13}$$

for some real $\alpha > 1$, then

$$\left|\frac{zf'(z)}{f(z)} - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}), \tag{2.14}$$

thus $f(z) \in \mathcal{S}_n^*(\alpha)$.

Proof. It follows from (2.13) that there exists an analytic function $w(z)$ in $\mathbb{U}$ with $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$, and

$$\frac{zf'(z)}{f(z)} = \frac{\alpha(1 + w(z)^{n-1})}{\alpha + (2-\alpha)w(z)^{n-1}} \qquad (z \in \mathbb{U}). \tag{2.15}$$

If we write that

$$F(z) = \frac{zf'(z)}{f(z)}, \tag{2.16}$$

then we obtain that

$$|w(z)^{n-1}| = \left|\frac{\alpha(F(z) - 1)}{\alpha - (2-\alpha)F(z)}\right| < 1 \qquad (z \in \mathbb{U}). \tag{2.17}$$

Since (2.17) gives us that

$$2|F(z)|^2 - \alpha(F(z) + \overline{F(z)}) < 0 \qquad (z \in \mathbb{U}), \tag{2.18}$$

we obtain (2.14). Noting that (2.14) implies that

$$0 < \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha \qquad (z \in \mathbb{U}), \tag{2.19}$$

we say that $f(z) \in \mathcal{S}_n^*(\alpha)$. □

For the class $\mathcal{K}_n(\alpha)$, we have

Theorem 2.6. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{\alpha(1 + z^{n-1})}{\alpha + (2-\alpha)z^{n-1}} \qquad (z \in \mathbb{U}) \tag{2.20}$$

for some real $\alpha > 1$, then

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}), \tag{2.21}$$

thus $f(z) \in \mathcal{K}_n(\alpha)$.

Next, we consider

Theorem 2.7. *If* $f(z) \in \mathcal{A}_n$ *satisfies*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) < \frac{(n-1)}{2(\alpha-1)} \qquad (z \in \mathbb{U}) \tag{2.22}$$

for some real $1 < \alpha \leqq 2$ *or*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) < \frac{(n-1)(\alpha-1)}{2} \qquad (z \in \mathbb{U}) \tag{2.23}$$

for some real $\alpha > 2$, *then*

$$\left|\frac{zf'(z)}{f(z)}-\frac{\alpha}{2}\right| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}), \tag{2.24}$$

therefore, $f(z) \in \mathcal{S}_n^*(\alpha)$.

Proof. Let us consider a function $w(z)$ which is analytic in $\mathbb{U}$, $w(0) = 0$, and given by

$$\frac{zf'(z)}{f(z)} = \frac{\alpha(1+w(z)^{n-1})}{\alpha+(2-\alpha)w(z)^{n-1}} \qquad (z \in \mathbb{U}) \tag{2.25}$$

for $f(z)$ satisfying (2.22) or (2.23). It follows from (2.25) that

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} = (n-1)\frac{zw'(z)}{w(z)}\left\{\frac{w(z)^{n-1}}{1+w(z)^{n-1}}-\frac{(2-\alpha)w(z)^{n-1}}{\alpha+(2-\alpha)w(z)^{n-1}}\right\}. \tag{2.26}$$

Let us suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1. \tag{2.27}$$

Then, Lemma 2.1 say that

$$z_0w'(z_0) = mw(z_0) \qquad (m \geqq 1). \tag{2.28}$$

Letting $w(z_0) = e^{i\theta}$, we have

$$\begin{aligned}\operatorname{Re}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0f'(z_0)}{f(z_0)}\right) &= (n-1)k\operatorname{Re}\left\{\frac{e^{i(n-1)\theta}}{1+e^{i(n-1)\theta}}-\frac{(2-\alpha)e^{i(n-1)\theta}}{\alpha+(2-\alpha)e^{i(n-1)\theta}}\right\}\\ &= (n-1)k\left\{\frac{1}{2}-\frac{(2-\alpha)(2-\alpha+\alpha\cos(n-1)\theta)}{\alpha^2+(2-\alpha)^2+2\alpha(2-\alpha)\cos(n-1)\theta}\right\}.\end{aligned} \tag{2.29}$$

Let us define the function $g(t)$ by

$$g(t) = \frac{2-\alpha+\alpha t}{\alpha^2+(2-\alpha)^2+2\alpha(2-\alpha)t} \qquad (t = \cos(n-1)\theta). \tag{2.30}$$

Then

$$g'(t) = \frac{4\alpha(\alpha-1)}{(\alpha^2+(2-\alpha)^2+2\alpha(2-\alpha)t)^2} > 0. \tag{2.31}$$

Since $g(t)$ is increasing for $t = \cos(n-1)\theta$, we obtain

$$\operatorname{Re}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0f'(z_0)}{f(z_0)}\right) \geqq \frac{(n-1)k}{2(\alpha-1)} \geqq \frac{n-1}{2(\alpha-1)}, \tag{2.32}$$

for $1 < \alpha \leqq 2$ and

$$\operatorname{Re}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}-\frac{z_0 f'(z_0)}{f(z_0)}\right) \geqq \frac{(n-1)(\alpha-1)k}{2} \geqq \frac{(n-1)(\alpha-1)}{2} \tag{2.33}$$

for $\alpha > 2$. This contradicts the conditions (2.22) and (2.23). Therefore, we say that there is no $w(z)$ such that $w(0) = 0$ and $|w(z_0)| = 1$ for $z_0 \in \mathbb{U}$. This means that $|w(z)| < 1$, for all $z \in \mathbb{U}$. From the above, we have

$$|w(z)^{n-1}| = \left| \frac{\alpha\left(\dfrac{zf'(z)}{f(z)}-1\right)}{\alpha-(2-\alpha)\dfrac{zf'(z)}{f(z)}} \right| < 1 \qquad (z \in \mathbb{U}) \tag{2.34}$$

and $f(z) \in \mathcal{S}_n^*(\alpha)$. □

3. Coefficient Inequalities

We consider some coefficient inequalities of functions $f(z)$ for classes $\mathcal{S}_n^*(\alpha)$ and $\mathcal{K}_n(\alpha)$.

Theorem 3.1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\sum_{k=n}^{\infty}(|2k-\alpha|+\alpha)|a_k| \leqq \alpha - |2-\alpha| \tag{3.1}$$

for some real $\alpha > 1$, then $f(z) \in \mathcal{S}_n^*(\alpha)$. The equality in (3.1) is attained for

$$f(z) = z + \sum_{k=n}^{\infty} \frac{(\alpha-|2-\alpha|)n\epsilon}{k(k+1)(|2k-\alpha|+\alpha)} z^k \qquad (|\epsilon| = 1). \tag{3.2}$$

Proof. We know that if $f(z) \in \mathcal{A}_n$ satisfies

$$\left|\frac{zf'(z)}{f(z)} - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}) \tag{3.3}$$

for $\alpha > 1$, then $f(z) \in \mathcal{S}_n^*(\alpha)$. The inequality (3.3) is equivalent to

$$|2zf'(z) - \alpha f(z)| < \alpha |f(z)|, \tag{3.4}$$

that is,

$$\left|(2-\alpha) + \sum_{k=n}^{\infty}(2k-\alpha)a_k z^{k-1}\right| < \alpha \left|1 + \sum_{k=n}^{\infty} a_k z^{k-1}\right| \tag{3.5}$$

for $z \in \mathbb{U}$. Therefore, if $f(z)$ satisfies

$$|2-\alpha| + \sum_{k=n}^{\infty} |2k-\alpha||a_k| \leqq \alpha - \alpha \sum_{k=n}^{\infty} |a_k|, \tag{3.6}$$

then $f(z) \in \mathcal{S}_n^*(\alpha)$. The inequality (3.6) is equivalent to (3.1). Further, if we consider a function $f(z)$ given by (3.2), then we have

$$\begin{aligned}\sum_{k=n}^{\infty}(|2k-\alpha|+\alpha)\,|a_k| &= \sum_{k=n}^{\infty}\frac{(\alpha-|2-\alpha|)n}{k(k+1)} \\ &= \sum_{k=n}^{\infty}(\alpha-|2-\alpha|)\,n\left(\frac{1}{k}-\frac{1}{k+1}\right)=\alpha-|2-\alpha|.\end{aligned} \tag{3.7}$$

This implies that $f(z)$ given by (3.2) satisfies the equality in (3.1). $\square$

For the class $\mathcal{K}_n(\alpha)$, we have

Theorem 3.2. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\sum_{k=n}^{\infty} k(|2k-\alpha|+\alpha)|a_k| \leqq \alpha-|2-\alpha| \tag{3.8}$$

for some real $\alpha > 1$, then $f(z) \in \mathcal{K}_n(\alpha)$. The equality in (3.8) is attained for $f(z)$ given by

$$f(z) = z + \sum_{k=n}^{\infty}\frac{(\alpha-|2-\alpha|)n\epsilon}{k^2(k+1)(|2k-\alpha|+\alpha)}z^k \qquad (|\epsilon|=1). \tag{3.9}$$

4. An Application for Superordinations

In this section, we consider an application of functions $f(z)$ for superordinations. Miller and Mocanu [6] have shown the following lemma for superordinations.

Lemma 4.1. *Let $h_1(z)$ and $h_2(z)$ be convex in $\mathbb{U}$ and $f(z)$ be univalent in $\mathbb{U}$ with $h_1(0) = h_2(0) = f(0)$. Let $\gamma \neq 0$ with* $\mathrm{Re}\gamma \geqq 0$. *If*

$$h_1(z) \prec f(z) \prec h_2(z) \qquad (z \in \mathbb{U}), \tag{4.1}$$

then

$$\frac{\gamma}{z^\gamma}\int_0^z h_1(t)t^{\gamma-1}dt \prec \frac{\gamma}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt \prec \frac{\gamma}{z^\gamma}\int_0^z h_2(t)t^{\gamma-1}dt, \tag{4.2}$$

where the middle integral is univalent.

Applying Lemma 4.1, we obtain

Theorem 4.2. *Let $f(z) \in \mathcal{A}_n$ satisfies*

$$\frac{\beta(1+z^{n-1})}{\beta+(2-\beta)z^{n-1}} \prec \frac{zf'(z)}{f(z)} \prec \frac{\alpha(1+z^{n-1})}{\alpha+(2-\alpha)z^{n-1}} \qquad (z \in \mathbb{U}) \tag{4.3}$$

for $1 < \beta < \alpha$. Then

$$h(z) \prec \frac{\gamma}{z^\gamma}\int_0^z \frac{t^\gamma f'(t)}{f(t)}dt \prec g(z) \qquad (z \in \mathbb{U}), \tag{4.4}$$

where $\gamma \neq 0$, $\mathrm{Re}\gamma \geqq 0$,

$$h(z) = \frac{\gamma}{z^\gamma}\int_0^z \frac{\beta(1+t^{n-1})t^{\gamma-1}}{\beta+(2-\beta)t^{n-1}}dt \tag{4.5}$$

and

$$g(z)=\frac{\gamma}{z^{\gamma}}\int_0^z \frac{\alpha(1+t^{n-1})t^{\gamma-1}}{\alpha+(2-\alpha)t^{n-1}}dt. \tag{4.6}$$

Proof. We define a function $h_1(z)$ by

$$h_1(z)=\frac{\beta(1+z^{n-1})}{\beta+(2-\beta)z^{n-1}} \qquad (z\in\mathbb{U}). \tag{4.7}$$

Then, we have

$$1+\frac{zh_1''(z)}{h_1'(z)}=(n-1)-\frac{2(n-1)(2-\beta)z^{n-1}}{\beta+(2-\beta)z^{n-1}}. \tag{4.8}$$

If we take $z=e^{i\theta}$ in (4.8), then we have that

$$\begin{aligned}\mathrm{Re}\left(1+\frac{zh_1''(z)}{h_1'(z)}\right) &= \mathrm{Re}\left((n-1)-\frac{2(n-1)(2-\beta)e^{i(n-1)\theta}}{\beta+(2-\beta)e^{i(n-1)\theta}}\right) \\ &= \mathrm{Re}\left((n-1)-\frac{2(n-1)(2-\beta)}{\beta e^{i(1-n)\theta}+2-\beta}\right) \\ &= (n-1)-\frac{2(n-1)(2-\beta)(\beta\cos(1-n)\theta+2-\beta)}{\beta^2+(2-\beta)^2+2\beta(2-\beta)\cos(1-n)\theta}.\end{aligned} \tag{4.9}$$

Letting

$$s(t)=\frac{\beta t+2-\beta}{\beta^2+(2-\beta)^2+2\beta(2-\beta)t} \qquad (t=\cos(1-n)\theta), \tag{4.10}$$

we see that

$$s'(t)=\frac{4\beta(\beta-1)}{(\beta^2+(2-\beta)^2+2\beta(2-\beta)t)^2}>0. \tag{4.11}$$

Therefore, $h_1(z)$ satisfies

$$\mathrm{Re}\left(1+\frac{zh_1''(z)}{h_1'(z)}\right)>0 \qquad (z\in\mathbb{U}). \tag{4.12}$$

This implies that $h_1(z)$ is convex in $\mathbb{U}$. The function

$$g_1(z)=\frac{\alpha(1+z^{n-1})}{\alpha+(2-\alpha)z^{n-1}} \qquad (z\in\mathbb{U}) \tag{4.13}$$

is also convex in $\mathbb{U}$. Furthermore, it is clear that

$$0<\mathrm{Re}h_1(z)<\beta \qquad (z\in\mathbb{U}) \tag{4.14}$$

and

$$0<\mathrm{Re}g_1(z)<\alpha \qquad (z\in\mathbb{U}). \tag{4.15}$$

Thus, the condition (4.3) of the theorem implies that $f(z)$ is starlike in $\mathbb{U}$, that is, that $f(z)$ is univalent in $\mathbb{U}$. Therefore, applying Lemma 4.1, we complete the proof of the theorem. □

Making $n=2$ and $\gamma=1$ in Theorem 4.1, we have

Corollary 4.3. Let $f(z)\in\mathcal{A}_2$ satisfies

$$\frac{\beta(1+z)}{\beta+(2-\beta)z} \prec \frac{zf'(z)}{f(z)} \prec \frac{\alpha(1+z)}{\alpha+(2-\alpha)z} \qquad (z\in\mathbb{U}) \tag{4.16}$$

for $0 < \beta < \alpha$. Then

$$h(z) \prec \frac{1}{z}\int_0^z \frac{tf'(t)}{f(t)}dt \prec g(z) \qquad (z \in \mathbb{U}), \tag{4.17}$$

where

$$h(z) = \begin{cases} \dfrac{\beta}{2-\beta}\left(1 + \dfrac{2(1-\beta)(\log(\beta + (2-\beta)z) - \log\beta)}{(2-\beta)z}\right) & (\beta \neq 2) \\ \\ 1 + \dfrac{1}{2}z \qquad (\beta = 2) \end{cases} \tag{4.18}$$

and

$$g(z) = \begin{cases} \dfrac{\alpha}{2-\alpha}\left(1 + \dfrac{2(1-\alpha)(\log(\alpha + (2-\alpha)z) - \log\alpha)}{(2-\alpha)z}\right) & (\alpha \neq 2) \\ \\ 1 + \dfrac{1}{2}z \qquad (\alpha = 2) \end{cases} \tag{4.19}$$

Finally, we consider

Theorem 4.4. *Let $f(z) \in \mathcal{A}_n$ be univalent in $\mathbb{U}$ and satisfy*

$$z + \frac{1}{m^2}z^m \prec f(z) \prec z + \frac{1}{n^2}z^n \qquad (z \in \mathbb{U}) \tag{4.20}$$

for natural numbers m and n such that $2 \leqq m < n$. Then, we have

$$h(z) \prec \frac{\gamma}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt \prec g(z) \qquad (z \in \mathbb{U}), \tag{4.21}$$

where $\gamma \neq 0$, $\mathrm{Re}\gamma \geqq 0$,

$$h(z) = \frac{\gamma}{\gamma+1}z + \frac{\gamma}{m^2(m+\gamma)}z^m \tag{4.22}$$

and

$$g(z) = \frac{\gamma}{\gamma+1}z + \frac{\gamma}{n^2(n+\gamma)}z^n. \tag{4.23}$$

Proof. If we write

$$h_1(z) = z + \frac{1}{m^2}z^m, \tag{4.24}$$

then, we have

$$\begin{aligned} \mathrm{Re}\left(1 + \frac{zh_1''(z)}{h_1'(z)}\right) &= \mathrm{Re}\left(m - \frac{m(m-1)}{m+z^{m-1}}\right) \\ &= m - \frac{m(m-1)(m + r\cos(m-1)\theta)}{m^2 + r^2 + 2mr\cos(m-1)\theta} \qquad (z = re^{i\theta}) \\ &> \frac{m(1-r)}{m-r} > 0. \end{aligned} \tag{4.25}$$

* This means that $h_1(z)$ and

$$g_1(z) = z + \frac{1}{n^2}z^n \tag{4.26}$$

are convex in $\mathbb{U}$. Noting that

$$\frac{n(1-r)}{n-r} < \frac{m(1-r)}{m-r} \tag{4.27}$$

for $0 \leqq r < 1$ and $2 \leqq m < n$, we say that $f(z), h_1(z)$ and $g_1(z)$ satisfy the conditions of Lemma 4.1. Further, since

$$\begin{aligned} \frac{\gamma}{z^\gamma}\int_0^z h_1(t)t^{\gamma-1}dt &= \frac{\gamma}{z^\gamma}\int_0^z \left(t^\gamma + \frac{1}{m^2}t^{m+\gamma-1}\right)dt \\ &= \frac{\gamma}{\gamma+1}z + \frac{\gamma}{m^2(m+\gamma)}z^m, \end{aligned} \tag{4.28}$$

we complete the proof of the theorem. □

References

[1] I. S. Jack: Functions starlike and convex of order alpha, J. London Math. Soc. **3** (1971), 469 – 474.

[2] S. S. Miller and P. T. Mocanu: On some classes of first order differential subordinations, Michigan Math. J. **32** (1985), 185 – 195.

[3] S. S. Miller and P. T. Mocanu: Differential subordinations and inequalities in the complex plane, J. Diff. Eqns. **67** (1987), 199 – 211.

[4] S. S. Miller and P. T. Mocanu: Briot-Bouquet differential subordinations, Complex Variables **33** (1997), 217 – 237.

[5] S. S. Miller and P. T. Monacu: Differential Subordinations, Theory and Applications, Marcel Dekker, 2000.

[6] S. S. Miller and P. T. Mocanu: Subordinations of differential superordinations, Complex Variables **48** (2003), 815 – 826.

[7] M. Obradović and S. Owa: On certain properties for some classes of starlike functions, J. Math. Anal. Appl. **145** (1987), 357 – 364.

[8] S. Owa and M. Obradović: An application of differential subordinations and some criteria for univalency, Bull. Austral. Math. Soc. **41** (1990), 487 – 494.

[9] S. Owa and H. M. Srivastava: Univalent and starlike generalized hypergeometric functions, Canad. J. Math. **39** (1987), 1057 – 1077.

[10] S. Owa and H. M. Srivastava: Analytic solution of a class of Briot-Bouquet differential equations, Current Topics in Analytic Function Theory, World Scientific, Singapore, London (1992), 252 – 259.

(Shigeyoshi Owa) Department of Mathematics, Faculty of Education, Yamato University, Katayama 2-5-1, Suita, Osaka 564-0082, Japan.

E-mail address: `owa.shigeyoshi@yamato-u.ac.jp`

(Junichi Nishiwaki) Department of Mathematics and Physics, Setsunan University, Neyagawa, Osaka 572-8508, Japan.

E-mail address: `jerjun2002@yahoo.co.jp`

(Daniel Breaz† and Nicoleta Breaz‡) Department of Mathematics-Informatics, Faculty of Sciences, "1 December 1918" University of Alba Iulia, 510009 Alba Iulia, Romania.

E-mail address: †`dbreaz@uab.ro`, ‡`nbreaz@uab.ro`