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Uniqueness and zeros of q-shift difference polynomials sharing one value

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Abstract. In this paper, we study uniqueness problems and zeros of q-shift difference polynomials of entire and meromorphic functions with zero order in the complex plane. The results of this paper extends the results obtained by Q. Zhao and J. Zhang [9].

1. Introduction and main results

The fundamental theorems and the standard notations of the Nevanlinna value distribution theory will be used in this article. By a meromorphic function we always mean a non-constant analytic function in the whole complex plane except at possible poles. If no poles occur, then it reduces to an entire function (see [1, 2, 3]).

We use $S(r, f)$ to denote any quantity satisfying $S(r, f) = o\{T(r, f)\}$, as $r \longrightarrow +\infty$, possibly outside a set of logarithmic density zero. Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions in the complex plane, $a \in \mathbb{C} \cup \{\infty\}$, we say that f and g share the value a IM (ignoring multiplicities) if $f - a$ and $g - a$ have the same zeros, they share the value a CM (counting multiplicities) if $f - a$ and $g - a$ have the same zeros with the same multiplicities. When $a = \infty$, the zeros of $f - a$ means the poles of f.

Let p be a positive integer and a be a complex constant, then we denote by $N_p(r, \frac{1}{f-a})$ the counting function of the zeros of $f - a$, where an m-fold zero is counted m times if $m \le p$ and p times if $m > p$.

In 1959, Hayman [4] proved that $f^n f^{'}$ takes every non-zero complex value infinitely often if $n \ge 3$. Yang and Hua [5] obtained some results about the uniqueness problems for entire functions. Since then the difference has become a subject of great interest (see [5, 6, 7, 8, 14]).

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In 2011, Liu and Cao [5] obtained results on the uniqueness and value distributions of q-shift difference polynomials. Here, we only state some results.

Theorem A. *Let $f(z)$ be a transcendental meromorphic (resp. entire) function with zero order and let m, n be positive integers and a, q be non-zero complex constants. If $n \geq 6$ (resp. $n \geq 2$), then $f(z)^n(f(z)^m - a)f(qz + c) - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a non-zero small function with respect to f. In particular, if $f(z)$ is a transcendental entire function and $\alpha(z)$ is a non-zero rational function, then m and n can be any positive integers.*

Theorem B. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order. If $n \geq m+5$, and $f(z)^n(f(z)^m - a)f(qz+c)$ and $g(z)^n(g(z)^m - a)g(qz+c)$ share a non-zero polynomial $p(z)$ CM, then $f(z) \equiv g(z)$.*

Recently, Zhao and Zhang [9] on the basis of Theorems A and B, studied the k-th derivative of q-shift difference polynomials and obtained the following results.

Theorem C. *Let $f(z)$ be a transcendental meromorphic function with zero order and let n, k be positive integers. If $n > k + 5$, then $(f(z)^n f(qz + c))^{(k)} - 1$ has infinitely many zeros.*

Theorem D. *Let $f(z)$ be a transcendental entire function with zero order and let n, k be positive integers, then $(f(z)^n f(qz + c))^{(k)} - 1$ has infinitely many zeros.*

Theorem E. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order and let n, k be positive integers. If $n > 2k + 5$, $(f(z)^n f(qz + c))^{(k)}$ and $(g(z)^n g(qz + c))^{(k)}$ share z CM, then $f = tg$ for a constant t with $t^{n+1} = 1$.*

Theorem F. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order and let n, k be positive integers. If $n > 2k+5$, and $(f(z)^n f(qz+c))^{(k)}$ and $(g(z)^n g(qz+c))^{(k)}$ share 1 CM, then $f = tg$ for a constant t with $t^{n+1} = 1$.*

Theorem G. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order and let n, k be positive integers. If $n > 5k+11$, and $(f(z)^n f(qz+c))^{(k)}$ and $(g(z)^n g(qz+c))^{(k)}$ share a value z IM, then $f = tg$ for a constant t with $t^{n+1} = 1$.*

Theorem H. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order and let n, k be positive integers. If $n > 5k+11$, and $(f(z)^n f(qz+c))^{(k)}$ and $(g(z)^n g(qz+c))^{(k)}$ share 1 IM, then $f = tg$ for a constant t with $t^{n+1} = 1$.*

In this paper, we study the zeros and uniqueness of q-shift difference polynomials of the form

$(f(z)^n(f(z)^m - 1)\prod_{j=1}^{d} f(q_j z + c_j)^{v_j})^{(k)}$ where, we assume q_j, $c_j \in \mathbb{C}\backslash\{0\}$ $(j = 1, 2, ..., d)$ are constants, n, m, d, v_j $(j = 1, ..., d)$ are positive integers, $\nu = \sum_{j=1}^{d} v_j = v_1 + v_2 + ... + v_d$ and hence obtain the following results which extends the results obtained by Zhao and

Zhang [9].

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function with zero order, let n, m, d, k and $v_j(j=1,...,d)$ be positive integers. If $n > k+2\nu+3$, then $(f(z)^n(f(z)^m-1)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j})^{(k)}-1$ has infinitely many zeros.*

Theorem 1.2. *Let $f(z)$ be a transcendental entire function with zero order, let n, m, d, k and $v_j(j=1,...,d)$ be positive integers. If $n > k+1$, then $(f(z)^n(f(z)^m-1)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j})^{(k)}-1$ has infinitely many zeros.*

Theorem 1.3. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, let n, m, d, k and $v_j(j=1,...,d)$ be positive integers. If $n > 2k+m+\nu+4$ and $(f(z)^n(f(z)^m-1)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j})^{(k)}$ and $(g(z)^n(g(z)^m-1)\prod_{j=1}^{d} g(q_jz+c_j)^{v_j})^{(k)}$ share z CM, then $f=tg$ for a constant t with $t^m=t^{n+\nu}=1$.*

Theorem 1.4. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, let n, m, d, k and $v_j(j=1,...,d)$ be positive integers. If $n > 2k+m+\nu+4$ and $(f(z)^n(f(z)^m-1)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j})^{(k)}$ and $(g(z)^n(g(z)^m-1)\prod_{j=1}^{d} g(q_jz+c_j)^{v_j})^{(k)}$ share 1 CM, then $f=tg$ for a constant t with $t^m=t^{n+\nu}=1$.*

Example 1.1. Let $f(z)=sin\, z$, and $g(z)=cos\, z$, $q_j=1$, $k=0$, $c_j=2\pi$, $n=7$, $\nu=1$ and $m=1$. Hence we have $n>1+\nu+4$ and $(f(z)^n(f(z)^m-1)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j})^{(k)}=(g(z)^n(g(z)^m-1)\prod_{j=1}^{d} g(q_jz+c_j)^{v_j})^{(k)}$. Therefore $(f(z)^n(f(z)^m-1)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j})^{(k)}$ and $(g(z)^n(g(z)^m-1)\prod_{j=1}^{d} g(q_jz+c_j)^{v_j})^{(k)}$ share 1 CM. Clearly, we get $f=tg$ for a constant t with $t^m=t^{n+\nu}=1$.

We also prove the following two theorems when sharing a single value IM.

Theorem 1.5. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, let n, m, d, k and $v_j(j=1,...,d)$ be positive integers. If $n > 5k+4m+4\nu+7$ and $(f(z)^n(f(z)^m-1)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j})^{(k)}$ and $(g(z)^n(g(z)^m-1)\prod_{j=1}^{d} g(q_jz+c_j)^{v_j})^{(k)}$ share z IM, then $f=tg$ for a constant t with $t^m=t^{n+\nu}=1$.*

Theorem 1.6. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, let n, m, d, k and $v_j(j=1,...,d)$ be positive integers. If $n > 5k+4m+4\nu+7$ and*

$(f(z)^n(f(z)^m-1)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j})^{(k)}$ and $(g(z)^n(g(z)^m-1)\prod_{j=1}^{d} g(q_jz+c_j)^{v_j})^{(k)}$ share 1 IM, then $f=tg$ for a constant t with $t^m=t^{n+\nu}=1$.

2. Preliminary Lemmas

The following q-shift difference analogue of the logarithmic derivative lemma is very important when considering q-shift difference polynomials.

Lemma 2.1 (see [10]). *Let $f(z)$ be a meromorphic function of zero order. Then on a set of logarithmic density 1*

$$m\left(r,\frac{f(qz+c)}{f(z)}\right)=o(T(r,f)).$$

Lemma 2.2 (see [11]). *If $f(z)$ is a non-constant zero order meromorphic function, then on a set of lower logarithmic density 1*

$$T(r,f(qz+c))=(1+o(1))T(r,f(z))+O(logr).$$

Lemma 2.3 (see [11]). *If $f(z)$ is a non-constant zero order meromorphic function, then on a set of lower logarithmic density 1*

$$N(r,f(qz+c))=(1+o(1))N(r,f(z))+O(logr).$$

When considering two non-constant meromorphic functions F and G that share at least one finite value CM, the following lemma plays a key role. In the original paper [4], $S(r,F)$ denotes any quantity satisfying $S(r,F)=o(T(r,F))$ as $r\longrightarrow\infty$ possibly outside a set of finite linear measure. So it holds when $S(r,F)=o(T(r,F))$ as $r\longrightarrow\infty$ possibly outside a set of logarithmic density 0.

Lemma 2.4 (see [4]). *Let F and G be two non-constant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds:*
(1) $max\{T(r,F),T(r,G)\}\leq N_2(r,1/F)+N_2(r,1/G)+N_2(r,F)+N_2(r,G)+S(r,F)+S(r,G)$,
(2) $FG=1$,
(3) $F=G$,
where $N_2(r,1/F)$ denotes the counting function of zeros of F such that the simple zeros are counted once and multiple zeros twice.

When two non-constant meromorphic functions share at least one finite value IM, then the following lemma is needed.

Lemma 2.5 (see [15]). *Let F and G be two non-constant meromorphic functions such that F and G share 1 IM and let*

$$H:=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right).$$

If $H \not\equiv 0$, *then*

$$\begin{aligned}T(r,F)+T(r,G) &\leq 2(N_2(r,1/F)+N_2(r,1/G)+N_2(r,F)+N_2(r,G))\\ &\quad +3(\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}(r,1/F)+\overline{N}(r,1/G))+S(r,F)+S(r,G).\end{aligned}$$

Lemma 2.6 (see [13]). *Let* $f(z)$ *be a non-constant meromorphic function and let* s, k *be two positive integers. Then*

$$\begin{aligned}&N_s(r,1/f^{(k)}) \leq T(r,f^{(k)})-T(r,f)+N_{s+k}(r,1/f)+S(r,f),\\ &N_s(r,1/f^{(k)}) \leq k\overline{N}(r,f)+N_{s+k}(r,1/f)+S(r,f).\end{aligned}$$

Clearly, $\overline{N}(r,1/f^{(k)})=N_1(r,1/f^{(k)})$.

Lemma 2.7 (see [12]). *Let* f *be an entire function of finite order and* $F=f(z)^n(f(z)^m-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}$,
Then, $T(r,F)=(n+m+\sigma)T(r,f)+S(r,f)$, $\sigma=\sum_{j=1}^{d} v_j$.

Lemma 2.8 (see [12]). *Suppose that* f *and* g *are transcendental entire functions of finite order,* $c_j (j=1,..,d)$ *are distinct finite complex numbers and* n, m, d, $v_j (j=1,..,d)$ *are integers. If* $n>m+5\sigma$ *and*
$f(z)^n(f(z)^m-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}=g(z)^n(g(z)^m-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}$.
Then, $f=tg$, *where* $t^m=t^{n+\sigma}=1$.

By applying the same method of Lemma 2.7 and 2.8, we can prove the following Lemmas respectively.

Lemma 2.9. *Let* f *be an entire function of finite order and* $F=f(z)^n(f(z)^m-1)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}$. *Then,* $T(r,F)=(n+m+\nu)T(r,f)+S(r,f)$, $\nu=\sum_{j=1}^{d} v_j = v_1+v_2+...+v_d$.

Lemma 2.10. *Suppose that* f *and* g *are transcendental entire functions of finite order,* $q_j, c_j (j=1,..,d)$ *are distinct finite complex numbers and* n, m, d, $v_j (j=1,..,d)$ *are integers. If* $n>m+5\nu$ *and*
$f(z)^n(f(z)^m-1)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}=g(z)^n(g(z)^m-1)\prod_{j=1}^{d} g(q_jz+c_j)^{v_j}$.
Then, $f=tg$, *where* $t^m=t^{n+\nu}=1$.

3. Proof of Theorems

Proof of Theorem 1.1 Let $F(z) = f(z)^n(f(z)^m - 1)\prod_{j=1}^{d} f(q_jz + c_j)^{v_j}$. Suppose that $F^{(k)} - 1$ has only finitely many zeros. Using the second main theorem, we have

$$T(r, F^{(k)}) \leq \overline{N}\left(r, \frac{1}{F^{(k)} - 1}\right) + \overline{N}\left(r, \frac{1}{F^{(k)}}\right) + \overline{N}\left(r, F^{(k)}\right) + S(r, F).$$

From Lemma 2.6, we have

$$T(r, F^{(k)}) \leq \overline{N}\left(r, \frac{1}{F^{(k)} - 1}\right) + T(r, F^{(k)}) - T(r, F) + N_{k+1}\left(r, \frac{1}{F}\right) + \overline{N}(r, F^{(k)}) + S(r, F).$$

Since $T(r, F) \leq (n+m+\nu)T(r, f)$, we have $S(r, F) = S(r, f)$. Thus the above inequality and Lemma 2.3 imply

$$\begin{aligned}
T(r, F) &\leq \overline{N}\left(r, \frac{1}{F^{(k)} - 1}\right) + N_{k+1}\left(r, \frac{1}{F}\right) + \overline{N}(r, F^{(k)}) + S(r, f) \\
&\leq \overline{N}\left(r, \frac{1}{F^{(k)} - 1}\right) + N_{k+1}\left(r, \frac{1}{f^n(f^m - 1)\prod_{j=1}^{d} f(q_jz + c_j)^{v_j}}\right) + \overline{N}(r, F) + S(r, f) \\
&\leq \overline{N}\left(r, \frac{1}{F^{(k)} - 1}\right) + (k + 1 + m + \nu)T(r, f) + 2T(r, f) + S(r, f) \\
&\leq \overline{N}\left(r, \frac{1}{F^{(k)} - 1}\right) + (k + 3 + m + \nu)T(r, f) + S(r, f). \qquad (3.1)
\end{aligned}$$

Also from Lemma 2.1, we have

$$\begin{aligned}
(n + m + \nu)T(r, f) &= T(r, f^{n+m+\nu}) = m(r, f^{n+m+\nu}) + N(r, f^{n+m+\nu}) \\
&\leq m\left(r, F(z) \cdot \frac{f(z)^\nu}{\prod_{j=1}^{d} f(q_jz + c_j)^{v_j}}\right) + N\left(r, F(z) \cdot \frac{f(z)^\nu}{\prod_{j=1}^{d} f(q_jz + c_j)^{v_j}}\right) + S(r, f) \\
&\leq T(r, F(z)) + 2\nu T(r, f) + S(r, f). \qquad (3.2)
\end{aligned}$$

From (1) and (2), we have $(n - k - 2\nu - 3)T(r, f) \leq \overline{N}\left(r, \frac{1}{F^{(k)} - 1}\right) + S(r, f)$.

Noting that $n > k + 2\nu + 3$, we conclude that $F^{(k)}(z) - 1$ has infinitely many zeros. Hence the proof of Theorem 1.1.

Proof of Theorem 1.2. Let $F(z)$ be stated as in proof of Theorem 1.1. Suppose that $F^{(k)}(z) - 1$ has only a finite number of zeros. Since by assumption, f is a transcendental entire function with zero order, there exists a polynomial $P(z)$ such that $F^{(k)}(z) - 1 = P(z)$. By integrating k times, we get from the above equation $F(z) = Q(z)$, where $Q(z)$ is a polynomial, given by $Q(z) = f(z)^n(f(z)^m - 1)\prod_{j=1}^{d} f(q_jz + c_j)^{v_j}$.Obviously, $Q(z) \not\equiv 0$.

Hence, we have

$$\begin{aligned}(n+m+\nu)T(r,f) &= T(r,f^{n+m+\nu}) = m(r,f^{n+m+\nu})\\ &\le m\left(r, F(z)\cdot\frac{f(z)^{\nu}}{\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}}\right) + S(r,f)\\ &\le T(r,F(z)) + S(r,f)\\ &= T(r,Q(z)) + S(r,f), \qquad (3.3)\end{aligned}$$

Also,

$$\begin{aligned}(n+m+\nu)T(r,f) &\le \overline{N}\left(r,\frac{1}{F^{(k)}-1}\right) + N_{k+1}\left(r,\frac{1}{f^n(f^m-1)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}}\right) + S(r,f)\\ &\le \overline{N}\left(r,\frac{1}{F^{(k)}-1}\right) + (k+m+\nu+1)T(r,f) + S(r,f)\end{aligned}$$

Hence, we have

$$(n-k-1)T(r,f) \le \overline{N}\left(r,\frac{1}{F^{(k)}-1}\right) + S(r,f).$$

Noting that $n > k+1$, we conclude that $F^{(k)}(z)-1$ has infinite zeros. Hence the proof of Theorem 1.2.

Proof of Theorem 1.5. Let $F(z)$ be defined as in Theorem 1.1, $G(z) = g(z)^n(g(z)^m - 1)\prod_{j=1}^{d} g(q_jz+c_j)^{v_j}$ and H be defined as in Lemma 2.5. Also, let $\Phi(z) = \frac{F^{(k)}(z)}{z}$, $\Psi(z) = \frac{G^{(k)}(z)}{z}$.

Then, $\Phi(z)$ and $\Psi(z)$ share 1 IM by the conditions. Since f is a transcendental entire function, from the definition of $\Phi(z)$ we deduce that $N_2(r,\Phi) = O(logr) = S(r,f)$. Using Lemmas 2.6 and 2.3, we have

$$\begin{aligned}N_2(r,1/\Phi) &\le N_2\left(r,\frac{1}{F^{(k)}}\right) + S(r,f)\\ &\le k\overline{N}(r,F) + N_{k+2}\left(r,\frac{1}{F}\right) + S(r,f)\\ &\le (k+2)\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^m-1}\right) + \sum_{j=1}^{d} N\left(r,\frac{1}{f(q_jz+c_j)^{\nu_j}}\right)\\ &\le (k+2)T(r,f) + mT(r,f) + \nu T(r,f) + S(r,f)\\ &\le (k+m+\nu+2)T(r,f) + S(r,f). \qquad (3.4)\end{aligned}$$

In the same manner, we get

$$\overline{N}(r,1/\Phi) \le (k+m+\nu+1)T(r,f)+S(r,f). \tag{3.5}$$

Therefore,

$$N_2(r,1/\Phi)+N_2(r,\Phi) \le (k+m+\nu+2)T(r,f)+S(r,f). \tag{3.6}$$

Similarly,

$$\overline{N}(r,1/\Psi) \le (k+m+\nu+1)T(r,g)+S(r,g). \tag{3.7}$$

Therefore,

$$N_2(r,1/\Psi)+N_2(r,\Psi) \le (k+m+\nu+2)T(r,g)+S(r,g). \tag{3.8}$$

By Lemmas 2.6 and 2.3, we get

$$\begin{aligned} N_2(r,1/\Phi) &\le N_2(r,1/F^{(k)})+S(r,f) \\ &\le T(r,F^{(k)})-T(r,F)+N_{k+2}\left(r,\frac{1}{F}\right)+S(r,f) \\ &\le T(r,F^{(k)})-T(r,F)+(k+2+m+\nu)T(r,f)+S(r,f) \\ &\le T(r,\Phi)-T(r,F)+(k+2+m+\nu)T(r,f)+S(r,f). \end{aligned} \tag{3.9}$$

From (3), we have

$$(n+m+\nu)T(r,f) \le T(r,F)+S(r,f). \tag{3.10}$$

Combining (9) and (10), we have

$$(n+m+\nu)T(r,f) \le T(r,\Phi)-N_2(r,1/\Phi)+(k+m+\nu+2)T(r,f)+S(r,f). \tag{3.11}$$

Similarly,

$$(n+m+\nu)T(r,g) \le T(r,\Psi)-N_2(r,1/\Psi)+(k+m+\nu+2)T(r,g)+S(r,g). \tag{3.12}$$

It follows from Lemma 2.5, that if $H \not\equiv 0$, then

$$T(r,\Phi)+T(r,\Psi) \le 2(N_2(r,1/\Phi)+N_2(r,1/\Psi))+3(\overline{N}(r,1/\Phi)+\overline{N}(r,1/\Psi))+S(r,\Phi)+S(r,\Psi). \tag{3.13}$$

Substituting (4)-(8), (11) and (12) in (13), we get

$$(n-5k-4m-4\nu-7)[T(r,f+T(r,g)] \le S(r,f)+S(r,g),$$

which is a contradiction, since $n > 5k+4m+4\nu+7$. Hence, we have $H \equiv 0$.
By integrating H twice, we have

$$\frac{1}{\Phi-1} = \frac{A}{\Psi-1}+B, \tag{3.14}$$

where $A \neq 0$ and B are constants. We can rewrite (14) as,

$$\Psi = \frac{(B-A)\Phi+(A-B-1)}{B\Phi-(B+1)} \tag{3.15}$$

Hence, we easily get $T(r,\Phi)=T(r,\Psi)+O(1)$. Thus, we have $S(r,f)=S(r,g)$.
Next, we discuss the following three cases.
Case 1. Suppose that $B\neq 0,\ -1$. From (15), we have

$$\overline{N}\left(r,1/\left(\Phi-\frac{B+1}{B}\right)\right)=\overline{N}(r,\Psi).$$

By second fundamental theorem and (5), we have

$$\begin{aligned}T(r,\Phi)&\leq \overline{N}(r,\Phi)+\overline{N}(r,1/\Phi)+\overline{N}\left(r,1/\left(\Phi-\frac{B+1}{B}\right)\right)+S(r,\Phi)\\&\leq (k+m+\nu+1)T(r,f)+S(r,f).\end{aligned}\tag{3.16}$$

In view of (9), (10) and (16), we have $(n-k-2)T(r,f)\leq T(r,\Phi)$, implying that $(n-2k-m-\nu-3)T(r,f)\leq S(r,f)$. This contradicts the assumption that $n>5k+4m+4\nu+7$.
Case 2. Suppose that $B=0$. From (15), we have

$$\Psi=A\Phi-(A-1).\tag{3.17}$$

If $A\neq 1$, then from (17), we can get $\overline{N}\left(r,1/\left(\Phi-\frac{A-1}{A}\right)\right)=\overline{N}(r,\frac{1}{\Psi})$. By second fundamental theorem and (8), we have

$$\begin{aligned}T(r,\Phi)&\leq \overline{N}(r,\Phi)+\overline{N}(r,1/\Phi)+\overline{N}\left(r,1/\left(\Phi-\frac{A-1}{A}\right)\right)+S(r,\Phi)\\&\leq (k+m+\nu+1)T(r,f)+(k+m+\nu+1)T(r,g)+S(r,f).\end{aligned}\tag{3.18}$$

Similarly, we have

$$T(r,\Psi)\leq (k+m+\nu+1)T(r,g)+(k+m+\nu+1)T(r,f)+S(r,g).\tag{3.19}$$

By (11), (12), (18) and (19), we have

$$(n-3k-2m-2\nu-4)(T(r,f)+T(r,g))\leq S(r,f)+S(r,g),$$

which is a contradiction, since by assumption $n>5k+4m+4\nu+7$. Thus, we have $A=1$, from (17), we get $\Phi=\Psi$, which implies that

$$(f(z)^n(f(z)^m-1)\prod_{j=1}^{d}f(q_jz+c_j)^{v_j})^{(k)}=(g(z)^n(g(z)^m-1)\prod_{j=1}^{d}g(q_jz+c_j)^{v_j})^{(k)}.$$

By integrating last inequality, we have

$$(f(z)^n(f(z)^m-1)\prod_{j=1}^{d}f(q_jz+c_j)^{v_j})^{(k-1)}=(g(z)^n(g(z)^m-1)\prod_{j=1}^{d}g(q_jz+c_j)^{v_j})^{(k-1)}+p(z),$$

where $p(z)$ is a polynomial of degree at most $k-1$. If $p(z)\not\equiv 0$, then from the second main theorem for small function, we have

$$\begin{aligned}(n+m+\nu)T(r,f)&\leq T(r,F)+S(r,f)\\&\leq \overline{N}(r,F)+\overline{N}(r,1/F)+\overline{N}(r,1/G)+S(r,f)\\&\leq (m+\nu+1)T(r,f)+(m+\nu+1)T(r,g)+S(r,f).\end{aligned}$$

Similarly,

$$(n+m+\nu)T(r,g) \leq (m+\nu+1)T(r,g) + (m+\nu+1)T(r,f) + S(r,g).$$

Therefore,

$$(n+m+\nu)[T(r,f)+T(r,g)] \leq (2m+2\nu+2)[T(r,f)+T(r,g)] + S(r,f) + S(r,g),$$

which is a contradiction, since by assumption $n > 5k+4m+4\nu+7$. Thus, $p(z) \equiv 0$. Repeating the integration $k-1$ times, we have

$$f(z)^n(f(z)^m-1)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j} = g(z)^n(g(z)^m-1)\prod_{j=1}^{d} g(q_jz+c_j)^{v_j}$$

Therefore by Lemma 2.10, we have $f = tg$, where $t^m = t^{n+\nu} = 1$.
Case 3. Suppose that $B = -1$. From (15), we have

$$\Psi = \frac{(A+1)\Phi - A}{\Phi}. \tag{3.20}$$

If $A \neq -1$, then from (20) we can deduce that $\overline{N}\left(r, 1/\left(\Phi - \dfrac{A}{A+1}\right)\right) = \overline{N}(r, \dfrac{1}{\Psi})$. Similar to the discussion as in Case 2, we obtain a contradiction. Hence, $A = -1$. From (20), we have $\Phi \cdot \Psi = 1$.

Hence,
$$(f(z)^n(f(z)^m-1)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j})^{(k)}.(g(z)^n(g(z)^m-1)\prod_{j=1}^{d} g(q_jz+c_j)^{v_j})^{(k)} = z^2. \tag{3.21}$$

Note that $n > 5k+4m+4\nu+7$, hence if z_0 is a zero of $f(z)$ with multiplicity p, then z_0 is a zero of $(f(z)^n(f(z)^m-1)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j})^{(k)}$ with multiplicity atleast $(n+m+\nu)p-k > 4k+4m+4\nu+7$ which is impossible by checking the right hand side of (21). Hence, zero is a Picard exceptional value of $f(z)$ and thus $f(z)$ is a constant, which is impossible. Hence the proof of Theorem 1.5.

Proof of Theorem 1.3. Let $\Phi(z)$ and $\Psi(z)$ be defined as in Theorem 1.5. Then $\Phi(z)$ and $\Psi(z)$ share 1 CM and from (9), we have

$$N_2(r,1/\Phi) \leq T(r,\Phi) - T(r,F) + (k+m+\nu+2)T(r,f) + S(r,f). \tag{3.22}$$

Similarly,

$$N_2(r,1/\Psi) \leq T(r,\Psi) - T(r,G) + (k+m+\nu+2)T(r,g) + S(r,g). \tag{3.23}$$

Assume that the Case(1) of Lemma 2.4 holds. Then, in view of Lemma 2.5 and (22), we can write

$$\begin{aligned}
T(r,\Phi) &\leq N_2(r,1/\Phi) + N_2(r,1/\Psi) + N_2(r,\Phi) + N_2(r,\Psi) + S(r,\Phi) + S(r,\Psi) \\
&\leq T(r,\Phi) - T(r,F) + (k+m+\nu+2)T(r,f) + N_2(r,1/G^{(k)}) + S(r,f) + S(r,g) \\
&\leq T(r,\Phi) - T(r,F) + (k+m+\nu+2)T(r,f) + k\overline{N}(r,,G) + N_{k+2}(r,1/G) + S(r,f) + S(r,g) \\
&\leq T(r,\Phi) - T(r,F) + (k+m+\nu+2)T(r,f) + (k+m+\nu+2)T(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

From the above inequality, we have

$$T(r,F) \leq (k+m+\nu+2)T(r,f)+(k+m+\nu+2)T(r,g)+S(r,f)+S(r,g). \quad (3.24)$$

Also from (3), we have

$$(n+m+\nu)T(r,f) \leq T(r,F)+S(r,f). \quad (3.25)$$

Combining (24) and (25), we have

$$(n-k-2)T(r,f) \leq (k+m+\nu+2)T(r,g)+S(r,f)+S(r,g).$$

Similarly,

$$(n-k-2)T(r,g) \leq (k+m+\nu+2)T(r,f)+S(r,f)+S(r,g).$$

Therefore,

$$(n-2k-m-\nu-4)[T(r,f)+T(r,g)] \leq S(r,f)+S(r,g),$$

which contradicts $n > 2k+m+\nu+4$. Hence, $\Phi(z)\cdot\Psi(z) \equiv 1$ or $\Phi(z) \equiv \Psi(z)$ by Lemma 2.4. Proceeding as in Theorem 1.5, we can get the proof of Theorem 1.3.
The proofs of Theorems 1.4 and 1.6 are similar to that of Theorems 1.3 and 1.5, hence we omit the proof.

4. Open Problems

Question 2.1. Can the zero order growth restriction in Theorems $1.1-1.6$ be extended to finite order?
Question 2.2. Can the difference polynomials in Theorems $1.1-1.6$ be replaced by difference polynomials of the form $f(z)^n(f(z)^m-1)\sum_{j=1}^{k} a_j f(q_j z+c_j)$?

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