*Poincare Journal of Analysis & Applications*
*Vol. 2017(2), 95-105*


# Wavelet packets and their vanishing moments

NIKHIL KHANNA[†], VARINDER KUMAR AND S. K. KAUSHIK

| | | |
|---|---|---|
| Date of Receiving | : | 26.05.2017 |
| Date of I$^{\text{st}}$ Revision | : | 21.07.2017 |
| Date of II$^{\text{nd}}$ Revision | : | 06.10.2017 |
| Date of Acceptance | : | 21.10.2017 |

***Abstract***. Sufficient condition for $\widehat{\omega}_{2n+1}(\xi)$ and $m_0(\xi)$ and their first $q-1$ derivatives to vanish respectively at $\xi = 0$ and $\xi = \pi$ has been obtained. A relationship between higher moments of $\omega_{2n}$ and $\omega_{2n+1}$ for wavelet packets associated with Haar wavelet has been given. Some results for Fourier transform of Hartley-like wavelet packets has been incurred and finally, sufficient conditions for higher vanishing moments of Hartley-like wavelet packets has been procured.

## 1. Introduction

In 1984, the combined effort of Grossmann and Morlet [5] directed to a complete mathematical study of the continuous wavelet transforms and their various applications. The wavelet theory has provided a new method for decomposing a function or a signal. In 1989, Mallat [10] and Meyer [13] presented the theory of multiresolution analysis. In 1996, Hernández and Weiss [6] studied higher vanishing moments of wavelets. Later in 2015, Khanna, Kumar and Kaushik [7, 8] studied vanishing moments of Hilbert transform of wavelets and give certain results to approximate the functions in $L^2(\mathbb{R})$. To fix the poor frequency localization of high frequency wavelet bases, its significant generalization was introduced by Coifman *et al*. [2] and thereby provide more adequate decomposition containing stationary and transient components. The wavelet packet method is a generalization of wavelet decomposition that offers a richer signal analysis. The wavelet packets can be used for numerous expansions of a given signal. Recently in 2016, Khanna, Kumar and Kaushik [9] introduced a notion of Hilbert transform of wavelet packets and Hartley-like wavelet packets. For various details related to wavelet packets, one may refer to [2, 6].

This paper deals with the study of wavelet packets. First, we give a sufficient condition under which $\widehat{\omega}_{2n+1}(\xi)$ and low pass filter $m_0(\xi)$ and their first $q-1$ derivatives vanishes at $\xi = 0$ and $\xi = \pi$ respectively. This is followed by a relationship between the $q^{th}$

2010 *Mathematics Subject Classification.* 42C40, 65T60.
*Key words and phrases.* Wavelet packets, Hartley-like wavelet packets, moments.
*Communicated by*: Sandra Saliani
[†]Corresponding author.

moment of $\omega_{2n}$ and $\omega_{2n+1}$ of wavelet packets $\{\omega_n\}$ associated with the Haar wavelet. Few results on Hartley-like wavelet packets and their higher vanishing moments have been obtained.

## 2. Preliminaries

A wavelet for $L^2(\mathbb{R})$ is a square integrable function $\psi$ such that the collection $\{\psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{R})$. Haar introduced an orthonormal basis which is now referred to as "Haar wavelet basis" with a box function $\phi = \chi[0, 1]$ and $\psi(x) = \phi(2x) - \phi(2x - 1)$ in 1909. The advantages of Haar wavelet being used are fewer intricacy of calculation which requires less hardware usage, quick, memory efficient and it is exactly reversible without fidelity loss but this has a major drawback of discontinuity, which restricts its practical applications. Good wavelets are usually constructed through Multiresolution analysis, a mathematical framework provided by Meyer and Mallat. Daubechies was the first to give explicit examples of scaling function and associated wavelets that have arbitrary high order of differentiability, have compact support and are orthonormal [3].

Wavelets are usually designed with higher vanishing moments which make them orthogonal to the low degree polynomials and therefore, they have the ability to compress non-oscillatory functions. The smoother $\psi$ is, the greater the number of vanishing moments. For any wavelet family, vanishing moments are necessary condition for the smoothness of the wavelet functions. Vanishing moments have far reaching implications for the efficient representation or approximation of functions. Specifically, the wavelet series of a smooth function will converge rapidly to the function as long as the wavelet has a lot of vanishing moments, resulting in a good approximation. For example, Haar wavelet has one vanishing moment, Daubechies $D4$ wavelet has two, etc.

Multiresolution analysis, MRA [6, 11], is an increasing sequence of closed subspaces $(V_j)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ satisfying

(i) $V_j \subseteq V_{j+1}$, for all $j \in \mathbb{Z}$,
(ii) $f \in V_j$ if and only if $f(2(\cdot)) \in V_{j+1}$, for all $j \in \mathbb{Z}$,
(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
(iv) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.
(v) There exists a function $\phi \in V_0$ such that $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for $V_0$.

The function $\phi$ whose existence is asserted in (v) is called a scaling function of the given MRA.
Scaling function $\phi$ solves the dilation equation

$$\phi(x) = 2 \sum_{k\in\mathbb{Z}} r_k \ \phi(2x + k)$$

with $|\widehat{\phi}(0)| = 1$ and the associated function $\psi$ is defined by

$$\psi(x) = 2 \sum_{k\in\mathbb{Z}} s_k \ \phi(2x + k).$$

It is well established ([6]), that a pair of quadrature mirror filter coefficients (QMF), $(r_k)_{k\in\mathbb{Z}}, (s_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, is associated to the MRA and the following relations

$$\widehat{\phi}(\xi) = m_0(\frac{\xi}{2})\ \widehat{\phi}(\frac{\xi}{2}),$$
$$\widehat{\psi}(\xi) = m_1(\frac{\xi}{2})\ \widehat{\phi}(\frac{\xi}{2})$$

are satisfied for a.e. $\xi \in \mathbb{R}$, where the symbols $m_0$ and $m_1$ are given by

$$m_0(\xi) = \sum_{k\in\mathbb{Z}} r_k\ e^{ik\xi},$$
$$m_1(\xi) = \sum_{k\in\mathbb{Z}} s_k\ e^{ik\xi} = e^{i\xi}\ \overline{m_0(\xi+\pi)} \tag{2.1}$$

which are known as low-pass and high-pass filters, respectively. Also, we have

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1,$$
$$|m_1(\xi)|^2 + |m_1(\xi+\pi)|^2 = 1. \tag{2.2}$$

Let $W_j$ be the corresponding wavelet subspaces: $W_j = \overline{span}\{\psi_{j,k} : k \in \mathbb{Z}\}$. Using the low-pass and high-pass filters associated with the MRA, the space $W_j$ can be split into two orthogonal subspaces, each of them can further be split into two parts. Repeating this process $j$ times, $W_j$ is decomposed into $2^j$ subspaces each generated by integer translates of a single function. If we apply this to each $W_j$ , then the resulting basis of $L^2(\mathbb{R})$, which will consist of integer translates of a countable number of functions (instead of all dilations and translations of the wavelet $\psi$), will give a better frequency localization. This basis is called the "wavelet packet basis."

A family of functions $\omega_n,\ n = 0, 1, 2, \ldots$ defined as

$$\omega_{2n}(x) = 2\ \sum_{k\in\mathbb{Z}} r_k\ \omega_n(2x+k),$$
$$\omega_{2n+1}(x) = 2\ \sum_{k\in\mathbb{Z}} s_k\ \omega_n(2x+k),$$

where $\psi = \omega_1$ and $\phi = \omega_0$ are often called mother and father wavelets, are called basic wavelet packets.

Fourier transform of wavelet packets $\omega_n$ are given by

$$\widehat{\omega}_{2n}(\xi) = m_0(\frac{\xi}{2})\ \ \widehat{\omega}_n(\frac{\xi}{2}), \tag{2.3}$$
$$\widehat{\omega}_{2n+1}(\xi) = m_1(\frac{\xi}{2})\ \ \widehat{\omega}_n(\frac{\xi}{2}). \tag{2.4}$$

Also, the set $\{\omega_n(x-k) : k \in \mathbb{Z},\ n = 0, 1, 2, \ldots\}$ is an orthonormal basis of $L^2(\mathbb{R})$. The family of wavelet packets $\{\omega_n\}$ define the family of subspaces of $L^2(\mathbb{R})$ corresponding to some orthonormal scaling function $\phi = \omega_0$ given by

$$U_j^n = \overline{span}\{\omega_n(2^j x - k) : k \in \mathbb{Z}\},\ j \in \mathbb{Z},\ n = 0, 1, 2, \ldots\ . \tag{2.5}$$

We notice that $U_j^0 = V_j,\ U_j^1 = W_j$ such that the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$ can be written as

$$U_{j+1}^0 = U_j^0 \oplus U_j^1,\ j \in \mathbb{Z}.$$

The generalized version of the above expression is

$$U^n_{j+1} = U^{2n}_j \oplus U^{2n+1}_j, \quad \text{for } n = 1, 2, 3, ...; j \in \mathbb{Z},$$

where $U^n_j$ is defined by (2.9).
For each $j = 1, 2, 3, ...$, the decomposition trick as given in [6] gives

$$\begin{cases} W_j & = U^1_j = U^2_{j-1} \oplus U^3_{j-1}, \\ W_j & = U^1_j = U^4_{j-2} \oplus U^5_{j-2} \oplus U^6_{j-2} \oplus U^7_{j-2}, \\ & \vdots \\ W_j & = U^1_j = U^{2^k}_{j-k} \oplus U^{2^k+1}_{j-k} \oplus ... \oplus U^{2^{k+1}-1}_{j-k}, \\ & \vdots \\ W_j & = U^1_j = U^{2^j}_0 \oplus U^{2^j+1}_0 \oplus ... \oplus U^{2^{j+1}-1}_0, \end{cases}$$

where $U^n_j$ is as defined in (2.9). Moreover, for each $j = 1, 2, ..., \ k = 1, 2, 3, ..., j$ and $m = 0, 1, ..., 2^k - 1$, the set $\{2^{\frac{j-k}{2}} \omega_p(2^{j-k}x - l) : l \in \mathbb{Z}\}$ is an orthonormal basis of $U^p_{j-k}$, where $p = 2^k + m$. However, all the elements of these bases have the general form given by $\omega_{j,n,k}(x) = 2^{\frac{j}{2}} \ \omega_n(2^j x - k), n = 0, 1, 2, ...; j, k \in \mathbb{Z}, x \in \mathbb{R}$.

## 3. Main results

Vanishing moments of Hilbert transform of wavelets were studied in [7]. Recall that a function $f(x)$ is said to have $q$ *vanishing moments* if

$$\int_{\mathbb{R}} x^r f(x) dx = 0, \ \text{for } 0 \leq r \leq q - 1,$$

where $Mom_r(f) = \int_{\mathbb{R}} x^r \ f(x) \ dx$ denotes the $r^{th}$ moment of $f$.

Motivated by Theorem 7.4 in [12], we give a sufficient condition for $\widehat{\omega}_{2n+1}(\xi)$ and low-pass filter $m_0(\xi)$ and their first $q-1$ derivatives to vanish at $\xi = 0$ and $\xi = \pi$ respectively.

**Theorem 3.1.** *Let $\omega_n, n = 0, 1, 2, ...$ be wavelet packets associated with an MRA and let $q \in \mathbb{N}_0$ such that $|\omega_0(t)| = O((1 + t^2)^{-\frac{q}{2}-1})$ and $|\omega_{2n+1}(t)| = O((1 + t^2)^{-\frac{q}{2}-1})$ and for any $0 \leq k < q, \ p_k(t) = \sum\limits_{m=-\infty}^{\infty} m^k \ \omega_{2n}(t - m)$ is a polynomial of degree $k$. Then for all $n \in \mathbb{N}_0, \ \widehat{\omega}_{2n+1}(\xi)$ and its first $q - 1$ derivatives are zero at $\xi = 0$ and the low-pass filter $m_0(\xi)$ given in (2.3) and its first $q - 1$ derivatives vanish at $\xi = \pi$.*

*Proof.* Since $\omega_{2n+1}$ is orthogonal to $\omega_{2n}$, it is also orthogonal to all translates of the polynomials $p_k, \ 0 \leq k < q$. This family forms a basis for the space of polynomials of degree atmost $q - 1$. Thus, $\omega_{2n+1}$ is orthogonal to any polynomial of degree $q - 1$ and in particular to $t^k$ for $0 \leq k < q$. This gives

$$Mom_r(\omega_{2n+1}) = 0 \text{ for } r = 0, 1, 2, ..., q - 1.$$

Now, since $|\omega_{2n+1}(t)| = O((1 + t^2)^{-\frac{q}{2}-1})$, we have

$$\int_{\mathbb{R}} |\omega_{2n+1}(-\xi)| \ (1 + |\xi|^q) d\xi < +\infty.$$

Thus, in view of Theorem 2.5 in [12], $\widehat{\omega}_{2n+1}(\xi)$ is $q$-times continuously differentiable. Similarly, one can prove that $\widehat{\omega}_0(\xi)$ is also $q$-times continuously differentiable.
Also, $\widehat{\omega}^{(k)}_{2n+1}(\xi) = \widehat{(-it)^k\omega_{2n+1}}(\xi), k = 0, 1, 2, ..., q-1$. This gives

$$\widehat{\omega}^{(k)}_{2n+1}(0) = \widehat{(-it)^k\omega_{2n+1}}(0) = \int_{\mathbb{R}} (-it)^k \ \omega_{2n+1}(t) \ dt = 0.$$

Thus, $\widehat{\omega}_{2n+1}(\xi)$ and its first $(q-1)$ derivatives are zero at $\xi = 0$.
Note that

$$\begin{aligned}\widehat{\omega}_{2n+1}(\xi) &= m_1\big(\frac{\xi}{2}\big) \ \widehat{\omega}_n\big(\frac{\xi}{2}\big) \\ &= e^{\frac{i\xi}{2}} \ \overline{m_0\big(\frac{\xi}{2}+\pi\big)} \ \widehat{\omega}_n\big(\frac{\xi}{2}\big).\end{aligned}$$

Since for all $n \in \mathbb{N}_0$, $\widehat{\omega}_{2n+1}(\xi)$ and its first $(q-1)$ derivatives are zero at $\xi = 0$ and $\widehat{\omega}_n(0) \neq 0$ $(n \neq 1)$, by differentiating the above expression $q$-times, we obtain

$$\widehat{\omega}^{(q)}_{2n+1}(\xi) = \sum_{l=0}^{q} {}^qC_l \ \frac{1}{2^q} \left[\sum_{m=0}^{l} {}^lC_m \ \frac{1}{2^l} e^{\frac{i\xi}{2}} \ i^m \ \overline{\{m_0\big(\frac{\xi}{2}+\pi\big)\}}^{(l-m)}\right] \widehat{\omega}^{(q-l)}_n\big(\frac{\xi}{2}\big).$$

This yields $m_0(\xi)$ and its first $(q-1)$ derivatives are zero at $\xi = \pi$. $\square$

The next result gives a relationship between the $q^{th}$ moments of $\omega_{2n}$ and $\omega_{2n+1}$ of wavelet packets $\{\omega_n\}$ associated with the Haar wavelet.

**Theorem 3.2.** *Let $\{\omega_n\}$ be a sequence of wavelet packets associated with the Haar wavelet. Then, the following formula gives a relationship between $Mom_q(\omega_{2n})$ and $Mom_q(\omega_{2n+1})$ as*

$$\begin{aligned}Mom_q(\omega_{2n+1}) = i\sum_{l=0}^{q} {}^qC_l \ \frac{1}{2^{2l}\pi^{l+1}} & \left[\frac{d^{l+1}}{d\xi^{l+1}}\left\{log\left(\Gamma\left(\frac{2\pi+\xi}{4\pi}\right)\right)\right\}\Bigg|_{\xi=0}\right. \\ & \left. + (-1)^{l+1} \ \frac{d^{l+1}}{d\xi^{l+1}}\left\{log\left(\Gamma\left(\frac{2\pi-\xi}{4\pi}\right)\right)\right\}\Bigg|_{\xi=0}\right] \times \ Mom_{q-l}(\omega_{2n}).\end{aligned}$$

*where $n, q \geq 0$ and $\Gamma(\xi)$ denotes the gamma function.*

*Proof.* The wavelet packets associated with the scaling function for the Haar wavelet are defined recursively by

$$\begin{aligned}\omega_{2n}(x) &= \omega_n(2x) + \omega_n(2x+1) \\ \omega_{2n+1}(x) &= -\omega_n(2x) + \omega_n(2x+1).\end{aligned}$$

Their Fourier transforms are given by (2.3) and (2.4) with $m_0(\xi) = \frac{e^{i\xi}+1}{2}$ and $m_1(\xi) = \frac{e^{i\xi}-1}{2}$. Then, we have

$$\begin{aligned}\widehat{\omega}_{2n+1}(\xi) &= \left(\frac{\frac{e^{i\frac{\xi}{2}}-1}{2}}{\frac{e^{i\frac{\xi}{2}}+1}{2}}\right) \widehat{\omega}_{2n}(\xi)\\ &= \left(\frac{i\ sin(\frac{\xi}{2})}{1+cos(\frac{\xi}{2})}\right) \widehat{\omega}_{2n}(\xi)\\ &= i\ tan(\frac{\xi}{4})\ \widehat{\omega}_{2n}(\xi).\end{aligned}$$

Differentiating both sides $q$-times and using $l^{th}$ derivative of function $tan(\xi)$ given in [1] , we have

$$\begin{aligned}\widehat{\omega}^{(q)}_{2n+1}(\xi) &= i\sum_{l=0}^{q} {}^{q}C_l\ \frac{1}{2^{2l}}\ tan^{(l)}\left(\frac{\xi}{4}\right)\ (\widehat{\omega}_{2n}(\xi))^{(q-l)}\\ &= i\sum_{l=0}^{q} {}^{q}C_l\ \frac{1}{2^{2l}}\ \left[\frac{\Gamma(l+1)}{\pi^{l+1}}\left\{(-1)^{l+1}\zeta\left(l+1, \frac{2\pi+\xi}{4\pi}\right)\right.\right.\\ &\qquad\left.\left. + \zeta\left(l+1, \frac{2\pi-\xi}{4\pi}\right)\right\}\right] \times\ (\widehat{\omega}_{2n}(\xi))^{(q-l)},\end{aligned}$$

where $\zeta$ and $\Gamma$ are Hurwitz zeta function and gamma functions, respectively and $\Gamma(l) = (l-1)!$.

Note that $\zeta(l+1,\xi) = \frac{(-1)^{l+1}}{l!}\ \vartheta^{(l)}(\xi)$, where $\vartheta^{(l)}$ is a polygamma function.
This gives

$$\begin{aligned}\widehat{\omega}^{(q)}_{2n+1}(\xi) &= i\sum_{l=0}^{q} {}^{q}C_l\ \frac{1}{2^{2l}\ \pi^{l+1}}\ \left[\vartheta^{(l)}\left(\frac{2\pi+\xi}{4\pi}\right)\right.\\ &\qquad\left. + (-1)^{l+1}\ \vartheta^{(l)}\left(\frac{2\pi-\xi}{4\pi}\right)\right] \times\ (\widehat{\omega}_{2n}(\xi))^{(q-l)}.\end{aligned}$$

Also, since $\vartheta^{(l)}(\xi) = \frac{d^{l+1}}{d\xi^{l+1}}\{log(\Gamma(\xi))\}$, we have

$$\begin{aligned}\widehat{\omega}^{(q)}_{2n+1}(\xi) &= i\sum_{l=0}^{q} {}^{q}C_l\ \frac{1}{2^{2l}\pi^{l+1}}\ \left[\frac{d^{l+1}}{d\xi^{l+1}}\left\{log\left(\Gamma\left(\frac{2\pi+\xi}{4\pi}\right)\right)\right\}\right.\\ &\qquad\left. + (-1)^{l+1}\ \frac{d^{l+1}}{d\xi^{l+1}}\left\{log\left(\Gamma\left(\frac{2\pi-\xi}{4\pi}\right)\right)\right\}\right] \times\ (\widehat{\omega}_{2n}(\xi))^{(q-l)}.\end{aligned}$$

Thus, we get

$$\begin{aligned}\widehat{\omega}^{(q)}_{2n+1}(0) &= i\sum_{l=0}^{q} {}^{q}C_l\ \frac{1}{2^{2l}\pi^{l+1}}\ \left[\frac{d^{l+1}}{d\xi^{l+1}}\left\{log\left(\Gamma\left(\frac{2\pi+\xi}{4\pi}\right)\right)\right\}\bigg|_{\xi=0}\right.\\ &\qquad\left. + (-1)^{l+1}\ \frac{d^{l+1}}{d\xi^{l+1}}\left\{log\left(\Gamma\left(\frac{2\pi-\xi}{4\pi}\right)\right)\right\}\bigg|_{\xi=0}\right] \times\ (\widehat{\omega}_{2n}(0))^{(q-l)}.\end{aligned}$$

If we denote the $q^{th}$ moment of wavelet packets $\omega_n$ by $Mom_q(\omega_n)$, then we have

$$Mom_q(\omega_n) = \mathcal{F}\{t^q\ \omega_n(t)\}(0) = i^q\ \widehat{\omega}_n^{(q)}(0).$$

This gives

$$\begin{aligned} Mom_q(\omega_{2n+1}) = i\sum_{l=0}^{q} {}^qC_l\ \frac{1}{2^{2l}\pi^{l+1}}\ & \left[\frac{d^{l+1}}{d\xi^{l+1}}\left\{log\left(\Gamma\left(\frac{2\pi+\xi}{4\pi}\right)\right)\right\}\Bigg|_{\xi=0}\right. \\ & \left. + (-1)^{l+1}\ \frac{d^{l+1}}{d\xi^{l+1}}\left\{log\left(\Gamma\left(\frac{2\pi-\xi}{4\pi}\right)\right)\right\}\Bigg|_{\xi=0}\right] \times\ Mom_{q-l}(\omega_{2n}). \end{aligned}$$

□

Recall from [9] that a family of functions $\widetilde{\mathcal{H}}\{\omega_n\}$, $n = 0, 1, 2, ...$ defined as $\widetilde{\mathcal{H}}\{\omega_{2n}\}(x) = 2 \sum_{k\in\mathbb{Z}} r_k\ \widetilde{\mathcal{H}}\{\omega_n\}(2x+k)$ and $\widetilde{\mathcal{H}}\{\omega_{2n+1}\}(x) = 2 \sum_{k\in\mathbb{Z}} s_k\ \widetilde{\mathcal{H}}\{\omega_n\}(2x+k)$, where $\widetilde{\mathcal{H}}\{\psi\} = \widetilde{\mathcal{H}}\{\omega_1\}$ and $\widetilde{\mathcal{H}}\{\phi\} = \widetilde{\mathcal{H}}\{\omega_0\}$ is said to be Hartley-like wavelet packets and the Hartley-like wavelet is defined as $\widetilde{\mathcal{H}}\{\psi\}(x) = \frac{1}{\sqrt{2}}(\psi(x) \mp \mathcal{H}\psi(x))$, where $\psi$ is a wavelet and $\mathcal{H}\psi$ is Hilbert transform of $\psi$.

Next, we give few results on Fourier transform of Hartley-like wavelet packets.

**Theorem 3.3.** *Let $\omega_n \in L^2(\mathbb{R})$ be wavelet packets associated with the scaling function $\omega_0$. Then,*

$$\sum_{k\in\mathbb{Z}}\sum_{n=0}^{2^r-1} |\widehat{\widetilde{\mathcal{H}}\{\omega_{2n}\}}(2^{r+1}(\xi+2k\pi))|^2 = |m_0(2^r\xi)|^2,$$

$$\sum_{k\in\mathbb{Z}}\sum_{n=0}^{2^r-1} |\widehat{\widetilde{\mathcal{H}}\{\omega_{2n+1}\}}(2^{r+1}(\xi+2k\pi))|^2 = |m_1(2^r\xi)|^2$$

*for almost every $\xi$ in $\mathbb{R}$ and $r \in \mathbb{N}_0$.*

*Proof.* In view of Theorem 7.4 in [9], we have

$$\sum_{k\in\mathbb{Z}}\sum_{n=0}^{2^r-1} |\widehat{\widetilde{\mathcal{H}}\{\omega_n\}}(2^r(\xi+2k\pi))|^2 = 1$$

for almost every $\xi \in \mathbb{R}$ and $r \in \mathbb{N}_0$.
This gives

$$\sum_{k\in\mathbb{Z}}\sum_{n=0}^{2^r-1} |\widehat{\widetilde{\mathcal{H}}\{\omega_{2n}\}}(2^{r+1}(\xi+2k\pi))|^2 + |\widehat{\widetilde{\mathcal{H}}\{\omega_{2n+1}\}}(2^{r+1}(\xi+2k\pi))|^2 = 1.$$

Since $\widehat{\widetilde{\mathcal{H}}\{\omega_{2n+1}\}}(\xi) = m_1(\frac{\xi}{2})\ \widetilde{\mathcal{H}}\{\omega_n\}(\frac{\xi}{2})$,

$$\sum_{k\in\mathbb{Z}}\sum_{n=0}^{2^r-1} |\widehat{\widetilde{\mathcal{H}}\{\omega_{2n}\}}(2^{r+1}(\xi+2k\pi))|^2 + \sum_{k\in\mathbb{Z}}\sum_{n=0}^{2^r-1} |m_1(2^r(\xi+2k\pi))|^2\ |\widehat{\widetilde{\mathcal{H}}\{\omega_n\}}(2^r(\xi+2k\pi))|^2 = 1.$$

This gives

$$\sum_{k\in\mathbb{Z}}\sum_{n=0}^{2^r-1}|\widehat{\widetilde{\mathcal{H}}\{\omega_{2n}}\}(2^{r+1}(\xi+2k\pi))|^2+|m_1(2^r\xi)|^2\sum_{k\in\mathbb{Z}}\sum_{n=0}^{2^r-1}|\widehat{\widetilde{\mathcal{H}}\{\omega_n}\}(2^r(\xi+2k\pi))|^2=1.$$

This further gives

$$\sum_{k\in\mathbb{Z}}\sum_{n=0}^{2^r-1}|\widehat{\widetilde{\mathcal{H}}\{\omega_{2n}}\}(2^{r+1}(\xi+2k\pi))|^2+|m_1(2^r\xi)|^2=1.$$

Since $|m_1(2^r\xi)|^2+|m_1(2^r\xi+\pi)|^2=1$, we have

$$\sum_{k\in\mathbb{Z}}\sum_{n=0}^{2^r-1}|\widehat{\widetilde{\mathcal{H}}\{\omega_{2n}}\}(2^{r+1}(\xi+2k\pi))|^2=|m_1(2^r\xi+\pi)|^2.$$

This gives

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{n=0}^{2^r-1}|\widehat{\widetilde{\mathcal{H}}\{\omega_{2n}}\}(2^{r+1}(\xi+2k\pi))|^2&=|e^{i(2^r\xi+\pi)}\ m_0(2^r\xi+2\pi)|^2\\&=|m_0(2^r\xi)|^2.\end{aligned}$$

Similarly, we obtain

$$\sum_{k\in\mathbb{Z}}\sum_{n=0}^{2^r-1}|\widehat{\widetilde{\mathcal{H}}\{\omega_{2n+1}}\}(2^{r+1}(\xi+2k\pi))|^2=|m_1(2^r\xi)|^2.$$

□

**Theorem 3.4.** *Let $\omega_n\in L^2(\mathbb{R})$ be wavelet packets associated with the scaling function $\omega_0$. Then,*

$$\sum_{p=1}^{\infty}\sum_{n=2^r}^{2^{r+1}-1}\sum_{k\in\mathbb{Z}}|\widehat{\widetilde{\mathcal{H}}\{\omega_{2n}}\}(2^{p+r+1}(\xi+2k\pi))|^2=|m_0(2^{p+r}\xi)|^2,$$

$$\sum_{p=1}^{\infty}\sum_{n=2^r}^{2^{r+1}-1}\sum_{k\in\mathbb{Z}}|\widehat{\widetilde{\mathcal{H}}\{\omega_{2n+1}}\}(2^{p+r+1}(\xi+2k\pi))|^2=|m_1(2^{p+r}\xi)|^2$$

*for almost every $\xi\in\mathbb{R}$ and $r=0,1,2,...,j$.*

*Proof.* Follows using Theorem 7.3 in [9] and the arguments given in the proof of Theorem 3.3. □

**Theorem 3.5.** *Let $\omega_1$ be a Lemarié-Meyer wavelet and let $\mathbf{X}=\{2^{\frac{j}{2}}\ \omega_n(2^jx-k),k\in\mathbb{Z},(n,j)\in F\}$ be a corresponding wavelet packet system, where $F=\bigcup_{L\in\mathbb{N}}F_L$ and $F_0=\big\{(n,j)\in\mathbb{N}\times\mathbb{Z}:j=-2r,r\in\mathbb{N}^*,n=\sum_{m=1}^{2r}\epsilon_m2^{m-1},\epsilon_1=0,\epsilon_{2l+1}=1,\ l=1,...,r-1\big\},F_1=\{(n+2^{-j},j)\in\mathbb{N}\times\mathbb{Z}:(n,j)\in F_0\},F_L=\{(L,0)\},L\geq2$. Then, a.e.*

*(i)* $\sum_{(n,j)\in F_0}|\widehat{\widetilde{\mathcal{H}}\{\omega_n}\}(2^{-j}\xi)|^2\geq|\widehat{\widetilde{\mathcal{H}}\{\omega_0}\}(\xi)|^2$ *and* $\sum_{(n,j)\in F_1}|\widehat{\widetilde{\mathcal{H}}\{\omega_n}\}(2^{-j}\xi)|^2\geq|\widehat{\widetilde{\mathcal{H}}\{\omega_1}\}(\xi)|^2,$

*(ii)* $\sum_{(n,j)\in F} |\widehat{\widetilde{\mathcal{H}}\{\omega_n\}}(2^{-j}\xi)|^2 \geq 1.$

*Proof.* (i) We have

$$\begin{aligned}\sum_{(n,j)\in F_0} |\widehat{\widetilde{\mathcal{H}}\{\omega_n\}}(2^{-j}\xi)|^2 &= \frac{1}{2}\sum_{(n,j)\in F_0} |1 \pm i\ sgn(2^{-j}\xi)|^2 |\widehat{\omega}_n(2^{-j}\xi)|^2 \\ &= \sum_{(n,j)\in F_0} |\widehat{\omega}_n(2^{-j}\xi)|^2.\end{aligned}$$

In view of Theorem 2.2 in [14], we have

$$\sum_{(n,j)\in F_0} |\widehat{\widetilde{\mathcal{H}}\{\omega_n\}}(2^{-j}\xi)|^2 \geq |\widehat{\widetilde{\mathcal{H}}\{\omega_0\}}(\xi)|^2.$$

Similarly $\sum_{(n,j)\in F_1} |\widehat{\widetilde{\mathcal{H}}\{\omega_n\}}(2^{-j}\xi)|^2 \geq |\widehat{\widetilde{\mathcal{H}}\{\omega_1\}}(\xi)|^2.$

(ii) Now, consider

$$\begin{aligned}\sum_{(n,j)\in F} |\widehat{\widetilde{\mathcal{H}}\{\omega_n\}}(2^{-j}\xi)|^2 &= \sum_{(n,j)\in F_0} |\widehat{\widetilde{\mathcal{H}}\{\omega_n\}}(2^{-j}\xi)|^2 + \sum_{(n,j)\in F_1} |\widehat{\widetilde{\mathcal{H}}\{\omega_n\}}(2^{-j}\xi)|^2 + \sum_{n=2}^{+\infty} |\widehat{\widetilde{\mathcal{H}}\{\omega_n\}}(\xi)|^2 \\ &\geq |\widehat{\widetilde{\mathcal{H}}\{\omega_0\}}(\xi)|^2 + |\widehat{\widetilde{\mathcal{H}}\{\omega_1\}}(\xi)|^2 + \sum_{n=2}^{+\infty} |\widehat{\widetilde{\mathcal{H}}\{\omega_n\}}(\xi)|^2 \\ &= \sum_{n=0}^{+\infty} |\widehat{\widetilde{\mathcal{H}}\{\omega_{2n}\}}(\xi)|^2 + \sum_{n=0}^{+\infty} |\widehat{\widetilde{\mathcal{H}}\{\omega_{2n+1}\}}(\xi)|^2 \\ &= \sum_{n=0}^{+\infty} \left|\widehat{\widetilde{\mathcal{H}}\{\omega_n\}}\left(\frac{\xi}{2}\right)\right|^2 \\ &= \sum_{n=0}^{+\infty} \left|\widehat{\widetilde{\mathcal{H}}\{\omega_n\}}\left(\frac{\xi}{2^q}\right)\right|^2, \text{ for any } q \in \mathbb{N}.\end{aligned}$$

Now, since $\{T_k\omega_n(\cdot), n \in \mathbb{N}, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, result follows as in the end of the proof of Theorem 2.2 in [14]. □

In the next result, we obtain a sufficient condition for higher vanishing moments for Hartley-like wavelet packets.

**Theorem 3.6.** *Let $\omega_n \in L^2(\mathbb{R})$ be wavelet packets such that for some $q \in \mathbb{N}$,*

*(i)* $x^q\omega_n(x) \in (L^1 \cap L^2)(\mathbb{R})$,

*(ii)* $\gamma^{q+1}\widehat{\omega_n}(\gamma) \in L^1(\mathbb{R})$.

*If $\{\omega_{j,n,k}\}_{j,k\in\mathbb{Z};n\in\mathbb{N}}$ is an orthogonal system on $\mathbb{R}$, then Hartley-like wavelet packets $\{\widetilde{\mathcal{H}}\{\omega_n\}\}_{n\in\mathbb{N}}$ has $q+1$ vanishing moments.*

*Proof.* In view of Theorem 9.3 in [15], one can prove that wavelet packets $\omega_n$ has $q+1$ vanishing moments.

Since $\omega_n \in L^2(\mathbb{R})$ and $x^q\omega_n(x) \in L^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}} |x|^{2p}\ |\omega_n(x)|^2\ dx \leq ||\omega_n||_2^2 + ||x^q\ \omega_n||_2^2 < +\infty,$$

for all $p = 0, 1, 2, ..., q$. Thus, $x^p\ \omega_n(x) \in L^2(\mathbb{R})$, for all $p = 0, 1, 2, ..., q$ and so, the moment formula for the Hilbert transform is well-defined, for all $p = 0, 1, 2, ..., q$.
Also, since $x^p\ \omega_n(x) \in L^2(\mathbb{R})$, for all $p = 0, 1, 2, ..., q$,

$$\int_{\mathbb{R}} x^p\ \mathcal{H}\omega_n(x)\ dx = 0, \text{ for all } p = 0, 1, 2, ..., q.$$

This gives $\mathcal{H}\omega_n$ has at least $(N+1)$ vanishing moments and therefore

$$\int_{\mathbb{R}} x^p\ \widetilde{\mathcal{H}}\{\omega_n\}(x)\ dx = \frac{1}{2}\int_{\mathbb{R}} x^p\ (\omega_n(x) \mp \mathcal{H}\omega_n(x))\ dx = 0.$$

□

The following result gives the relationship between the regularity of orthonormal wavelet packets $\omega_n$ and the vanishing moments of Hartley-like wavelet packets $\widetilde{\mathcal{H}}\{\omega_n\}, n \in \mathbb{N}$.

**Theorem 3.7.** *Let* $\omega_{j,n,k}(x) = 2^{\frac{-j}{2}}\omega(2^{-j}x - k), j, k \in \mathbb{Z}, n \in \mathbb{N}$ *constitutes an orthonormal set in* $L^2(\mathbb{R})$ *such that*

*(i)* $|\omega_n(x)| \leq K(1+|x|)^{-q-1-\epsilon}$, *where* $\epsilon > 0$,
*(ii)* $\omega_n \in \mathcal{C}^q(\mathbb{R})$ *such that* $(\omega_n)^{(p)}$ *is bounded for* $p \leq q$,
*(iii)* $x^q\omega_n(x) \in L^2(\mathbb{R})$.

*Then* $\int_{\mathbb{R}} x^p\widetilde{\mathcal{H}}\{\omega_n\}(x)dx = 0$ *for* $p = 0, 1, 2, ...., q$.

*Proof.* Follows using Corollary 5.5.2 in [4] and the arguments given in the proof of Theorem 3.6. □

**Acknowledgement**: The authors pay their sincere thanks to the reviewer for his/her critical remarks and suggestions for the improvement of this paper. Also, we are thankful to him/her for suggesting the reformulation of the statements of some results.

## References

[1] M. Benghorbal and R. M. Corless, The $n$th Derivative, *ACM SIGSAM Bulletin*, 36(1):10-14, 2002.
[2] R. R. Coifman, Y. Meyer, S. Quake, and M. V. Wickerhauser, Signal processing and compression with wavelet packets, Techn. Report, New Haven: Yale Univ., 1990.
[3] I. Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.*, 41:909-996, 1988.
[4] I. Daubechies, *Ten lectures on wavelets*, SIAM, Philadelphia, 1992.
[5] A. Grossmann and J. Morlet, Decomposition of Hardy functions into square integrable wavelets of constant shape, *SIAM J. Math. Anal.*, 15:723–-736, 1984.
[6] E. Hernández and G. Weiss, *A first course on wavelets*, CRC Press, Boca Raton, FL, 1996.
[7] N. Khanna, V. Kumar and S. K. Kaushik, Vanishing moments of Hilbert transform of wavelets, *Poincare J. Anal. Appl., Special Issue (IWWFA-II, Delhi)*, 2015(2):115-127, 2015.
[8] N. Khanna, V. Kumar and S. K. Kaushik, Approximations using Hilbert transform of wavelets, *J. Classical Anal.*, 7(2):83-91, 2015.

[9] N. Khanna, V. Kumar and S. K. Kaushik, Wavelet packet approximation, *Integ. Transf. Special Funct.*, 27(9):698-714, 2016.

[10] S. Mallat, Multiresolution approximations and wavelet orthonormal basis of $L^2(\mathbb{R})$, *Trans. Amer. Math. Soc.*, 315:69-–87, 1989.

[11] S. Mallat, A theory for multiresolution signal decomposition: The wavelet representation, *IEEE Trans. Patt. Recog. And Mach. Intell.*, 11:674–-693, 1989.

[12] S. Mallat, *A wavelet tour of signal processing*, Academic Press, 2009.

[13] Y. Meyer, *Wavelets and operators*, Cambridge University Press, Cambridge, 1992.

[14] S. Saliani, The solution of a problem of Coifman, Meyer, and Wickerhauser on wavelet packets, *Constr. Approx.*, 33:15-–39, 2011.

[15] D. F. Walnut, *An introduction to wavelet analysis*, Boston, MA: Birkhäuser, 2002.

(Nikhil Khanna) Department of Mathematics, University of Delhi, Delhi 110007, India

*E-mail address*: nikkhannak232@gmail.com

(Varinder Kumar) Department of Mathematics, Shaheed Bhagat Singh College, University of Delhi, Delhi 110017, India

*E-mail address*: varinder1729@gmail.com

(S. K. Kaushik) Department of Mathematics, Kirori Mal college, University of Delhi, Delhi 110007, India

*E-mail address*: shikk2003@yahoo.co.in