

REGULARITY AND CONSTRUCTION OF BOUNDARY MULTIWAVELETS

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Date of Receiving : 04. 03. 2015
 Date of Acceptance : 09. 03. 2015

Abstract. The conventional way of constructing boundary functions for wavelets on a finite interval is to form linear combinations of boundary-crossing scaling functions. In this article we focus instead on boundary functions defined by recursion relations. We show that the number of boundary functions at each end is uniquely determined, and derive conditions for determining regularity from the boundary recursion coefficients. We then develop an algorithm based on linear algebra which can be used to construct boundary functions with maximal regularity.

1. Introduction

The Discrete Wavelet Transform (DWT) is designed to act on infinitely long signals. For finite signals, the algorithm breaks down near the boundaries. This can be dealt with by extending the data by zero padding, extrapolation, symmetry, or other methods [4, 9, 11, 12], or by constructing special boundary functions [3, 6, 7].

Boundary functions can be constructed as linear combinations of boundary-crossing interior functions, or from boundary recursion relations. The standard examples in the literature have both properties, but the recursion relation approach can produce new boundary functions which cannot be derived using linear combinations. There are also linear combinations which are not refinable, but these are of little practical value, since the DWT algorithm requires recursion coefficients.

One way to find suitable boundary recursion coefficients is based on linear algebra. The infinite banded Toeplitz matrix representing the DWT is replaced by a finite matrix, by suitable end-point modifications.

A particular such method for scalar wavelets is presented in Madych [9]. The Madych approach can be generalized to multiwavelets under an additional assumption, which may or may not be satisfied for a given multiwavelet. We present a modified method which does not require this extra assumption.

Linear algebra completions are not unique; they all include multiplication by an arbitrary orthogonal matrix. A random choice of matrix does not in general produce

2010 *Mathematics Subject Classification.* 42C40.

Key words and phrases. wavelets on an interval, multiwavelets, discrete wavelet transform, boundary handling.

Communicated by. Palle Jorgensen

coefficients that correspond to any actual boundary function. Random choice also does not provide any approximation orders at the boundary.

Thus, we investigate the question of continuity and approximation order at the boundary in some detail. We then show how to impose these constraints in the algorithm, and in the process remove much or all of the non-uniqueness.

The results given below are based on the papers [1] and [2]. For more details, and most of the proofs, we refer to the original papers.

2. Wavelets and Multiwavelets

We assume that the reader is familiar with the basics of wavelet theory, so we only review a few basic concepts that we need in this paper.

A *multiresolution approximation* (MRA) of $L^2(\mathbb{R})$ is a chain of closed subspaces $\{V_n\}$, $n \in \mathbb{Z}$,

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R})$$

satisfying

- (i) $V_n \subset V_{n+1}$ for all $n \in \mathbb{Z}$;
- (ii) $f(x) \in V_n \iff f(2x) \in V_{n+1}$ for all $n \in \mathbb{Z}$;
- (iii) $f(x) \in V_n \implies f(x - 2^{-n}k) \in V_n$ for all $n, k \in \mathbb{Z}$;
- (iv) $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$;
- (v) $\bigcup_{n \in \mathbb{Z}} V_n = L^2(\mathbb{R})$;
- (vi) there exists a function vector

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_r(x) \end{pmatrix}, \quad \phi_i \in L^2(\mathbb{R}),$$

such that $\{\phi_j(x - k) : 1 \leq j \leq r, k \in \mathbb{Z}\}$ is an orthonormal basis for V_0 [10].

The function ϕ is called the *multiscaling function* of the given MRA. r is called the *multiplicity*.

ϕ satisfies a recursion relation

$$\phi(x) = \sqrt{2} \sum_k H_k \phi(2x - k). \quad (2.1)$$

The recursion coefficients H_k are $r \times r$ matrices.

The orthogonal projection P_n of a function $s \in L^2$ into V_n is given by

$$P_n s = \sum_{k \in \mathbb{Z}} \langle s, \phi_{nk} \rangle \phi_{nk},$$

where $\phi_{nk} := 2^{n/2} \phi(2^n x - k)$. $P_n s$ is interpreted as an approximation to s at scale 2^{-n} .

$Q_n = P_{n+1} - P_n$ is also an orthogonal projection onto a closed subspace W_n which is the orthogonal complement of V_n in V_{n+1} :

$$V_{n+1} = V_n \oplus W_n.$$

$Q_n s$ is interpreted as the fine detail in s at resolution 2^{-n} .

An orthonormal basis of W_0 is generated from the integer translates of a single function vector $\boldsymbol{\psi} \in L^2(\mathbb{R})$, called a *multiwavelet function*. Since $W_0 \subset V_1$, $\boldsymbol{\psi}$ can be represented as

$$\boldsymbol{\psi}(x) = \sqrt{2} \sum_n G_n \boldsymbol{\phi}(2x - n) \quad (2.2)$$

for some coefficient matrices G_n .

The *Discrete Wavelet Transform* (DWT) takes a function $s \in V_n$ for some n , and decomposes it into a coarser approximation at level $m < n$, plus the fine detail at the intermediate levels.

$$s = P_n s = P_m s + \sum_{k=m}^{n-1} Q_k s.$$

It suffices to describe the step from level n to level $n-1$. Since the signal $s \in V_n = V_{n-1} \oplus W_{n-1}$, we can represent it by its coefficients $\mathbf{s}_{nk}^* = \langle s, \boldsymbol{\phi}_{nk} \rangle$, $\mathbf{d}_{nk}^* = \langle s, \boldsymbol{\psi}_{nk} \rangle$ as

$$s = \sum_k \mathbf{s}_{nk}^* \boldsymbol{\phi}_{nk} = \sum_j \mathbf{s}_{n-1,j}^* \boldsymbol{\phi}_{n-1,j} + \sum_j \mathbf{d}_{n-1,j}^* \boldsymbol{\psi}_{n-1,j}.$$

Here $\mathbf{s}^*$ denotes the transpose of $\mathbf{s}$.

We find that

$$\mathbf{s}_{n-1,j} = \sum_k H_{k-2j} \mathbf{s}_{nk}, \quad \mathbf{d}_{n-1,j} = \sum_k G_{k-2j} \mathbf{s}_{nk}.$$

If we interleave the coefficients at level $n-1$

$$(\mathbf{sd})_{n-1} = \begin{pmatrix} \vdots \\ \mathbf{s}_{n-1,0} \\ \mathbf{d}_{n-1,0} \\ \mathbf{s}_{n-1,1} \\ \mathbf{d}_{n-1,1} \\ \vdots \end{pmatrix},$$

the DWT can be written as

$$(\mathbf{sd})_{n-1} = \Delta \mathbf{s}_n,$$

where

$$\Delta = \begin{pmatrix} \cdots & \cdots & \cdots & & & & \\ \cdots & T_{-1} & T_0 & T_1 & \cdots & & \\ & \cdots & T_{-1} & T_0 & T_1 & \cdots & \\ & & \cdots & T_{-1} & T_0 & T_1 & \cdots \\ & & & \cdots & \cdots & \cdots & \end{pmatrix}, \quad T_k = \begin{pmatrix} H_{2k} & H_{2k+1} \\ G_{2k} & G_{2k+1} \end{pmatrix}. \quad (2.3)$$

The matrix Δ is orthogonal. Signal reconstruction corresponds to

$$\mathbf{s}_n = \Delta^* (\mathbf{sd})_{n-1}.$$

A multiscaling function ϕ has *approximation order* p if all polynomials of degree less than p can be expressed locally as linear combinations of integer shifts of ϕ . That is,

there exist row vectors $\mathbf{c}_{jk}^*$, $j = 0, \dots, p-1$, $k \in \mathbb{Z}$, so that

$$x^j = \sum_k \mathbf{c}_{jk}^* \phi(x-k). \quad (2.4)$$

For orthogonal wavelets

$$\mathbf{c}_{jk}^* = \sum_{\ell=0}^j \binom{j}{\ell} k^{j-\ell} \boldsymbol{\mu}_\ell^*, \quad (2.5)$$

where $\boldsymbol{\mu}_j$ is the j th continuous moment of ϕ ,

$$\boldsymbol{\mu}_j = \int x^j \phi(x) dx.$$

A high approximation order is desirable in applications. A minimum approximation order of 1 is a required condition in many theorems.

3. Boundary Functions

The original DWT is defined for infinitely long signals. In practice, we can only transform a signal of finite length. What do we do near the boundaries?

A frequently used approach is *data extension*, for example by zeros, by symmetry, or by linear or higher order extrapolation [4, 9, 11, 12]. We will be using instead special *boundary wavelets*. We make the following assumptions:

- The original functions ϕ , ψ are orthogonal, continuous, provide approximation order $p \geq 1$, and are supported on $[0, N]$.
- The interval $[0, M]$ on which the signal and the wavelets are defined is long enough so that the left and right boundary functions do not overlap.
- We use all of the interior functions, that is, all $\phi(x-n)$, $\psi(x-n)$ whose support lies in $[0, M]$.
- There are L (scalar) left boundary scaling functions, R right boundary functions, collected into function vectors $\boldsymbol{\phi}^L$, $\boldsymbol{\phi}^R$. Likewise, we have L , R left and right boundary wavelet functions $\boldsymbol{\psi}^L$, $\boldsymbol{\psi}^R$. The left boundary functions have support on $[0, N-1]$, the right boundary functions have support on $[M-N+1, M]$.
- The boundary functions are *regular*, which means they are orthogonal, continuous, and provide approximation order ≥ 1 . This implies that $\boldsymbol{\phi}^L$, $\boldsymbol{\phi}^R$ are not identically zero at the endpoints.

The numbers L and R of boundary functions are the same at every level. Note that L and R are generally not equal, and they are not in general multiples of $r = \text{multiplicity of interior functions}$.

Lemma 3.1. [1] *Under our assumptions, L and R are uniquely determined.*

We sketch the idea of the proof, since some of the results are needed later.

Assume at first that there are only four recursion coefficients, so only two matrices T_0 , T_1 . Orthogonality of Δ means

$$\begin{aligned} T_0 T_0^* + T_1 T_1^* &= I, \\ T_0 T_1^* &= 0. \end{aligned}$$

From this one can show that $\rho_j = \text{rank}(T_j)$ satisfy $\rho_0 + \rho_1 = 2r$, and that T_0, T_1 have a common SVD

$$T_0 = U \begin{pmatrix} I_{\rho_0} & 0 \\ 0 & 0 \end{pmatrix} V^*, \quad T_1 = U \begin{pmatrix} 0 & 0 \\ 0 & I_{\rho_1} \end{pmatrix} V^*.$$

Here I_k denotes the $k \times k$ identity matrix.

With an argument due to Strang [11] one can show $L = \rho_1, R = \rho_0$.

If there are more than two matrices T_j , we form block matrices. For example if we have $T_0, \dots, T_3$, we use

$$\hat{T}_0 = \begin{pmatrix} T_0 & T_1 & T_2 \\ 0 & T_0 & T_1 \\ 0 & 0 & T_0 \end{pmatrix}, \quad \hat{T}_1 = \begin{pmatrix} T_3 & 0 & 0 \\ T_2 & T_3 & 0 \\ T_1 & T_2 & T_3 \end{pmatrix}$$

There are two systematic ways to construct boundary functions. We show the details for the left endpoint only, since the notation is easier there.

The first method uses linear combinations of boundary-crossing functions (see e. g. [7]).

$$\phi^L(x) = \sum_{k=-N+1}^{-1} C_k \phi(x-k) \quad \text{for } x > 0.$$

Each C_k is of size $L \times r$.

The second method uses recursion relations. For the interior functions, these are given by (2.1) and (2.2). For the boundary functions at the left end, they are

$$\begin{aligned} \phi^L(x) &= \sqrt{2} A \phi^L(2x) + \sqrt{2} \sum_{k=0}^{N-2} B_k \phi(2x-k), \\ \psi^L(x) &= \sqrt{2} E \phi^L(2x) + \sqrt{2} \sum_{k=0}^{N-2} F_k \phi(2x-k). \end{aligned}$$

A and E are of size $L \times L$, each B_k and F_k is of size $L \times r$.

Under our assumptions, the decomposition matrix on the interval $[0, M]$ takes the form

$$\Delta_M = \left(\begin{array}{c|cccccccc|c} L_0 & L_1 & \cdots & L_K & 0 & \cdots & \cdots & 0 & 0 \\ \hline 0 & T_0 & T_1 & \cdots & T_K & 0 & \cdots & 0 & 0 \\ \vdots & 0 & T_0 & T_1 & \ddots & T_K & 0 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & 0 & T_0 & T_1 & \cdots & T_K & 0 \\ \hline 0 & 0 & \cdots & \cdots & 0 & R_0 & R_1 & \cdots & R_K \end{array} \right). \quad (3.1)$$

Here

$$L_0 = \begin{pmatrix} A \\ E \end{pmatrix}, \quad L_k = \begin{pmatrix} B_{2k-2} & B_{2k-1} \\ F_{2k-2} & F_{2k-1} \end{pmatrix}, \quad k = 1, \dots, K, \quad K = \left\lfloor \frac{N-1}{2} \right\rfloor,$$

and similarly at the right end.

Theorem 3.2. [1] *Assume that the boundary functions are both refinable and linear combinations. Then A , B_k , C_k are related by*

$$\begin{aligned} CV &= AC, \\ CW &= B, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} B &= (B_0 \ B_1 \ \cdots \ B_{N-2}), \\ C &= (C_{-N+1} \ C_{-N+2} \ \cdots \ C_{-1}), \\ V &= \begin{pmatrix} H_{N-1} & H_N & 0 & \cdots & \cdots & 0 \\ H_{N-3} & H_{N-2} & H_{N-1} & H_N & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & H_3 \\ 0 & \cdots & \cdots & 0 & H_0 & H_1 \end{pmatrix}, \\ W &= \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ H_4 & \cdots & \cdots & H_N & 0 & 0 \\ H_2 & \cdots & \cdots & H_{N-2} & H_{N-1} & H_N \end{pmatrix}. \end{aligned}$$

Some observations about the two approaches are as follows.

- Eq. (3.2) is a form of eigenvalue problem. There are only a small number of choices for C , A , B . In fact, if there is a single boundary scaling function (i. e. $L = 1$), (3.2) is a standard eigenvalue problem.
- Not every refinable boundary function is a linear combination of boundary-crossing functions. Conversely, not every linear combination of boundary-crossing functions is refinable.
- Refinability is required for applications. Without the existence of A , B_k there is no DWT algorithm. Linear combinations are not required.
- If boundary functions as constructed as linear combinations, then continuity is automatic, and approximation order is easy to enforce. If the approximation order is greater than or equal to the number of boundary scaling functions, refinability is automatic. Orthogonality is hard, and this is where most of the effort lies.
- If boundary functions are constructed from recursion relations, then orthogonality is easy. Continuity and approximation order are harder to impose, but a suitable algorithm is detailed in this paper.

4. Construction of Boundary Functions Using Linear Algebra

In [9], Madych proposed an algorithm for constructing a decomposition matrix Δ_M of the form (3.1). His algorithm was for scalar wavelets. It turns out that it generalizes to multiwavelets only if $L = R$. Our algorithm does not require this assumption.

Assume again initially that there are only four recursion coefficients. The general case is handled as before, by forming block matrices.

It is sufficient to demonstrate the idea on the 3×3 orthogonal block matrix

$$\tilde{\Delta}_3 = \begin{pmatrix} T_0 & T_1 & 0 \\ 0 & T_0 & T_1 \\ T_1 & 0 & T_0 \end{pmatrix}.$$

We use pre- and postmultiplication by orthogonal matrices to make T_1 in the lower left disappear.

Let U and V be the orthogonal matrices from the joint SVD of T_0 and T_1 . Let

$$U_3^* = \begin{pmatrix} U^* & 0 & 0 \\ 0 & U^* & 0 \\ 0 & 0 & U^* \end{pmatrix}, \quad V_3 = \begin{pmatrix} V & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & V \end{pmatrix},$$

then

$$U_3^* \tilde{\Delta}_3 V_3 = \left(\begin{array}{cc|cc|cc} I_{\rho_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\rho_1} & 0 & 0 \\ \hline 0 & 0 & I_{\rho_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\rho_1} \\ \hline 0 & 0 & 0 & 0 & I_{\rho_0} & 0 \\ 0 & I_{\rho_1} & 0 & 0 & 0 & 0 \end{array} \right) \quad (4.1)$$

By inspection, a technique that produces the correct structure is to move the first ρ_0 columns to the end, and then interchange the first ρ_0 rows with the last ρ_1 rows. That amounts to multiplying from the right and left with appropriate permutation matrices.

Then

$$P_L U_3^* \tilde{\Delta}_3 V_3 P_R = \left(\begin{array}{c|cc|cc|c} I_{\rho_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{\rho_1} & 0 & 0 & 0 \\ \hline 0 & I_{\rho_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{\rho_1} & 0 \\ \hline 0 & 0 & 0 & I_{\rho_0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\rho_0} \end{array} \right)$$

Now let Q_L , Q_R be arbitrary orthogonal matrices of size $2\rho_1 \times 2\rho_1, 2\rho_0 \times 2\rho_0$, respectively. We multiply from the left with

$$U_L = \begin{pmatrix} Q_L & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & Q_R \end{pmatrix}$$

and from the right with

$$U_R = \begin{pmatrix} I_{\rho_1} & 0 & 0 & 0 \\ 0 & V^* & 0 & 0 \\ 0 & 0 & V^* & 0 \\ 0 & 0 & 0 & I_{\rho_0} \end{pmatrix}.$$

Let

$$Q_L = (Q_{L,0} | Q_{L,1}), \quad Q_R = (Q_{R,0} | Q_{R,1}), \quad V = (V_0 | V_1) \quad (4.2)$$

with $Q_{L,0}$, $Q_{L,1}$ of size $2\rho_1 \times \rho_1$, $Q_{R,0}$, $Q_{R,1}$ of size $2\rho_0 \times \rho_0$, and V_0 , V_1 of size $2r \times \rho_0$, $2r \times \rho_1$, respectively.

Then

$$\Delta_3 = U_L P_L U_3^* \tilde{\Delta}_3 V_3 P_R U_R = \begin{pmatrix} L_0 & L_1 & 0 & 0 \\ 0 & T_0 & T_1 & 0 \\ 0 & 0 & R_0 & R_1 \end{pmatrix}$$

where

$$L_0 = Q_{L,0}, \quad L_1 = Q_{L,1} V_1^*, \quad R_0 = Q_{R,0} V_0^*, \quad R_1 = Q_{R,1}.$$

If we consider Madych's approach in our notation, we find that it amounts to moving the second set of columns in (4.1) to the end, so that we end up with

$$\left(\begin{array}{c|cc|cc|c} I_{\rho_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{\rho_1} & 0 & 0 & 0 \\ \hline 0 & I_{\rho_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{\rho_1} & 0 \\ \hline 0 & 0 & 0 & I_{\rho_0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\rho_1} \end{array} \right)$$

This only yields matrices of the correct size if $\rho_0 = \rho_1$. Madych only considered scalar wavelets, where this condition is automatic.

The drawback of the algorithm is that Q_L , Q_R are arbitrary, and the recursion coefficients produced by most random choices do not correspond to actual boundary functions, nor do they provide any approximation orders.

In the following, we first investigate how to determine continuity and approximation orders from the boundary recursion coefficients. After that, we will show how to find suitable matrices Q_L , Q_R to impose these conditions.

5. Regularity Properties of Refinable Boundary Functions

First we consider continuity. The cascade algorithm at the left boundary is given by

$$\phi^{L,n+1}(x) = \sqrt{2}A\phi^{L,n}(2x) + \sqrt{2} \sum_{k=0}^{N-2} B_k \phi(2x-k)$$

Starting with an initial function $\phi^{L,0}$ with support $[0, N-1]$, we see that $\phi^{L,1}$ is given on $[(N-1)/2, N-1]$ by a linear combination of translates of $\phi(2x)$. This part of $\phi^{L,1}$ does not change in future steps. After step 2, $\phi^{L,2}$ is given on $[(N-1)/4, (N-1)/2]$ by a linear combination of translates of $\phi(4x)$, and so on. Thus, the cascade algorithm converges to a continuous limit on $(0, N-1]$. The only question is what happens at $x=0$.

It is easy to show that if the spectral radius $\rho(\sqrt{2}A) < 1$, the cascade algorithm will converge, but $\phi^L(0) = \mathbf{0}$. If $\rho(\sqrt{2}A) > 1$, the cascade algorithm will diverge for most starting guesses. It is conceivable but unlikely that there will be continuous solutions. We will not pursue this further.

We say that $\sqrt{2}A$ satisfies *Condition E(m)* if it is diagonalizable, has an m -fold eigenvalue 1, and all other eigenvalues are less than in magnitude. In this case,

$$(\sqrt{2}A)^n \rightarrow \Gamma = (\mathbf{r}_1, \dots, \mathbf{r}_m) \begin{pmatrix} \mathbf{l}_1^* \\ \vdots \\ \mathbf{l}_m^* \end{pmatrix}$$

where $\mathbf{r}_j, \mathbf{l}_j$ are the right and left eigenvectors to eigenvalue 1, normalized to $\mathbf{l}_j^* \mathbf{r}_j = 1$.

Theorem 5.1. [1] *If $\sqrt{2}A$ satisfies condition $E(m)$, then*

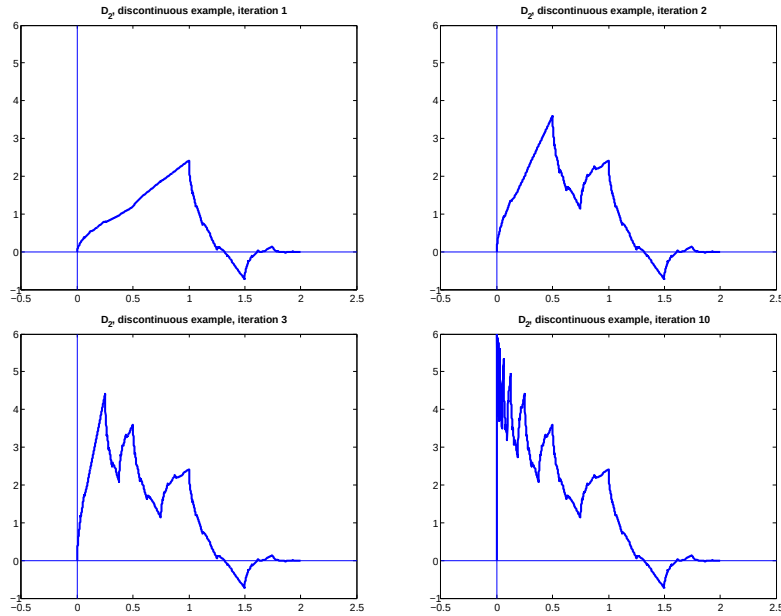
$$\phi^L(2^{-n}x_0) \rightarrow \Gamma \left[\phi^L(x_0) + (\sqrt{2}B_0)(I - \sqrt{2}H_0)^{-1} \phi(x_0) \right]$$

as $n \rightarrow \infty$.

In other words: Starting with any point $x_0 > 0$, the sequence of point values $\phi^L(2^{-n}x_0)$ converges. Thus, continuity of ϕ^L at 0 is equivalent to the condition that the limit is independent of x_0 . This appears hard to verify, but we will see below that this is actually a consequence of approximation order 1.

It can be shown that our regularity assumptions imply that $(I - \sqrt{2}H_0)^{-1}$ exists.

The following graph shows an example where the limit of $\phi(2^{-n}x_0)$ exists for every x_0 , but the resulting function is not continuous at 0. It is based on the Daubechies D_2 wavelet, with $A = \sqrt{2}/2$ and $B = (\sqrt{6}/4, 2)$.



We next consider approximation order. The approximation order conditions for interior functions are stated in (2.4). For left boundary functions, approximation order p means

$$x^j = \mathbf{l}_j^* \phi^L(x) + \sum_{k \geq 0} \mathbf{c}_{jk}^* \phi(x - k), \quad j = 0, \dots, p-1,$$

for suitable vectors $\mathbf{l}_j$.

Theorem 5.2. [1] *The left boundary functions have approximation order p if the interior functions do, and if there exist vectors $\mathbf{l}_j^*$ so that for $j = 0, \dots, p-1$*

$$\begin{aligned} \mathbf{l}_j^* (\sqrt{2}A) &= 2^{-j} \mathbf{l}_j^*, \\ \mathbf{l}_j^* (\sqrt{2}B_m) &= \gamma_{jm}^*, \quad m = 0, \dots, N-2, \end{aligned} \tag{5.1}$$

where the γ_{jm}^* are known vectors:

$$\gamma_{jm}^* = 2^{-j} \mathbf{c}_{jm}^* - \sqrt{2} \sum_{k=0}^{\lfloor m/2 \rfloor} \mathbf{c}_{jk}^* H_{m-2k}.$$

Corollary 5.3. If $\sqrt{2}A$ satisfies condition $E(1)$, then approximation order 1 implies continuity.

Proof. Approximation order 1 at $x = 0$ means

$$1 = \mathbf{l}_0^* \phi^L(x) + \mu_0^* \sum_{k \geq 0} \phi(x - k), \quad x \geq 0,$$

which for $x \in [0, 1]$ reduces to

$$1 = \mathbf{l}_0^* \phi^L(x) + \mu_0^* \phi(x).$$

The approximation order conditions (5.1) for $j = 0$ begin with

$$\begin{aligned} \mathbf{l}_0^* (\sqrt{2}A) &= \mathbf{l}_0^*, \\ \mathbf{l}_0^* (\sqrt{2}B_0) &= \mu_0^* - \mu_0^* \sqrt{2}H_0, \\ \mathbf{l}_0^* (\sqrt{2}B_1) &= \mu_0^* - \mu_0^* \sqrt{2}H_1, \\ &\dots \end{aligned}$$

The second of these implies

$$\mu_0 = \mathbf{l}_0^* (\sqrt{2}B_0) (I - \sqrt{2}H_0)^{-1},$$

so that

$$\begin{aligned} &\Gamma \left[\phi^L(x) + (\sqrt{2}B_0)(I - \sqrt{2}H_0)^{-1} \phi(x) \right] \\ &= \mathbf{r}_0 \cdot \left[\mathbf{l}_0^* \phi^L(x) + \mathbf{l}_0^* \sqrt{2}B_0 (I - \sqrt{2}H_0)^{-1} \right] \\ &= \mathbf{r}_0 \cdot \left[\mathbf{l}_0^* \phi^L(x) + \mu_0^* \phi(x) \right] = \mathbf{r}_0. \end{aligned}$$

□

It is not immediately clear whether continuity and $\phi^L(0) \neq \mathbf{0}$ implies approximation order 1. We conjecture that this is not the case.

6. Imposing Regularity

We now describe how to construct suitable orthogonal matrices Q_L, Q_R for use in the algorithm described in section 4, to impose continuity and approximation orders on the boundary functions. This will also remove some or all of the arbitrariness in the construction.

Let Λ be the matrix with columns $\mathbf{l}_j$ (see theorem 5.2).

It is easy to see that if we replace the boundary function vector ϕ^L by $M\phi^L$ for an invertible matrix M , the new boundary functions still span the same space, and remain refinable. They also remain orthogonal if M is orthogonal.

The effect of M on the coefficients A , B , C and the matrix Λ is

$$\begin{aligned} A &\rightarrow \tilde{A} = MAM^{-1}, \\ B &\rightarrow \tilde{B} = MB, \\ C &\rightarrow \tilde{C} = MC, \\ \Lambda &\rightarrow \tilde{\Lambda} = \Lambda M^{-1}. \end{aligned}$$

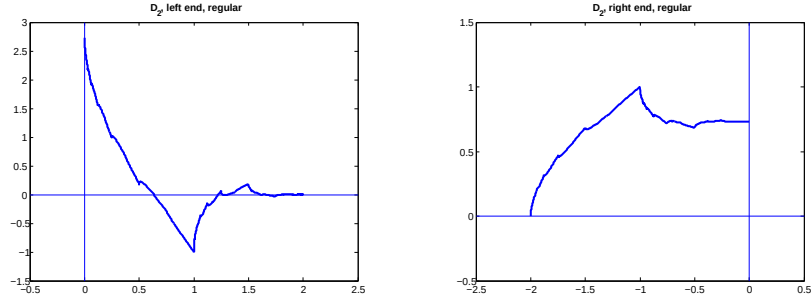
The key observation is that by using a suitable M , we can assume that Λ is upper triangular. We don't actually need to find this M .

One can show that it is possible to construct vectors $\mathbf{l}_0 = \mathbf{e}_1$, $\mathbf{l}_1 \in \text{span}(\mathbf{e}_1, \mathbf{e}_2), \dots$, and a matrix Q_L which enforce the approximation order conditions one by one. The details are rather lengthy, and can be found in [2].

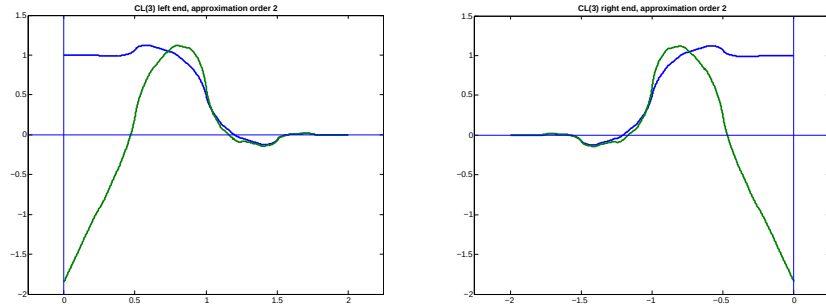
This algorithm will build one boundary function per approximation order, up to the approximation order provided by the interior functions. If the number of boundary functions is higher than the interior approximation order, we can fill up the remaining rows of recursion coefficients arbitrarily.

7. Examples

Example 7.1. For the Daubechies scaling function with two vanishing moments we get one boundary function at each end. We can impose approximation order 1 at each end.



Example 7.2. The Chui-Lian multiwavelet CL(3) [5] has approximation order 3. We need 2 boundary functions at each end, so we can impose approximation order 2 at the ends. This multiwavelet is symmetric, and the boundary functions are mirror images of each other.

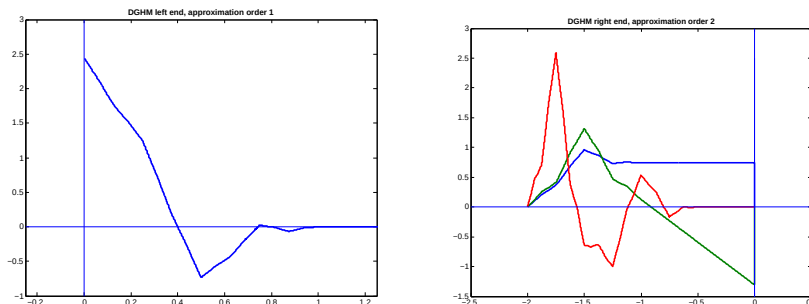


Example 7.3. The DGHM multiwavelet [8] has approximation order 2. We find $\rho_0 = 3$, $\rho_1 = 1$, so we have only a single boundary function at the left end, but three at the right end.

At the left end we can only enforce approximation order 1.

At the right end we can enforce approximation order 2. This determines two of the boundary functions. The third function (determined by the third row of Q_R) is arbitrary, except that the diagonal entry in Z should be smaller than $\sqrt{2}/2$ to ensure that the completion is regular.

As an example, we set the third column of Z to zero, and let the QR-factorization built into Matlab choose an appropriate third row of Y . The result was



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