

ON PERTURBATION OF LOCAL ATOMS FOR SUBSPACES

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Abstract. A family of local atoms is a collection of vectors which are analysis and synthesis systems with frame-like properties for closed subspaces of a separable Hilbert space $\mathcal{H}$. In this paper, we present some perturbation results for local atoms in a subspace of a Hilbert space. Some algebraic properties of one of derivatives of local atoms are given.

1. Background

Frames in a Hilbert space are a redundant system of vectors which provides a series representation for each vector in the space. Duffin and Schaeffer [10] in 1952, introduced frames for Hilbert spaces, in the context of nonharmonic Fourier series. Let $\mathcal{H}$ be an infinite dimensional separable real (or complex) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ linear in first entry. A countable sequence $\{f_k\} \subset \mathcal{H}$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$ if there exist numbers $0 < m_o \leq M_o < \infty$ such that

$$m_o \|f\|^2 \leq \sum_k |\langle f, f_k \rangle|^2 \leq M_o \|f\|^2 \text{ for all } f \in \mathcal{H}. \quad (1.1)$$

The numbers m_o and M_o are called *lower* and *upper frame bounds*, respectively. They are not unique. If it is possible to choose $m_o = M_o$, then the frame $\{f_k\}$ is called *Parseval frame* (or *tight frame*). If the upper inequality in (1.1) satisfied, then we say that $\{f_k\}$ is a *Bessel sequence* in $\mathcal{H}$. Frames were revived by Daubechies, Grossmann and Meyer in [9]. For the utility of frames in different directions in applied mathematics an interested may refer to [2, 6].

Let $\{f_k\}$ be a frame for $\mathcal{H}$. The operator $S : \mathcal{H} \rightarrow \mathcal{H}$ given by

$$Sf = \sum_k \langle f, f_k \rangle f_k$$

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is called the *frame operator* associated with the frame $\{f_k\}$. The frame operator S is a positive, self-adjoint and invertible operator on $\mathcal{H}$. This gives the *reconstruction formula* for all $f \in \mathcal{H}$,

$$f = SS^{-1}f = \sum_k \langle S^{-1}f, f_k \rangle f_k = \sum_k \langle f, S^{-1}f_k \rangle f_k.$$

The scalars $\{\langle S^{-1}f, f_k \rangle\}$ are called *frame coefficients* of the vector $f \in \mathcal{H}$. The representation of f in the reconstruction formula not need to be unique.

In order to handle frame-like properties for closed subspaces in Hilbert spaces, Feichtinger and Werther [11] proposed *atomic system* (or *local atoms*) in separable Hilbert spaces. The sampling theory is responsible for this new system. More precisely, one of the fundamental property of the atomic system is that it can produce a proper subspace even though it is not a part of it !

Definition 1.1. [11] Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{H}_0$ be a closed subspace of $\mathcal{H}$. A sequence $\{f_k\} \subset \mathcal{H}$ is called a *family of local atoms* (or *atomic system*) for $\mathcal{H}_0$, if

- (i) $\{f_k\}$ is Bessel sequence in $\mathcal{H}$, that is there exists a real number $B > 0$ such that

$$\sum_k |\langle f, f_k \rangle|^2 \leq B \|f\|^2 \text{ for all } f \in \mathcal{H},$$

- (ii) there exists a sequence of linear functionals $\{c_k\}$ and a real number $C > 0$ such that

$$\|\{c_k(f)\}\|_{\ell^2}^2 \leq C \|f\|^2 \text{ for all } f \in \mathcal{H}_0 \text{ and } f = \sum_k c_k(f) f_k \text{ for all } f \in \mathcal{H}_0.$$

Remark 1.2. The linear functionals $\{c_k\}$ need only to be defined on the subspace $\mathcal{H}_0$. We say that c_k are *associated functionals* of the local atoms $\{f_k\}$. The constant C is called the *atomic bound*. Furthermore, the partial sum $\sum_{k=1}^N c_k(f) f_k$ of the series in (1.2) can be converges to f from “outside” of $\mathcal{H}_0$.

Recently, L. Găvruta introduced and studied K -frames in Hilbert spaces in [13]. He tried to study atomic systems with respect to a bounded linear operator K on Hilbert spaces and proved some new results.

Definition 1.3. [13] Let $\mathcal{H}$ be a Hilbert space and let K be a bounded linear operator on $\mathcal{H}$. A sequence $\{f_k\} \subset \mathcal{H}$ is called a K -frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A \|K^* f\|^2 \leq \sum_k |\langle f, f_k \rangle|^2 \leq B \|f\|^2 \text{ for all } f \in \mathcal{H}. \quad (1.2)$$

By definition of K -frames it is easy to see that they are more general than standard frames.

1.1. About the Contribution: This paper is organized as follows: In Section 2, we give some Paley-Wiener type perturbation results for local atoms in a subspace of a Hilbert space. We not claim that these perturbation ideas are new, however these are important in the setting of a family of local atoms in subspaces. For applications of perturbation theory in frames, atomic decompositions and the reconstruction property for Banach spaces, we refer papers by Casazza and Christensen [3, 5], Christensen and Heil [7], Favier and Zalik [12] and Kaushik et al. [14]. We present some algebraic properties of one of derivatives of local atoms, namely K -frames in Section 3.

2. Local Atoms for Subspaces

We start with one of the variants of the Paley-Wiener type perturbation for local atoms where the perturbed sequence is a Bessel sequence.

Theorem 2.1. *Assume that $\{f_n\}_{n=1}^\infty$ is a family of local atoms for a closed subspace $\mathcal{H}_o \subset \mathcal{H}$ with associated functionals $\{c_n\}_{n=1}^\infty$ and atomic bound C . Let $\{g_n\}_{n=1}^\infty \subset \mathcal{H}_o$ be a Bessel sequence and let $\lambda \in (0, 1)$ be such that*

$$\left\| \sum_{n=1}^\infty c_n(f)(f_n - g_n) \right\| \leq \lambda \|f\| \text{ for all } f \in \mathcal{H}_o.$$

Let $U : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator such that $U(f_n) = g_n$ for all $n \in \mathbb{N}$. Then, $\{g_n\}_{n=1}^\infty$ is a family of local atoms for $\mathcal{H}_o$.

Proof. Define $L : \mathcal{H}_o \rightarrow \mathcal{H}_o$ as $L(f) = \sum_{n=1}^\infty c_n(f)g_n$ for all $f \in \mathcal{H}_o$. Then, L is well defined bounded linear operator. Indeed, for all $k \geq m$, we have

$$\begin{aligned} \left\| \sum_{n=1}^k c_n(f)g_n - \sum_{n=1}^m c_n(f)g_n \right\| &= \left\| \sum_{n=m+1}^k c_n(f)g_n \right\| \\ &= \left\| U \left(\sum_{n=m+1}^k c_n(f)f_n \right) \right\| \\ &\leq \|U\| \left\| \sum_{n=m+1}^k c_n(f)f_n \right\| \rightarrow 0 \text{ as } n, m \rightarrow \infty. \end{aligned}$$

Furthermore L is invertible. To see this, let $f \in \mathcal{H}_o$. Then, we compute

$$\begin{aligned} \|(I - L)f\| &= \|f - L(f)\| \\ &= \left\| \sum_{n=1}^\infty c_n(f)f_n - \sum_{n=1}^\infty c_n(f)g_n \right\| \\ &\leq \lambda \|f\|. \end{aligned}$$

Therefore, $\|I - L\| < 1$. Thus, L is an invertible operator. Choose $d_n = (L^{-1})^* c_n$, $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \sum_{n=1}^\infty d_n(f)(g_n) &= \sum_{n=1}^\infty (L^{-1})^* c_n(f)g_n \\ &= \sum_{n=1}^\infty c_n(L^{-1}(f))g_n \\ &= L(L^{-1}f) \\ &= f \text{ for all } f \in \mathcal{H}_o. \end{aligned}$$

Next we compute

$$\begin{aligned} \sum_{n=1}^\infty |d_n(f)|^2 &= \sum_{n=1}^\infty |(L^{-1})^* c_n(f)|^2 \\ &\leq C \|L^{-1}(f)\|^2 \\ &\leq C \|L^{-1}\|^2 \|f\|^2 \text{ for all } f \in \mathcal{H}_o. \end{aligned}$$

Hence $\{g_n\}_{n=1}^\infty$ is a family of local atoms for $\mathcal{H}_o$ with associated sequence of functionals $\{d_n\}$. $\square$

In Theorem 2.1, the perturbed sequence is equivalent to the given local family of atoms in some sense! Recall that two sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ in a Banach space are said to be *equivalent* if the mapping $f_n \rightarrow g_n$ can be extended to a well defined bounded linear map of $[f_n]$ onto $[g_n]$, where $[f_n]$ denotes the closed linear span of the sequence $\{f_n\}_{n=1}^\infty$. Casazza and Christensen [3] proved a Paley-Wiener type perturbation result in the context of the reconstruction property in separable Banach spaces, that does not require the perturbed family to be equivalent to the original reconstruction sequence. A similar result for the perturbation of local atoms is given in the following theorem.

Theorem 2.2. *Assume that $\{f_n\}_{n=1}^\infty$ is a family of local atoms for a closed subspace $\mathcal{H}_o \subset \mathcal{H}$ with associated functionals $\{c_n\}_{n=1}^\infty$ and atomic bound C . Assume that $R : \mathcal{H}_o \rightarrow \ell^2(\mathbb{N})$ given by $R(f) = \{c_n(f)\}_{n=1}^\infty$ is a bounded operator. Let $\{g_n\}_{n=1}^\infty \subset \mathcal{H}_o$ be a sequence in $\mathcal{H}$ such that there exist $\lambda, \mu > 0$ with $\lambda + \mu\|R\| < 1$ and*

$$\left\| \sum_{n=1}^\infty \alpha_n (f_n - g_n) \right\| \leq \lambda \left\| \sum_{n=1}^\infty \alpha_n f_n \right\| + \mu \|\{\alpha_n\}\|_{\ell^2(\mathbb{N})},$$

for all finitely non-zero scalar sequences $\{\alpha_n\}_{n=1}^\infty$. Then, $\{g_n\}_{n=1}^\infty$ is a family of local atoms for $\mathcal{H}_o$

Proof. First we show that $\{g_n\}_{n=1}^\infty$ is a Bessel sequence. For any finite sequence $\{\alpha_n\}_{n=1}^\infty$, we have

$$\begin{aligned} \left\| \sum_{n=1}^\infty \alpha_n g_n \right\| &\leq \left\| \sum_{n=1}^\infty \alpha_n (f_n - g_n) \right\| + \left\| \sum_{n=1}^\infty \alpha_n f_n \right\| \\ &\leq (1 + \lambda) \left\| \sum_{n=1}^\infty \alpha_n f_n \right\| + \mu \|\{\alpha_n\}\|_{\ell^2(\mathbb{N})}. \end{aligned}$$

Thus, for any $\{\alpha_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$ and for all $k > m$ in $\mathbb{N}$, we have

$$\begin{aligned} \left\| \sum_{n=1}^k \alpha_n g_n - \sum_{n=1}^m \alpha_n g_n \right\| &= \left\| \sum_{n=m+1}^k \alpha_n g_n \right\| \\ &= (1 + \lambda) \left\| \sum_{n=m+1}^k \alpha_n f_n \right\| + \mu \left\| \sum_{n=m+1}^k \alpha_n e_n \right\|_{\ell^2(\mathbb{N})}. \end{aligned}$$

Since $\{f_n\}_{n=1}^\infty$ is a Bessel sequence and $\{\alpha_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$, the series $\sum_{n=1}^\infty \alpha_n g_n$ converges. Thus, $\{g_n\}_{n=1}^\infty$ is a Bessel sequence.

Next, define $L : \mathcal{H}_o \rightarrow \mathcal{H}_o$ as $L(f) = \sum_{n=1}^\infty c_n(f)g_n$ for all $f \in \mathcal{H}_o$. Using the fact that $R(f) = \{c_n(f)\}_{n=1}^\infty \in \ell^2(\mathbb{N})$ and $\sum_{n=1}^\infty \alpha_n g_n$ for all $\{\alpha_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$, L becomes a well

defined bounded linear operator. For any $f \in \mathcal{H}_o$ we have

$$\begin{aligned}
\|(I - L)f\| &= \|f - L(f)\| \\
&= \left\| \sum_{n=1}^{\infty} c_n(f) f_n - \sum_{n=1}^{\infty} c_n(f) g_n \right\| \\
&\leq \lambda \left\| \sum_{n=1}^{\infty} c_n(f) f_n \right\| + \mu \|\{c_n(f)\}\|_{\ell^2(\mathbb{N})} \\
&= \lambda \|f\| + \mu \|R(f)\|_{\ell^2(\mathbb{N})} \\
&\leq \lambda \|f\| + \mu \|R\| \|f\| \\
&= (\lambda + \mu \|R\|) \|f\| < \|f\|.
\end{aligned}$$

Therefore, $\|I - L\| < 1$. Thus, L is an invertible operator. As in the proof of Theorem 2.1, we can show that $\{g_n\}_{n=1}^{\infty}$ is a family of local atoms for $\mathcal{H}_o$. $\square$

Remark 2.3. Theorem 2.2 is a particular case of Theorem 3.13 in [15]. More precisely, families of local atoms are particular cases of K -frames, when K is an orthogonal projection.

The following proposition provides existence of a compact linear operator on the closed subspace $\mathcal{H}_o$ in terms of local atoms. The proof is based on a technique given in [14].

Proposition 2.4. Let $\{f_n\}_{n=1}^{\infty}$ be a family of local atoms for a closed subspace $\mathcal{H}_o$ with associated functionals $\{c_n\}_{n=1}^{\infty}$. If Θ is a bounded linear operator on $\mathcal{H}_o^*$ such that

$$\sum_{n=1}^{\infty} \|\Theta(c_n) - c_n\| \|f_n\| < \infty,$$

then $f \mapsto \sum_{n=1}^{\infty} [(\Theta(c_n) - c_n)(f)] f_n$ defines a compact operator on $\mathcal{H}_o$.

Proof. Define $T : \mathcal{H}_o \rightarrow \mathcal{H}_o$ by

$$T(f) = \sum_{n=1}^{\infty} [(\Theta(c_n) - c_n)(f)] f_n, \quad f \in \mathcal{H}_o.$$

Then, T is well defined bounded linear operator. Indeed, for $k \geq m$, we have

$$\begin{aligned}
&\left\| \sum_{n=1}^k [(\Theta(c_n) - c_n)(f)] f_n - \sum_{n=1}^m [(\Theta(c_n) - c_n)(f)] f_n \right\| \\
&= \left\| \sum_{n=m+1}^k [(\Theta(c_n) - c_n)(f)] f_n \right\| \\
&\leq \left| \sum_{n=m+1}^k [(\Theta(c_n) - c_n)(f)] \right| \|f\| \\
&\leq \left(\sum_{n=m+1}^k \|\Theta(c_n) - c_n\| \|f_n\| \right) \|f\| \rightarrow 0 \text{ as } k, m \rightarrow \infty.
\end{aligned}$$

Therefore, T is a well defined bounded linear operator on $\mathcal{H}_o$. For each $k \in \mathbb{N}$, define $T_k : \mathcal{H}_o \rightarrow \mathcal{H}_o$ by

$$T_k(f) = \sum_{n=1}^k [(\Theta(c_n) - c_n)(f)] f_n.$$

Then, $\|T - T_k\| \leq \sum_{n \geq k+1} \|\Theta(c_n) - c_n\| \|f_n\| \rightarrow 0$ as $k \rightarrow \infty$. Since each T_k is a finite dimensional, T is compact. $\square$

3. Linear Combination of Frames Revisited

Motivated by the work of Aldroubi in [1], a part of which gives sufficient conditions for linear combination of frames in a separable Hilbert $\mathcal{H}$, in this section we discuss combination of K -frames. First we recall that Aldroubi considered the following problem: Let $\{f_k\}$ be a frame for a separable Hilbert space $\mathcal{H}$. Define a sequence $\{g_j\} \subset \mathcal{H}$ as follows:

$$g_j = \sum_{k=1}^{\infty} \alpha_{j,k} f_k, \quad (j \in \mathbb{N}) \quad (3.1)$$

where $\alpha_{j,k}$ are scalars. A natural question arises about the conditions on the coefficients in the linear combination (in (3.1)), so that the sequence $\{g_j\}$ constitutes a frame for $\mathcal{H}$? Aldroubi gave answer positively and proved sufficient conditions on $\{\alpha_{j,k}\}$ such that $\{g_j\}$ constitutes a frame for $\mathcal{H}$. Christensen in [6] reviewed linear combinations of frames and proved sufficient conditions which are different from those given by Aldroubi. The following result is given in [6] regarding conditions on the scalars $\{\alpha_{s,n}\}_{s,n \in \mathbb{Z}}$ such that the new sequence $\{g_j\}_{j \in \mathbb{Z}}$ constitutes a frame for $\mathcal{H}$. This is also true for K -frame for $\mathcal{H}$.

Theorem 3.1. [6, p. 107]. *Suppose that $\{f_n\}_{n \in \mathbb{Z}}$ is a frame for $\mathcal{H}$ with bounds A, B and let $\{\alpha_{s,n}\}_{n \in \mathbb{Z}} \subseteq \ell^2(\mathbb{Z})$ for each $s \in \mathbb{Z}$. If*

$$a := \inf_{n \in \mathbb{Z}} \left[\sum_{s \in \mathbb{Z}} |\alpha_{s,n}|^2 - \sum_{\substack{n' \in \mathbb{Z} \\ n \neq n'}} \left| \sum_{s \in \mathbb{Z}} \alpha_{s,n} \overline{\alpha_{s,n'}} \right| \right] > 0,$$

$$b := \sup_{n \in \mathbb{Z}} \sum_{n' \in \mathbb{Z}} \left| \sum_{s \in \mathbb{Z}} \alpha_{s,n} \overline{\alpha_{s,n'}} \right| < \infty,$$

then the sequence $\{g_s\}_{s \in \mathbb{Z}} \equiv \left\{ \sum_{n \in \mathbb{Z}} \alpha_{s,n} f_n \right\}_{s \in \mathbb{Z}}$ is a frame for $\mathcal{H}$ with frame bounds aA, bB .

The conditions given in Theorem 3.1 are sufficient but not necessary for K -frames in $\mathcal{H}$. This is justified in the following example.

Example 3.2. Consider the discrete signal space $\mathcal{H} = L^2(\mathbb{Z}, \mu)$, where μ is the counting measure. Let $\{\chi_n\}_{n \in \mathbb{Z}}$ be the sequence of canonical unit vectors in $\mathcal{H}$. Let $K : \mathcal{H} \rightarrow \mathcal{H}$ be an operator given by

$$K(\{..., x_{-2}, x_{-1}, x_0, x_1, ..\}) = \{..., x_{-2}, x_{-1}, 0, x_1, ..\}.$$

It is easy to see that $K^* = K$.

Choose $f_n = \chi_n$ for all $n \in \mathbb{Z}$. Then, $\{f_s\}_{s \in \mathbb{Z}}$ is a K -frame with frame bounds

$A = B = 1$.

Define $\{\alpha_{s,n}\}_{n \in \mathbb{Z}}$ as follows:

$$\alpha_{s,n} = \begin{cases} \chi_{\{s\}}(n), & \text{if } s \neq 0 \\ \chi_{\{0,1\}}(n), & \text{if } s = 0. \end{cases}$$

Then, by definition of g_s given in (3.1), we have

$$g_s = \begin{cases} f_s, & s \neq 0 \\ f_0 + f_1, & s = 0. \end{cases}$$

To show that $\{g_s\}_{s \in \mathbb{Z}}$ is a K -frame for $\mathcal{H}$. First we compute for the upper K -frame bound:

$$\begin{aligned} \sum_{s \in \mathbb{Z}} |\langle f, g_s \rangle|^2 &= \sum_{s \in \mathbb{Z} \setminus \{0\}} |\langle f, f_s \rangle|^2 + |\langle f, f_0 + f_1 \rangle|^2 \\ &\leq \sum_{s \in \mathbb{Z} \setminus \{0\}} |\langle f, f_s \rangle|^2 + 2|\langle f, f_0 \rangle|^2 + 2|\langle f, f_1 \rangle|^2 \\ &= \sum_{s \in \mathbb{Z}} |\langle f, f_s \rangle|^2 + |\langle f, f_0 \rangle|^2 + 2|\langle f, f_1 \rangle|^2 \\ &\leq 3 \sum_{s \in \mathbb{Z}} |\langle f, f_s \rangle|^2 \\ &\leq 3\|f\|^2 \text{ for all } f \in \mathcal{H}. \end{aligned}$$

For the lower K -frame inequality, we compute

$$\begin{aligned} \sum_{s \in \mathbb{Z}} |\langle f, g_s \rangle|^2 &= \sum_{s \in \mathbb{Z} \setminus \{0\}} |\langle f, f_s \rangle|^2 + |\langle f, f_0 + f_1 \rangle|^2 \\ &\geq \sum_{s \in \mathbb{Z} \setminus \{0\}} |\langle f, f_s \rangle|^2 \\ &= \|K^*f\|^2 \text{ for all } f \in \mathcal{H}. \end{aligned}$$

Hence $\{g_s\}_{s \in \mathbb{Z}}$ is a K -frame for $\mathcal{H}$.

Next we compute

$$\begin{aligned} \left[\sum_{s \in \mathbb{Z}} |\alpha_{s,0}|^2 - \sum_{\substack{n' \in \mathbb{Z} \\ 0 \neq n'}} \left| \sum_{s \in \mathbb{Z}} \alpha_{s,0} \overline{\alpha_{s,n'}} \right| \right] &= |\alpha_{0,0}|^2 - |\alpha_{0,0}\alpha_{0,1}| \\ &= 1 - 1 = 0. \end{aligned}$$

For $n \neq 0 \in \mathbb{Z}$, we have

$$\left[\sum_{s \in \mathbb{Z}} |\alpha_{s,n}|^2 - \sum_{\substack{n' \in \mathbb{Z} \\ n \neq n'}} \left| \sum_{s \in \mathbb{Z}} \alpha_{s,n} \overline{\alpha_{s,n'}} \right| \right] = 1.$$

Therefore

$$a := \inf_{n \in \mathbb{Z}} \left[\sum_{s \in \mathbb{Z}} |\alpha_{s,n}|^2 - \sum_{\substack{n' \in \mathbb{Z} \\ n \neq n'}} \left| \sum_{s \in \mathbb{Z}} \alpha_{s,n} \overline{\alpha_{s,n'}} \right| \right] = 0.$$

Hence conditions given in Theorem 3.1 are not necessary.

Remark 3.3. Casazza proved in [4] that every frame in a complex Hilbert space is a linear combination of three orthonormal bases. More precisely, it is multiple of a sum of three orthonormal bases. If $\mathcal{H}$ is a complex Hilbert space and $\{f_n\}_{n=1}^\infty$ is a sequence in $\mathcal{H}$ such that there exist a linear bounded operator $U : \ell^2(\mathbb{N}) \rightarrow \mathcal{H}$ such that $U(\delta_n) = f_n$ for all n , where $\{\delta_n\}_{n=1}^\infty$ is the canonical orthonormal basis of $\ell^2(\mathbb{N})$. Then, for every $\epsilon \in (0, 1)$ there exist three orthonormal bases $\{e_n^1\}_{n=1}^\infty$, $\{e_n^2\}_{n=1}^\infty$ and $\{e_n^3\}_{n=1}^\infty$ for $\mathcal{H}$ such that

$$f_n = \frac{\|U \circ V\|}{1 - \epsilon} (e_n^1 + e_n^2 + e_n^3) \quad (n \in \mathbb{N}),$$

where V is an isometric isomorphism from $\mathcal{H}$ to $\ell^2(\mathbb{N})$ which maps e_n to δ_n . Since every K -frame $\{f_n\}_{n=1}^\infty$ is a Bessel sequence in $\mathcal{H}$, therefore there exist a linear bounded operator U which maps δ_n to f_n , for all $n \in \mathbb{N}$, one of the choice for U is the pre-frame operator associated with $\{f_n\}_{n=1}^\infty$. Hence $\{f_n\}_{n=1}^\infty$ can be expressed as a sum of three orthonormal bases in $\mathcal{H}$. Thus, every K -frame can be expressed as a sum of three orthonormal bases in $\mathcal{H}$.

To conclude the paper we show that the image of a K -frame for $\mathcal{H}$ under a bounded linear operator U forms a UK -frame for $\mathcal{H}$. First we recall that the image of a frame under a bounded linear operator need not be a frame for the underlying space, see [8]. The following proposition shows that the situation is different for K -frames.

Proposition 3.4. *Let $\{f_n\}_{n \in \mathbb{Z}}$ be a K -frame for $\mathcal{H}$ and U be a bounded linear operator on $\mathcal{H}$. Then, $\{U(f_n)\}_{n \in \mathbb{Z}}$ is a UK -frame for $\mathcal{H}$.*

Proof. Let $\{f_n\}_{n \in \mathbb{Z}}$ be a K -frame for $\mathcal{H}$ with frame bounds A and B . Then

$$A\|K^*f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, f_n \rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in \mathcal{H}. \quad (3.2)$$

By using upper inequality in (3.2), we have

$$\begin{aligned} \sum_{n \in \mathbb{Z}} |\langle f, U(f_n) \rangle|^2 &= \sum_{n \in \mathbb{Z}} |\langle U^*f, f_n \rangle|^2 \\ &\leq B\|U^*f\|^2 \\ &\leq B\|U^*\|^2\|f\|^2 \quad \text{for all } f \in \mathcal{H}. \end{aligned} \quad (3.3)$$

By using the fact that A is one of the choice for lower K -frame bound for $\{f_n\}_{n \in \mathbb{Z}}$, we compute

$$\begin{aligned} \sum_{n \in \mathbb{Z}} |\langle f, U(f_n) \rangle|^2 &= \sum_{n \in \mathbb{Z}} |\langle U^*f, f_n \rangle|^2 \\ &\geq A\|K^*U^*f\|^2 \\ &= A\|(UK)^*f\|^2 \quad \text{for all } f \in \mathcal{H}. \end{aligned} \quad (3.4)$$

The outcomes in inequalities (3.3) and (3.4) gives

$$A\|(UK)^*f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, U(f_n) \rangle|^2 \leq B\|U^*\|^2\|f\|^2 \quad \text{for all } f \in \mathcal{H}.$$

Hence $\{U(f_n)\}_{n \in \mathbb{Z}}$ is a UK -frame for $\mathcal{H}$ with frame bounds A , $B\|U^*\|^2$. $\square$

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