

CONSTRUCTIONS OF MRA-BASED WAVELETS AND FRAMES IN WALSH ANALYSIS

Yu. A. Farkov

Date of Receiving : 03. 03. 2015
 Date of Revision : 14. 05. 2015
 Date of Acceptance : 18. 05. 2015

Abstract. In this paper, we review the basic constructions of MRA-based orthogonal wavelets and tight frames related to the generalized Walsh functions. Several differences between the constructed wavelets and the classical wavelets are indicated. Further, for any integer $p \geq 2$, all compactly supported step refinable functions on the Vilenkin group G_p are characterized. In addition, we include references to the following topics: biorthogonal wavelets on G_p , approximation properties of compactly supported orthogonal wavelets and frames on the Vilenkin/Cantor groups, unconditional convergence of wavelet expansions, nonstationary, periodic, and discrete wavelets in Walsh analysis, p -multiresolution analysis on the positive half-line, and related subjects.

1. Introduction

Given an integer $p \geq 2$, the Vilenkin group G_p consists of sequences $x = (x_j)$, where $x_j \in \{0, 1, \dots, p-1\}$ for $j \in \mathbb{Z}$ and with at most finite number of negative j such that $x_j \neq 0$. The zero sequence is denoted by θ . If $x \neq \theta$, then there exists a unique $k = k(x)$ such that $x_k \neq 0$ and $x_j = 0$ for all $j < k$. The group operation $\oplus$ on G is defined as the coordinatewise addition modulo p ,

$$(z_j) = (x_j) \oplus (y_j) \iff z_j = x_j + y_j \pmod{p} \quad \text{for all } j \in \mathbb{Z};$$

the topology on G_p is introduced via the complete system of neighbourhoods of zero

$$U_l = \{(x_j) \in G_p : x_j = 0 \text{ for all } j \leq l\}, \quad l \in \mathbb{Z}.$$

The equality $z = x \ominus y$ means that $z \oplus y = x$. For $p = 2$ we have $x \oplus y = x \ominus y$ and the group G_2 coincides with the locally compact Cantor group $\mathcal{C}$.

Notice that for $p = 2$ the subgroup U_0 is isomorphic to the compact Cantor group $\mathcal{C}_0$; i.e., the topological Cartesian product of a countable set of cyclic groups with discrete

2010 *Mathematics Subject Classification.* 42C40, 42C10, 42C15, 43A15, 65T60 .

Key words and phrases. wavelets, frames, Walsh functions, Cantor group, Vilenkin groups, local fields.

The author thanks E. A. Rodionov for writing computer programs to draw graphs of wavelets and frames.

Communicated by. V. Yu. Protasov

topology. It is well known that $\mathcal{C}_0$ is a perfect nowhere dense totally disconnected metrizable space, and therefore U_0 is homeomorphic to the Cantor ternary set. An account of elements of harmonic analysis on the Vilenkin/Cantor groups and some their applications to problems in signal processing may be found in [1]–[3]. For applications these groups in the theory of Fourier series we refer the reader to [4, Ch. 14 and 15].

Wavelet theory for functions defined on the Vilenkin/Cantor groups has been extensively studied in the last three decades. The continuous wavelet transform on $L^2(\mathcal{C})$ was examined in [5, 6], where the case of Vilenkin groups was also considered. The family of compactly supported orthogonal MRA-based wavelets on $\mathcal{C}$ was constructed by Lang [7, 8]. Moreover, Lang [9] has found wavelets that generate unconditional bases for the spaces $L^q(\mathcal{C})$, $1 < q < \infty$. Some wavelets on the Cantor group were treated independently by Sendov [10]. For all integers $p, n \geq 2$, necessary and sufficient conditions for the solutions of the corresponding refinement equations with p^n coefficients to generate an MRA in $L^2(G_p)$ were put forward in [13–16]. Further results were concerned with the following topics:

- algorithms for constructing orthogonal and biorthogonal compactly supported wavelets on G_p (see [17]–[21]);
- algorithm for constructing an MRA with a Riesz basis for $L^2(G_p)$ [22];
- smoothness estimates of Lang’s wavelets [23];
- unconditional convergence of wavelet expansions in the dyadic Hardy space [24];
- approximation properties of dyadic wavelet expansions [25, 26];
- periodic wavelets in Walsh analysis [27, 28];
- discrete wavelets related to the Vilenkin–Chrestenson transform [29, 30];
- algorithms for constructing wavelet frames on G_p (see [31–35]);
- nonstationary wavelets related to the Walsh functions [36];
- wavelets associated with non-uniform multiresolution analysis [37];
- wavelet packets related to the Walsh polynomials (see [38]–[40]);
- Uncertainty Principle in Walsh analysis [41];
- wavelet methods for differential equations on the Cantor group [42];
- applications for digital signal processing (see [43]–[47]).

J. J. Benedetto and R. L. Benedetto [48, 49] developed a wavelet theory for a locally compact abelian group $\mathcal{G}$ with compact open subgroup $\mathcal{H}$. The best known example of such a group is $\mathcal{G} = \mathbb{Q}_p$, the field of p -adic numbers (as a group under addition), which has compact open subgroup $\mathcal{G} = \mathbb{Z}_p$, the ring of p -adic integers. Wavelet bases for $L^2(\mathcal{G})$ are constructed by means of an iterative method giving rise to so-called wavelet sets in the dual group $\widehat{\mathcal{G}}$. As shown in [48], for the Haar wavelets this construction is broader than those of Lang [7] and Kozyrev [50, Theorem 2]. There were several attempts to develop wavelet theory in a more general setting, e.g., for zero-dimensional groups (see [22, 51, 52]). Recently, Evdokimov and Skopina [53] described the situation for the field $\mathbb{Q}_p$ and proved that any orthogonal wavelet basis for $L^2(\mathbb{Q}_p)$ that consists of band-limited (periodic) functions, is a modification of the Haar basis. In [54], Novikov and Skopina discuss the question of why Haar bases in various structures are the same. In [55], Haar multiresolution analysis and Haar bases on the ring of rational adeles are constructed. A characterization of MRA-based wavelets on local fields of positive characteristic including the Vilenkin/Cantor groups is suggested in [56].

In this paper, we describe the basic constructions of MRA-based orthogonal wavelets and tight frames on the Vilenkin/Cantor groups. The set of all compactly supported step refinable functions on the group G_p are characterized (see Theorem 3.9). Some significant differences of the properties of wavelets in the Walsh analysis from those of the classical wavelets are connected with approximation theory problems (see e.g. Remarks 2.3, 2.6 and the end of § 4). Notice also that some numerical experiments (see [36], [43]–[47]) demonstrate an effectiveness of the corresponding wavelet transforms for image coding and signal processing. For example, in [28], the periodic wavelets associated with the Walsh-Dirichlet type kernel are defined. In [47], applications of these wavelets to coding of the Riemann, Weierstrass, Schwarz, Van der Waerden, Hankel, and Takagi fractal functions are given (cf. [36]).

In the sequel, the Vilenkin group G_p will be denoted by G . One can show that G is self-dual. The duality pairing on G takes $x, \omega \in G$ to

$$\chi(x, \omega) = \exp \left(\frac{2\pi i}{p} \sum_{j \in \mathbb{Z}} x_j \omega_{1-j} \right).$$

The Lebesgue spaces $L^q(G)$, $1 \leq q \leq \infty$, are defined with the help of the Haar measure μ on Borel subsets of G normalized by $\mu(U) = 1$ with $U = U_0$. The Fourier transform of a function $f \in L^1(G) \cap L^2(G)$,

$$\widehat{f}(\omega) = \int_G f(x) \overline{\chi(x, \omega)} d\mu(x), \quad \omega \in G,$$

admits a standard extension to the space $L^2(G)$. Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}_+$, and $\mathbb{R}_+$ the sets of positive integers, nonnegative integers, and nonnegative real numbers, respectively. The characteristic function of a set E is denoted by $\mathbf{1}_E$.

Let us consider the mapping $\lambda : G \rightarrow \mathbb{R}_+$,

$$\lambda(x) = \sum_{j \in \mathbb{Z}} x_j p^{-j}, \quad x = (x_j) \in G.$$

It is clear that the image of the discrete subgroup $H := \{(x_j) \in G : x_j = 0 \text{ for } j > 0\}$ under λ is the set of nonnegative integers: $\lambda(H) = \mathbb{Z}_+$. For each $\alpha \in \mathbb{Z}_+$, let $h_{[\alpha]}$ denote the element of H such that $\lambda(h_{[\alpha]}) = \alpha$; in particular, $h_{[0]} = \theta$.

The *generalized Walsh functions* for the group G can be defined by

$$W_\alpha(x) = \chi(x, h_{[\alpha]}), \quad \alpha \in \mathbb{Z}_+, x \in G.$$

These functions are continuous on G and satisfy the orthogonality relations

$$\int_U W_\alpha(x) \overline{W_\beta(x)} d\mu(x) = \delta_{\alpha, \beta}, \quad \alpha, \beta \in \mathbb{Z}_+,$$

where $\delta_{\alpha, \beta}$ is the Kronecker delta. Moreover, the orthonormal system $\{W_\alpha\}$ is complete in $L^2(U)$ (e.g. [3, § 2.8]).

We define an automorphism $A : G \rightarrow G$ by letting $(Ax)_j = x_{j+1}$ for all $x = (x_j) \in G$. In [13], a scaling function $\varphi \in L^2(G)$ is defined such that:

(1) φ satisfies a refinement equation of the type

$$\varphi(x) = p \sum_{\alpha=0}^{p^n-1} a_\alpha \varphi(Ax \ominus h_{[\alpha]}), \quad x \in G; \quad (1.1)$$

(2) φ is the sum of Walsh series;

(3) $\{\varphi(\cdot \ominus h) : h \in H\}$ is an orthonormal system in $L^2(G)$;

(4) $\text{supp } \varphi \subset U_{n-1}$;

(5) φ generates an MRA in $L^2(G)$.

To calculate the coefficients in (1.1) one chooses $p^n - p$ complex parameters satisfying a certain ‘orthogonality condition’, complements them by $p - 1$ zeros, and applies the discrete Vilenkin–Chrestenson transform; *cf.* with Algorithm A below. As distinct from the classical case (see [57, § 2.6]), the mask of equation (1.1) assumes a finite number of values, which simplifies many proofs and algorithms. Moreover, as distinct from wavelet constructions in $L^2(\mathbb{R}^d)$, from the above property (2) it follows that the scaling functions φ can be given analytically.

There is a standard interpretation of G on $\mathbb{R}_+$ which can be obtained using the above-defined mapping λ (see [2, § 9], [3, § 1.2]). In doing so, instead of $\oplus$ one uses the operation $\dot{+}$, the subgroup H is replaced by the set $\mathbb{Z}_+$, the subgroup U corresponds to the interval $[0, 1)$, the functions $W_\alpha(x)$ are replaced by the functions $w_l(x)$, instead of the Haar measure μ one uses the Lebesgue measure on $\mathbb{R}_+$, and instead of the Fourier transform one applies the Walsh–Fourier transform.

Let us recall the definitions of the operation $\dot{+}$, the functions $w_l(x)$ and the Walsh–Fourier transform on $L^2(\mathbb{R}_+)$. As usual, the integer and fractional parts of a number x are denoted, respectively, by $[x]$ and $\{x\}$. For any $s \in \mathbb{Z}$, let $\langle s \rangle_p$ be the remainder of s divided by p . Next, for $x \in \mathbb{R}_+$, we set

$$x_j = \langle [p^j x] \rangle_p, \quad x_{-j} = \langle [p^{1-j} x] \rangle_p, \quad j \in \mathbb{N}. \quad (1.2)$$

These numbers are the digits of the p -ary expansion

$$x = \sum_{j < 0} x_j p^{-j-1} + \sum_{j > 0} x_j p^{-j}$$

(for a p -adic rational x we obtain an expansion with finitely many nonzero terms). It is clear that

$$[x] = \sum_{j=1}^{\infty} x_{-j} p^{j-1}, \quad \{x\} = \sum_{j=1}^{\infty} x_j p^{-j},$$

and, for each $x \neq 0$, there exists $k = k(x)$ such that $x_k \neq 0$ and $x_{-j} = 0$ for all $j > k$.

Consider the p -adic addition defined on $\mathbb{R}_+$ as follows: if $z = x \dot{+} y$, then

$$z = \sum_{j < 0} \langle x_j + y_j \rangle_p p^{-j-1} + \sum_{j > 0} \langle x_j + y_j \rangle_p p^{-j},$$

where x_j, y_j are given by (1.2). As usual, we will denote by $\dot{-}$ the inverse operation of $\dot{+}$.

Letting $\varepsilon_p = \exp(2\pi i/p)$, we define a function w_1 on $[0, 1)$ by

$$w_1(x) = \begin{cases} 1, & x \in [0, 1/p), \\ \varepsilon_p^l, & x \in [lp^{-1}, (l+1)p^{-1}), l \in \{1, \dots, p-1\}, \end{cases}$$

and extend it to $\mathbb{R}_+$ by periodicity: $w_1(x+1) = w_1(x)$ for all $x \in \mathbb{R}_+$. Then the functions w_l are defined by

$$w_0(x) \equiv 1, \quad w_l(x) = \prod_{j=1}^k (w_1(p^{j-1}x))^{l_{-j}}, \quad l \in \mathbb{N}, \quad x \in \mathbb{R}_+,$$

where the l_{-j} are the digits of the p -ary expansion of l :

$$l = \sum_{j=1}^k l_{-j} p^{j-1}, \quad l_{-j} \in \{0, 1, \dots, p-1\}, \quad l_{-k} \neq 0, \quad k = k(l).$$

For any $x, y \in \mathbb{R}_+$, we set

$$\psi(x, y) = \varepsilon_p^{t(x, y)}, \quad t(x, y) = \sum_{j=1}^{\infty} (x_j y_{-j} + x_{-j} y_j),$$

where x_j, y_j are given by (1.2). The *Walsh-Fourier transform* of a function $f \in L^1(\mathbb{R}_+)$ is defined by

$$\widehat{f}(\omega) = \int_{\mathbb{R}_+} f(x) \overline{\psi(x, \omega)} dx, \quad \omega \in \mathbb{R}_+.$$

If $f \in L^2(\mathbb{R}_+)$ and

$$J_a f(\omega) = \int_0^a f(x) \overline{\psi(x, \omega)} dx, \quad a > 0,$$

then $\widehat{f}$ is the limit of $J_a f$ in $L^2(\mathbb{R}_+)$ as $a \rightarrow +\infty$. Notice that for $p = 2$ we have

$$x \dot{+} y = x \dot{-} y = \sum_{j < 0} |x_j - y_j| 2^{-j-1} + \sum_{j > 0} |x_j - y_j| 2^{-j}, \quad (1.3)$$

while the functions w_l coincide with the classical Walsh functions.

Some wavelet constructions on $(\mathbb{R}_+, \dot{+})$ were studied in [8, 14, 18], [34]–[39], [43]. Chapter 8 of the book [44] contains a detailed description of the results of [15, 16] and [18]. In particular, the conditions under which a compactly supported refinable function on $\mathbb{R}_+$ is stable and has a linearly independent system of integer shifts in $L^2(\mathbb{R}_+)$ are discussed in [44, § 8.2] (cf. [14], [16, Theorem 2]).

2. Lang's wavelets on $\mathbb{R}_+$

For $p = 2$, orthogonal wavelets with compact supports on $\mathbb{R}_+$ can be defined in terms of the scaling functions φ satisfying equations of the form

$$\varphi(x) = \sum_{k=0}^{2^n-1} c_k \varphi(2x \dot{+} k), \quad x \in \mathbb{R}_+, \quad (2.1)$$

where $\dot{+}$ is defined by (1.3). From [14, Theorem 1] it follows that if a function $\varphi \in L^2(\mathbb{R}_+)$ has compact support, satisfies equation (2.1), and $\widehat{\varphi}(0) = 1$, then

$$\text{supp } \varphi \subset [0, 2^{n-1}] \quad \text{and} \quad \widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(2^{-j}\omega), \quad (2.2)$$

where

$$m_0(\omega) = \frac{1}{2} \sum_{k=0}^{2^n-1} c_k w_k(\omega), \quad \omega \in \mathbb{R}_+. \quad (2.3)$$

It is also known that if a system $\{\varphi(\cdot \dot{+} k) : k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$, then

$$|m_0(\omega)|^2 + |m_0(\omega + 1/2)|^2 = 1 \quad \text{for all } \omega \in \mathbb{R}_+. \quad (2.4)$$

An interval $I \subset \mathbb{R}_+$ is a *dyadic interval of range n* if $I = I_l^{(n)} = [l2^{-n}, (l+1)2^{-n})$ for some $l \in \mathbb{Z}_+$. Since w_k is a constant on $I_l^{(n)}$ whenever $0 \leq k, l \leq 2^n - 1$, it is clear that the mask m_0 is a dyadic step function. The coefficients of equation (2.1) are calculated from the values $b_l = m_0(l2^{-n})$, $0 \leq l \leq 2^n - 1$, by the discrete Walsh transform:

$$c_k = \frac{1}{2^{n-1}} \sum_{l=0}^{2^n-1} b_l w_l(k2^{-n}), \quad 0 \leq k \leq 2^n - 1.$$

Notice that the product in (2.2) is convergent at each point $\omega \in \mathbb{R}_+$, because $m_0(\omega) = 1$ for all $\omega \in [0, 2^{-n})$. Moreover, in view of (2.4) we have

$$b_0 = 1, \quad |b_l|^2 + |b_{l+2^{n-1}}|^2 = 1, \quad 0 \leq l \leq 2^{n-1} - 1. \quad (2.5)$$

Note also that $\widehat{\varphi}$ is a constant on each dyadic interval of range $n - 1$, which is clear from (2.2).

If a solution φ of equation (2.1) generates an MRA, then an orthogonal wavelet ψ in $L^2(\mathbb{R}_+)$ can be defined by φ . More precisely [14, § 4], if we define a function ψ by setting

$$\psi(x) = \sum_{k=0}^{2^n-1} (-1)^k \bar{c}_{k+1} \varphi(2x \dot{+} k), \quad (2.6)$$

then the functions

$$\psi_{jk}(x) := 2^{j/2} \psi(2^j x \dot{+} k), \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z}_+,$$

form an orthonormal basis for $L^2(\mathbb{R}_+)$.

Example 2.1. For $n = 2$, Lang's function φ corresponds to the values $b_0 = 1$, $b_1 = a$, $b_2 = 0$, $b_3 = b$, where $0 < |a| < 1$, $|b| = \sqrt{1 - |a|^2}$. It was shown in [7, 8] that this function generates an MRA, satisfies the equation

$$\varphi(x) = \sum_{k=0}^3 c_k \varphi(2x \dot{+} k) \quad (2.7)$$

with coefficients

$$c_0 = \frac{1+a+b}{2}, \quad c_1 = \frac{1+a-b}{2}, \quad c_2 = \frac{1-a-b}{2}, \quad c_3 = \frac{1-a+b}{2},$$

and possesses the following fractal property:

$$\varphi(x) = \begin{cases} (1+a-b)/2 + b\varphi(2x), & 0 \leq x < 1, \\ (1-a+b)/2 - b\varphi(2x-2), & 1 \leq x \leq 2. \end{cases}$$

Moreover, from (2.2) and (2.7) it follows that φ is represented by a lacunary Walsh series

$$\varphi(x) = (1/2)\mathbf{1}_{[0,1)}(x/2)(1 + a \sum_{j=0}^{\infty} b^j w_{2^{j+1}-1}(x/2)), \quad x \in \mathbb{R}_+. \quad (2.8)$$

The corresponding orthogonal wavelet ψ is defined by

$$\psi(x) = c_1\varphi(2x) - c_0\varphi(2x+1) + c_3\varphi(2x+2) - c_2\varphi(2x+3). \quad (2.9)$$

Plottings of these φ and ψ are given in [7] for certain values of a and b . As noted in [8], the vector functions

$$\vec{\varphi}(x) = \begin{pmatrix} \varphi(x) \\ \varphi(x+1) \end{pmatrix}, \quad \vec{\psi}(x) = \begin{pmatrix} \psi(x) \\ \psi(x+1) \end{pmatrix},$$

satisfy the equalities

$$\vec{\varphi}(x/2) = P_0 \vec{\varphi}(x) + P_2 \vec{\varphi}(x+2), \quad \vec{\psi}(x/2) = Q_0 \vec{\varphi}(x) + Q_2 \vec{\varphi}(x+2),$$

where

$$P_0 = (1/\sqrt{2}) \begin{pmatrix} c_0 & c_1 \\ c_2 & c_3 \end{pmatrix}, \quad P_2 = (1/\sqrt{2}) \begin{pmatrix} c_2 & c_3 \\ c_0 & c_1 \end{pmatrix}, \\ Q_0 = (1/\sqrt{2}) \begin{pmatrix} c_1 & -c_0 \\ c_3 & -c_2 \end{pmatrix}, \quad Q_2 = (1/\sqrt{2}) \begin{pmatrix} c_3 & -c_2 \\ c_1 & -c_0 \end{pmatrix}.$$

Thus, the wavelet ψ defined by (2.9) can be interpreted [8, 58] as a multiwavelet on $\mathbb{R}_+$. Next, in accordance with [9, 24], when $|b| < 1/2$, the system $\{\psi_{jk}\}$ is an unconditional basis for the dyadic Hardy space $H^1(\mathbb{R}_+)$ and for the spaces $L^q(\mathbb{R}_+)$, $1 < q < \infty$. We also note that for $a = 1$ and $a = -1$ the Haar function $\varphi(x) = \mathbf{1}_{[0,1)}(x)$ and the shifted Haar function $\varphi(x) = \mathbf{1}_{[0,1)}(x+1)$ are, respectively, solutions of equation (2.7). For $a = 0$, we have $\varphi(x) = (1/2)\mathbf{1}_{[0,1)}(x/2)$, $\varphi(x+1) = \varphi(x)$, and $\{\psi_{jk}\}$ is a Parseval frame for $L^2(\mathbb{R}_+)$ (cf. [31, Theorem 3.4]).

Remark 2.2. The Walsh–Fourier transform of the Haar function $\varphi = \mathbf{1}_{[0,1)}$ coincides with the function itself: $\widehat{\varphi} = \varphi$. Hence, in the Walsh analysis the Haar wavelets coincide with the Kotelnikov–Shannon wavelets (a similar effect is also observed in the p -adic analysis, see [48]).

Remark 2.3. It is noted in [13, Example 4.3] that the dyadic smoothness of the Lang function (2.8) is unbounded. Moreover, the dyadic smoothness exponent of this function φ can be calculated by $\alpha_\varphi = \log_2(1/|b|)$ (see [14, 23]). At the same time, for any b we have $\text{supp } \varphi \subset [0, 2]$. This is one of the striking differences from Daubechies' wavelets (see e.g. [57, § 7.5], [59, § 7.2], [60, 61]).

As above, suppose that $\varphi \in L^2(\mathbb{R}_+)$ has compact support, satisfies equation (2.1) and $\widehat{\varphi}(0) = 1$. Assume that m_0 is its mask. Denote by a_m the smallest nonzero value of the modulus of m_0 ; that is

$$a_m = \min \{ |m_0(\omega)| : \omega \in \mathbb{R}_+, m_0(\omega) \neq 0 \}.$$

Since m_0 has period 1 and is a constant on each dyadic interval of range n , the value of a_m coincides with the smallest nonzero number in the set $\{|m_0(2^{-n}k)|, k = 0, \dots, 2^{n-1}\}$. It is shown in [14] that the following alternative holds:

- i) φ is constant on each dyadic interval of range n ;
- ii) $a_m < 1$ and $\alpha_\varphi \leq \log_2(1/a_m)$.

All compactly supported step functions φ satisfying equation (2.1) are characterized in [14, § 6].

Example 2.4. For the case $n = 3$, the following step functions are solutions of equation (2.1):

- 1) $\varphi(x) = (1/4)\mathbf{1}_{[0,1)}(x/4)$ ($b_1 = 0$),
- 2) $\varphi(x) = (1/4)\mathbf{1}_{[0,1)}(x/4)(1 + b_1w_1(x/4))$ ($b_2 = b_3 = 0$),
- 3) $\varphi(x) = (1/4)\mathbf{1}_{[0,1)}(x/4)(1 + b_1w_1(x/4) + b_1b_3w_3(x/4))$ ($b_2 = b_6 = b_7 = 0$),
- 4) $\varphi(x) = (1/4)\mathbf{1}_{[0,1)}(x/4)(1 + b_1w_1(x/4) + b_1b_2w_2(x/4))$ ($b_3 = b_4 = b_5 = 0$),
- 5) $\varphi(x) = (1/4)\mathbf{1}_{[0,1)}(x/4)(1 + b_1w_1(x/4) + b_1b_2w_2(x/4) + b_1b_3w_3(x/4) + b_1b_3b_6w_6(x/4))$ ($b_4 = b_5 = b_7 = 0$).

Now we define a one-parameter family of step refinable functions on $\mathbb{R}_+$.

Example 2.5. For arbitrary $n \geq 3$, we set

$$m_0(\omega) = \begin{cases} 1, & \omega \in [0, 1/2 - 1/2^{n-1}) \cup [1/2 - 1/2^n, 1/2), \\ \alpha, & \omega \in [1/2 - 1/2^{n-1}, 1/2 - 1/2^n), \\ 0, & \omega \in [1/2, 1/2 - 1/2^n) \cup [1 - 1/2^n, 1), \\ \beta, & \omega \in [1 - 1/2^{n-1}, 1 - 1/2^n), \end{cases}$$

where $0 \leq |\alpha| < 1$, $|\beta| = \sqrt{1 - |\alpha|^2}$. Then it follows from (2.2) that

$$\widehat{\varphi}(\omega) = \mathbf{1}_{[0, 1-1/2^{n-2})}(\omega) + \alpha \cdot \mathbf{1}_{[1-1/2^{n-2}, 1-1/2^{n-1})}(\omega) + \mathbf{1}_{[1-1/2^{n-1}, 1)}(\omega) + \beta \cdot \mathbf{1}_{[2-1/2^{n-2}, 2-1/2^{n-1})}(\omega),$$

and therefore

$$\begin{aligned} \varphi(x) = \frac{1}{2^{n-1}} \mathbf{1}_{[0,1)}(x/2^{n-1}) & \left(1 + \sum_{l=1}^{2^{n-1}-3} w_l(x/2^{n-1}) + \alpha w_{2^{n-1}-2}(x/2^{n-1}) \right. \\ & \left. + w_{2^{n-1}-1}(x/2^{n-1}) + \beta w_{2^{n-1}-2}(x/2^{n-1}) \right). \end{aligned}$$

By [14, Example 6], this function generates an MRA in $L^2(\mathbb{R}_+)$. Hence, the corresponding wavelet ψ can be defined by (2.6).

Remark 2.6. A number of significant differences of the approximation properties of Lang's wavelets from those of the classical wavelets were pointed out in [25], where the relation between the smoothness of wavelets and their order of approximation is analysed in various function spaces. In particular, according to [25, Theorem 3], for Lang's wavelets the defect of approximation is determined by the greatest integer k , for which the mask m_0 of equation (2.1) vanishes on the interval $[1/2, 1/2 + 2^{k-n})$ (compare with [57, Theorem 3.3.2]).

The Jackson and Bernstein approximation theorems for Lang's wavelets are proved in [26].

3. Orthogonal wavelets on Vilenkin's group

As above, we denote by G the Vilenkin group G_p . Recall that $(Ax)_j = x_{j+1}$ for all $x = (x_j) \in G$. The sets

$$U_{n,s} := A^{-n}(h_{[s]}) \oplus A^{-n}(U), \quad 0 \leq s \leq p^n - 1,$$

are cosets of the subgroup $A^{-n}(U)$ in the group U . It is easy to check that for every $0 \leq k \leq p^n - 1$ the function W_k is constant on each set $U_{n,s}$.

A function $\varphi \in L^2(G)$ is called a *refinable function* if it satisfies an equation of the type

$$\varphi(x) = p \sum_{\alpha=0}^{\infty} a_{\alpha} \varphi(Ax \ominus h_{[\alpha]}).$$

Denote by $L_c^2(G)$ the set of all compactly supported functions in $L^2(G)$. If $\varphi \in L_c^2(G)$ is a refinable function, then there exists a positive integer n such that φ satisfies equation (1.1) (see [32, Theorem 1]). Using the Fourier transform, from (1.1) we have

$$\widehat{\varphi}(\omega) = m_0(A^{-1}\omega) \widehat{\varphi}(A^{-1}\omega), \quad (3.1)$$

where

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_{\alpha} \overline{W_{\alpha}(\omega)} \quad (3.2)$$

is called the *mask* of equation (1.1) (or the mask of its solution φ).

The coefficients of equation (1.1) are related to the values b_s of m_0 on $U_{n,s}$ by means of the direct and the inverse Vilenkin-Chrestenson transforms:

$$a_{\alpha} = \frac{1}{p^n} \sum_{s=0}^{p^n-1} b_s W_{\alpha}(A^{-n}h_{[s]}), \quad 0 \leq \alpha \leq p^n - 1, \quad (3.3)$$

$$b_s = \sum_{\alpha=0}^{p^n-1} a_{\alpha} \overline{W_{\alpha}(A^{-n}h_{[s]})}, \quad 0 \leq s \leq p^n - 1. \quad (3.4)$$

They can be realized by fast algorithms (see, for instance, [2, p.463], [62]). Thus, any choice of the values of m_0 on $U_{n,s}$ also defines the coefficients of equation (1.1) (note that if $\widehat{\varphi}(\theta) \neq 0$ then $b_0 = m_0(\theta) = 1$). Further, in addition to (2.2), the following theorem holds (see [14, 15]):

Theorem 3.1. *Let $\varphi \in L_c^2(G)$ be a solution of equation (1.1) such that $\widehat{\varphi}(\theta) = 1$. Then*

$$\sum_{\alpha=0}^{p^n-1} a_{\alpha} = 1 \quad \text{and} \quad \text{supp } \varphi \subset U_{1-n}.$$

This solution is unique, is given by the formula

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(A^{-j}\omega),$$

and satisfies the following properties:

- 1) $\widehat{\varphi}(h) = 0$ for all $h \in H \setminus \{\theta\}$;
- 2) $\sum_{h \in H} \varphi(x \oplus h) = 1$ for almost every $x \in G$.

We recall that a *multiresolution analysis* (MRA) in $L^2(G)$ is a sequence of subspaces $V_j \subset L^2(G)$, $j \in \mathbb{Z}$, with the following properties:

- (i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
- (ii) $\bigcup V_j$ is dense in $L^2(G)$ and $\bigcap V_j = \{\theta\}$;
- (iii) $f \in V_j$ if and only if $f(A^{-j}\cdot) \in V_0$;
- (iv) there is a function $\varphi \in L^2(G)$ such that $\{\varphi(\cdot \ominus h) : h \in H\}$ is an orthonormal basis for V_0 .

The function φ in condition (iv) is called a *scaling function* of the MRA.

Given a function $f \in L^2(G)$, we set

$$f_{j,h}(x) = p^{j/2} f(A^j x \ominus h), \quad j \in \mathbb{Z}, h \in H.$$

We say that a function φ generates an MRA if, first, the family $\{\varphi(\cdot \ominus h) : h \in H\}$ is an orthonormal system in $L^2(G)$ and, second, the subspaces

$$V_j = \text{clos}_{L^2(G)} \text{span} \{\varphi_{j,h} \mid h \in H\}, \quad j \in \mathbb{Z},$$

form an MRA in $L^2(G)$. If a refinable function φ generates an MRA, then, for every j , the system $\{\varphi_{j,h} : h \in H\}$ is an orthonormal basis of V_j and one can define *orthogonal wavelets* $\psi^{(1)}, \dots, \psi^{(p-1)}$ in such a way that the functions

$$\psi_{j,h}^{(\nu)}(x) = p^{j/2} \psi^{(\nu)}(A^j x \ominus h), \quad 1 \leq \nu \leq p-1, j \in \mathbb{Z}, h \in H,$$

form an orthonormal basis for $L^2(G)$ (see Algorithm A below).

Example 3.2. Let $a_0 = \dots = a_{p-1} = 1/p$ and $a_\alpha = 0$ for all $\alpha \geq p$. Then $\varphi = \mathbf{1}_{U_{n-1}}$ is a solution of equation (1.1); in particular, the Haar function $\mathbf{1}_U$ satisfies this equation with $n = 1$. The corresponding orthogonal wavelets can be written as

$$\psi^{(\nu)}(x) = \sum_{\alpha=0}^{p-1} \varepsilon_p^{\nu\alpha} \varphi(Ax \ominus h_{[\alpha]}), \quad 1 \leq \nu \leq p-1,$$

where $\varepsilon_p = \exp(2\pi i/p)$ (cf. [16, § 3], [48, § 5.1], [50, 54, 63]).

As claimed in [13, § 3], if a function $\varphi \in L_c^2(G)$ is a solution of equation (1.1) such that $\widehat{\varphi}(\theta) = 1$ and $\{\varphi(\cdot \ominus h) \mid h \in H\}$ is an orthonormal system, then

$$b_0 = 1, \quad |b_l|^2 + |b_{l+p^{n-1}}|^2 + \dots + |b_{l+(p-1)p^{n-1}}|^2 = 1, \quad 0 \leq l \leq p^{n-1} - 1. \quad (3.5)$$

As above, let m_0 be the mask of equation (1.1). Using the H -periodicity of m_0 (that is, $m_0(\omega \oplus h) = m_0(\omega)$ for all $h \in H$), we can rewrite (3.5) as

$$m_0(\theta) = 1, \quad \sum_{l=0}^{p-1} |m_0(\omega \oplus \delta_l)|^2 = 1, \quad \omega \in G, \quad (3.6)$$

where $\delta_l \in U$ and $\lambda(\delta_l) = l/p$ for $l \in \{0, 1, \dots, p-1\}$.

For $M \subset U$, we set

$$T_p M = \bigcup_{l=0}^{p-1} \left\{ A^{-1} h_{[l]} + A^{-1}(\omega) : \omega \in M \right\}.$$

Suppose that M either coincides with one of $U_{n-1,s}$, $1 \leq s \leq p^{n-1} - 1$, or is a union of some of these sets. Such a set M is said to be a *blocked set* for m_0 if

$$T_p M \subset M \cup \{\omega \in U : m_0(\omega) = 0\}.$$

It is clear that each mask can have only a finite number of blocked sets.

A compact subset of G is said to be *congruent to U modulo H* if $\mu(E) = 1$ and, for each $\omega \in E$, there is an element $h \in H$ such that $\omega \oplus h \in U$. The following theorem, which was proved in [15] (see also [14, Theorem 3] for the case $p = 2$), includes an analog of the well-known Cohen criterion:

Theorem 3.3. *Let $\varphi \in L_c^2(G)$ be a solution of equation (1.1) such that $\widehat{\varphi}(\theta) = 1$. Suppose that its mask m_0 satisfies (3.6). Then the following are equivalent:*

- (a) φ generates an MRA in $L^2(G)$.
- (b) m_0 has no blocked sets.
- (c) There exists a compact subset E of G congruent to U modulo H , containing a neighbourhood of zero, such that

$$\inf_{j \in \mathbb{N}} \inf_{\omega \in E} |m_0(A^{-j}\omega)| > 0.$$

Remark 3.4. By definition, the mask m_0 is an H -periodic step function, which is completely determined by the values b_s (see (3.4) and (3.5)). Since E is a compact set and $m_0(\theta) = 1$, there exists $j_0 \in \mathbb{N}$ such that $m_0(A^{-j}\omega) = 1$ for all $j > j_0$, $\omega \in E$. Therefore, condition (c) holds if m_0 does not vanish on the sets $A^{-1}(E), \dots, A^{-j_0}(E)$.

We have the following algorithm for construction of orthogonal wavelets in $L^2(G)$:

Algorithm A.

- **Step 1.** Choose numbers b_s , $0 \leq s \leq p^n - 1$, so that (3.5) is true.
- **Step 2.** Compute a_α , $0 \leq \alpha \leq p^n - 1$, by (3.3) and verify that

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W_\alpha(\omega)}$$

satisfies conditions (b) or (c) in Theorem 3.3.

- **Step 3.** Define $\varphi \in L^2(G)$ such that

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(A^{-j}\omega), \quad \omega \in G.$$

- **Step 4.** Find

$$m_\nu(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha^{(\nu)} \overline{W_\alpha(\omega)}, \quad 1 \leq \nu \leq p-1,$$

such that $(m_\nu(\omega \oplus \delta_k))_{\nu,k=0}^{p-1}$ is a unitary matrix.

- **Step 5.** Define $\psi^{(1)}, \dots, \psi^{(p-1)}$ as

$$\psi^{(\nu)}(x) = p \sum_{\alpha=0}^{p^n-1} a_\alpha^{(\nu)} \varphi(Ax \ominus h_{[\alpha]}), \quad 1 \leq \nu \leq p-1. \quad (3.7)$$

Note that condition (c) in Step 2 is fulfilled with $E = U$ if $m_0(\omega) \neq 0$ for all $\omega \in A^{-1}(U)$, or, equivalently, if in (3.5) we have $b_j \neq 0$ for all $j \in \{1, 2, \dots, p^{n-1} - 1\}$. For small values of p and n condition (b) can be checked numerically (see examples in [14, 15, 19]). In connection with Step 3 see Proposition 4.3 below and the expansion of φ in a Walsh series in [13].

Further, let us set

$$b_s^{(\nu)} = m_\nu(A^{-n}h_{[s]}), \quad \nu = 0, 1, \dots, p-1, \quad s = 0, 1, \dots, p^n - 1.$$

According to (3.5), we have

$$b_0^{(0)} = 1, \quad |b_l^{(0)}|^2 + |b_{l+p^{n-1}}^{(0)}|^2 + \dots + |b_{l+(p-1)p^{n-1}}^{(0)}|^2 = 1, \quad l = 1, 2, \dots, p^{n-1} - 1.$$

In Step 4, we should find the coefficients

$$b_0^{(\nu)}, b_1^{(\nu)}, \dots, b_{p^n-1}^{(\nu)}, \quad \nu = 1, 2, \dots, p-1,$$

such that

$$\begin{pmatrix} b_l^{(0)} & b_{l+p^{n-1}}^{(0)} & \dots & b_{l+(p-1)p^{n-1}}^{(0)} \\ b_l^{(1)} & b_{l+p^{n-1}}^{(1)} & \dots & b_{l+(p-1)p^{n-1}}^{(1)} \\ \dots & \dots & \dots & \dots \\ b_l^{(p-1)} & b_{l+p^{n-1}}^{(p-1)} & \dots & b_{l+(p-1)p^{n-1}}^{(p-1)} \end{pmatrix}, \quad l = 0, 1, \dots, p^{n-1} - 1,$$

will be unitary matrices. To do so, we can use the following statement based on the Householder transform (see [57, § 2.6]):

Proposition 3.5. *Let $c_k \in \mathbb{C}$, $k = 1, \dots, r$, $c_0 \neq 1$, $\sum_{k=0}^r |c_k|^2 = 1$, and let*

$$c_{k0} = c_k, \quad c_{0j} = \bar{c}_{j0} \frac{1 - c_{00}}{1 - \bar{c}_{00}}, \quad c_{kj} = \delta_{kj} - \frac{c_{k0}\bar{c}_{j0}}{1 - \bar{c}_{00}}, \quad j, k = 1, \dots, r. \quad (3.8)$$

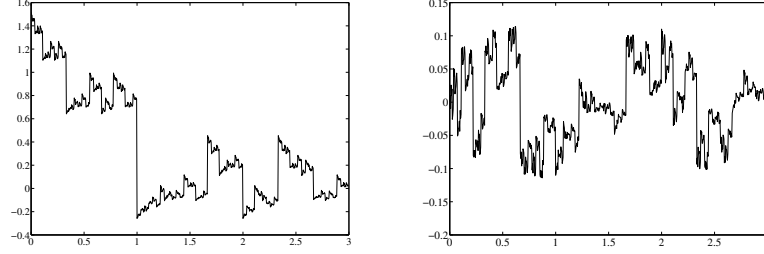
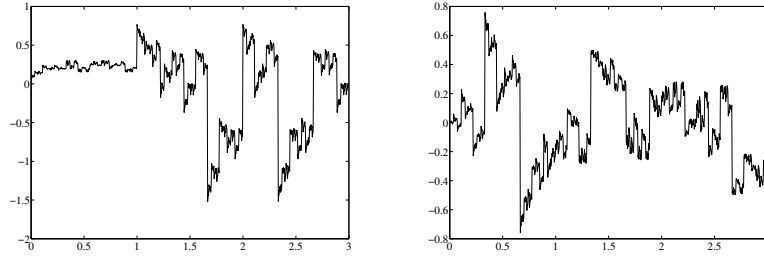
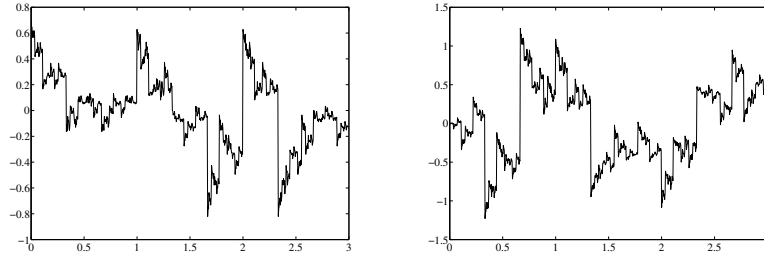
Then $(c_{jk})_{j,k=0}^r$ is a unitary matrix.

Some other methods of unitary extension of numerical matrices are given in [19], [32, Theorem 9], [64, Theorem 3.4].

Remark 3.6. A generalization of Algorithm A for constructing biorthogonal compactly supported wavelets on Vilenkin's group G is given in [19]; see also [30] for biorthogonal discrete bases. A general construction of periodic wavelets on the group G is presented in [28] (some particular cases were considered earlier in [27, 47]). A modification of Algorithm A for tight wavelet frames will be presented in § 4.

Remark 3.7. The values b_s are universal in the sense that they occur not only in the construction of wavelet bases for $L^2(G)$ by Algorithm A, but also for the spaces $\ell^2(\mathbb{Z}_N)$ and $\ell^2(\mathbb{Z})$. At the same time, in constructing discrete wavelets we have more freedom in choosing values of parameters, since the condition of absence of blocking sets is insignificant for sequence spaces (see [29, 65, 66] for details; compare with Algorithm B below).

Example 3.8. (*cf.* [19, Example 1]). Let $p = 3$, $n = 2$. Suppose that m_0 takes the value 1 on $U_{2,0}$, vanishes on $U_{2,3} \cup U_{2,6}$, and is defined on the remaining part of U as


 FIGURE 1. $\text{Re } \Phi$ (left) and $\text{Im } \Phi$ (right) from Example 3.8.

 FIGURE 2. $\text{Re } \Psi^{(1)}$ (left) and $\text{Im } \Psi^{(1)}$ (right) from Example 3.8.

 FIGURE 3. $\text{Re } \Psi^{(2)}$ (left) and $\text{Im } \Psi^{(2)}$ (right) from Example 3.8.

follows:

$$\begin{aligned} m_0(\omega) &= a, & \omega \in U_{2,1}, & & m_0(\omega) &= \alpha, & \omega \in U_{2,2}, \\ m_0(\omega) &= b, & \omega \in U_{2,4}, & & m_0(\omega) &= \beta, & \omega \in U_{2,5}, \\ m_0(\omega) &= c, & \omega \in U_{2,7}, & & m_0(\omega) &= \gamma, & \omega \in U_{2,8}, \end{aligned}$$

where

$$|a|^2 + |b|^2 + |c|^2 = |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1.$$

In this case, we have by (3.3)

$$\begin{aligned}
a_0 &= \frac{1}{9}(1 + a + b + c + \alpha + \beta + \gamma), \\
a_1 &= \frac{1}{9}(1 + a + \alpha + (b + \beta)\varepsilon_3^2 + (c + \gamma)\varepsilon_3), \\
a_2 &= \frac{1}{9}(1 + a + \alpha + (b + \beta)\varepsilon_3 + (c + \gamma)\varepsilon_3^2), \\
a_3 &= \frac{1}{9}(1 + (a + b + c)\varepsilon_3^2 + (\alpha + \beta + \gamma)\varepsilon_3), \\
a_4 &= \frac{1}{9}(1 + c + \beta + (a + \gamma)\varepsilon_3^2 + (b + \alpha)\varepsilon_3), \\
a_5 &= \frac{1}{9}(1 + b + \gamma + (a + \beta)\varepsilon_3^2 + (c + \alpha)\varepsilon_3), \\
a_6 &= \frac{1}{9}(1 + (a + b + c)\varepsilon_3 + (\alpha + \beta + \gamma)\varepsilon_3^2), \\
a_7 &= \frac{1}{9}(1 + b + \gamma + (a + \beta)\varepsilon_3 + (c + \alpha)\varepsilon_3^2), \\
a_8 &= \frac{1}{9}(1 + c + \beta + (a + \gamma)\varepsilon_3 + (b + \alpha)\varepsilon_3^2),
\end{aligned}$$

where $\varepsilon_3 = \exp(2\pi i/3)$. The blocked sets are: 1) $U_{1,1}$ for $a = c = 0$, 2) $U_{1,2}$ for $\alpha = \beta = 0$, 3) $U_{1,1} \cup U_{1,2}$ for $a = \alpha = 0$. Now, we let

$$\Lambda = \{1, 2\} \cup \left\{ \sum_{j=0}^k l_j 3^j : l_j \in \{1, 2\}, k \in \mathbb{Z}_+ \right\} = \{1, 2, 4, 5, 7, 8, 13, \dots\}.$$

For $s \in \{1, 2, 4, 5, 7, 8\}$, we set $\gamma(i_1, i_2) = b_s$ if $s = i_1 + 3i_2$, $i_1, i_2 \in \{0, 1, 2\}$; i.e.,

$$\gamma(1, 0) = a, \quad \gamma(2, 0) = \alpha, \quad \gamma(1, 1) = b, \quad \gamma(2, 1) = \beta, \quad \gamma(1, 2) = c, \quad \gamma(2, 2) = \gamma.$$

Further, for $l \in \Lambda$ we write

$$l = \sum_{j=0}^k l_j 3^j, \quad l_j \in \{0, 1, 2\}, \quad k = k(l) \in \mathbb{Z}_+,$$

and define

$$\begin{aligned}
\gamma_l &= \gamma(l_0, 0) \quad \text{if } k = k(l) = 0; \\
\gamma_l &= \gamma(l_1, 0)\gamma(l_0, l_1) \quad \text{if } k = k(l) = 1; \\
&\dots \\
\gamma_l &= \gamma(l_k, 0)\gamma(l_{k-1}, l_k) \dots \gamma(l_0, l_1) \quad \text{if } k = k(l) \geq 1.
\end{aligned}$$

$$l = \sum_{j=0}^k l_j 3^j, \quad l_j \in \{0, 1, 2\}, \quad k = k(l) \in \mathbb{Z}_+,$$

Then as in [13] (see also [16, § 4]) we have

$$\varphi(x) = (1/3)\mathbf{1}_U(A^{-1}x)(1 + \sum_{l \in \Lambda} \gamma_l W_l(A^{-1}x)), \quad x \in G. \quad (3.9)$$

Taking into account the above given blocked sets, we conclude from Theorem 3.3 that φ generates an MRA in the following cases: 1) $a \neq 0, \alpha \neq 0$, 2) $a = 0, \alpha \neq 0, c \neq 0$, 3)

$\alpha = 0, a \neq 0, \beta \neq 0$. We point out two cases where the series in (3.9) becomes a finite sum:

i) if $|\alpha| = 1, |a|^2 + |c|^2 = 1, b = \beta = \gamma = 0$, then

$$\varphi(x) = (1/3)\mathbf{1}_U(A^{-1}x)(1 + aW_1(A^{-1}x) + \alpha W_2(A^{-1}x) + \alpha cW_7(A^{-1}x)); \quad (3.10)$$

ii) if $|a| = 1, |\alpha|^2 + |\beta|^2 = 1, b = c = \gamma = 0$, then

$$\varphi(x) = (1/3)\mathbf{1}_U(A^{-1}x)(1 + aW_1(A^{-1}x) + \alpha W_2(A^{-1}x) + a\beta W_5(A^{-1}x)). \quad (3.11)$$

Now we choose

$$\begin{aligned} a &= 0.900000, & \alpha &= 0.900000, & b &= 0.435889, \\ \beta &= 0.2064742, & c &= 0.000000, & \gamma &= 0.383886. \end{aligned}$$

Using (3.9), we find the real and imaginary parts of φ . The function $\Phi : \mathbb{R}_+ \rightarrow \mathbb{C}$ is related to φ by the equality $\varphi(x) = \Phi[\lambda(x)]$ for almost every $x \in G$. The graph of this function is shown in Fig. 1. Further, by Proposition 3.5, we obtain

$$\begin{aligned} b_0^{(1)} &= 0.000000, & b_1^{(1)} &= 0.435890, & b_2^{(1)} &= 0.206474, & b_3^{(1)} &= 1.000000, & b_4^{(1)} &= -0.900000, \\ b_5^{(1)} &= 0.573684, & b_6^{(1)} &= 0.000000, & b_7^{(1)} &= 0.000000, & b_8^{(1)} &= -0.792625, \\ b_0^{(2)} &= 0.000000, & b_1^{(2)} &= 0.000000, & b_2^{(2)} &= 0.383886, & b_3^{(2)} &= 0.000000, & b_4^{(2)} &= 0.000000, \\ b_5^{(2)} &= -0.792625, & b_6^{(2)} &= 1.000000, & b_7^{(2)} &= 1.000000, & b_8^{(2)} &= -0.473684. \end{aligned}$$

Then

$$\psi^{(\nu)}(x) = 3 \sum_{\alpha=0}^8 a^{(\nu)} \varphi(Ax \ominus h_{[\alpha]}), \quad \nu = 1, 2,$$

where $a^{(\nu)}$ are calculated by (3.3). In Figs 2 and 3, the functions $\Psi^{(1)}$ and $\Psi^{(2)}$ are defined by analogy with Φ and related to $\psi^{(1)}$ and $\psi^{(2)}$, respectively.

For any positive integer n let $E_n(G)$ denote the collection of all functions on G that are constant on $U_{n,s} = A^{-n}(h_{[s]}) \oplus A^{-n}(U)$ for each $\alpha \in \mathbb{Z}_+$. By Proposition 2 of [15] (see also [3, § 6.2]), the following properties hold:

- (a) if $f \in L^1(G) \cap E_n(G)$, then $\text{supp } \hat{f} \subset U_{-n}$;
- (b) if $f \in L^1(G)$ and $\text{supp } f \subset U_{-n}$, then $\hat{f} \in E_n(G)$.

The following theorem gives conditions on the mask of equation (1.1) ensuring that the refinable function φ will not be an element of the set $E(G) = \cup_n E_n(G)$:

Theorem 3.9. *Let $\varphi \in L_c^2(G)$ be a solution of equation (1.1) such that $\widehat{\varphi}(\theta) = 1$. Then φ does not belong to $E(G)$ if and only if there exists a finite sequence $\{d_k\}_{k=1}^N$, $N \leq p^{n-1} + n - 1$, $d_k \in \{0, 1, \dots, p - 1\}$, such that*

- (i) $d_1 = \dots = d_{n-1} = 0, d_n \neq 0$;
- (ii) *there exists an interger j , $n - 1 \leq j \leq N - 1$, such that $d_{j-s} = d_{N-s}$ for $s = 0, \dots, n - 2$;*
- (iii) $m_0(\omega^{(0)})m_0(\omega^{(1)})\dots m_0(\omega^{(N-n)}) \neq 0$, where m_0 is the mask of equation (1.1) and $\omega^{(k)} \in U$ are determined by the sequence $\{d_k\}_{k=1}^N$ such that

$$\lambda(\omega^{(k)}) = \sum_{\nu=1}^n d_{\nu+k} p^{-\nu}, \quad k = 0, 1, \dots, N - n.$$

The proof of this theorem is essentially the same as in the case $p = 2$ (see [14, § 6]).

The main result of [21] is the following

Theorem 3.10. *Let $l \in \{1, 2, \dots, p-2\}$, where $p > 2$ is a prime number. Suppose that sets $E_l^{(0)}$ and $E_l^{(1)}$ form a partition of $\{0, 1, \dots, p-1\}$ such that $E_l^{(0)} = \{j_1, j_2, \dots, j_l\}$ and $0 \in E_l^{(1)}$. Fix $j_0 \in E_l^{(1)}$, $j_0 \neq 0$, and choose numbers b_s , $s = s_1 + s_2 p$, $s_1, s_2 \in \{0, 1, \dots, p-1\}$, such that*

- 1) $b_0 = 1$,
- 2) $|b_s| = 1$ for all $s \in E_l^{(1)} \setminus \{0\}$,
- 3) $|b_{j_1+j_0p}| = |b_{j_2+j_1p}| = \dots = |b_{j_l+j_{l-1}p}| = 1$,
- 4) $b_s = 0$ in the remaining cases.

Further, assume that the coefficients of the mask

$$m_0(\omega) = \sum_{\alpha=0}^{p^2-1} a_\alpha \overline{W_\alpha(\omega)}$$

are defined by (3.3) with $n = 2$. Then the function φ , which is defined by

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(A^{-j}\omega),$$

is a solution of the equation

$$\varphi(x) = p \sum_{\alpha=0}^{p^2-1} a_\alpha \varphi(Ax \ominus h_{[\alpha]})$$

and satisfies the following:

- (i) $\text{supp } \varphi \subset U_{-1}$,
- (ii) $\varphi \in E_l(G)$,
- (iii) φ generate an MRA in $L^2(G)$.

Remark 3.11. Suppose that m_0 and φ are defined as in Theorem 3.10. Let us choose a mask

$$\widetilde{m}_0(\omega) = \sum_{\alpha=0}^{p^2-1} \widetilde{a}_\alpha \overline{W_\alpha(\omega)}$$

for which the values $\widetilde{b}_s$ on $U_{n,s}$ satisfy

$$\widetilde{b}_0 = 1, \quad |\widetilde{b}_k|^2 + |\widetilde{b}_{k+p}|^2 + \dots + |\widetilde{b}_{k+(p-1)p}|^2 = 1, \quad 0 \leq k \leq p-1,$$

and $\widetilde{m}_0(\omega) \neq 0$ for all ω such that $|m_0(\omega)| = 1$. Then, as in Step 3 of Algorithm A, we define a function $\widetilde{\varphi}$ by

$$\widehat{\widetilde{\varphi}}(\omega) = \prod_{j=1}^{\infty} \widetilde{m}_0(A^{-j}\omega).$$

According to Theorem 3.3, this function $\widetilde{\varphi}$ generate an MRA in $L^2(G)$ (indeed, if condition (c) of Theorem 3.3 is fulfilled for m_0 with a set E , then the same is true for $\widetilde{m}_0$). In the case $p = 3$ Theorem 3.10 leads to the cases

- 1) $b_0 = 1$, $|b_1| = |b_5| = 1$, and $b_s = 0$ for the rest of s ($E_1^{(0)} = \{2\}$, $E_1^{(1)} = \{0, 1\}$, $j_0 = 1$);
- 2) $b_0 = 1$, $|b_2| = |b_7| = 1$, and $b_s = 0$ for the rest of s ($E_1^{(0)} = \{1\}$, $E_1^{(1)} = \{0, 2\}$, $j_0 = 2$).

As in Example 3.8, we set $\tilde{b}_1 = a$, $\tilde{b}_2 = \alpha$, $\tilde{b}_4 = b$, $\tilde{b}_5 = \beta$, $\tilde{b}_7 = c$, $\tilde{b}_8 = \gamma$. Then we conclude from (3.10) and (3.11) that, in the first case,

$$\tilde{\varphi}(x) = (1/3)\mathbf{1}_U(A^{-1}x)(1 + aW_1(A^{-1}x) + a\beta W_5(A^{-1}x)),$$

and, in the second case,

$$\tilde{\varphi}(x) = (1/3)\mathbf{1}_U(A^{-1}x)(1 + \alpha W_2(A^{-1}x) + \alpha c W_7(A^{-1}x)).$$

Notice that (3.11) includes one more case: if $b_0 = 1$, $|b_1| = |b_2| = 1$, then

$$\tilde{\varphi}(x) = (1/3)\mathbf{1}_U(A^{-1}x)(1 + aW_1(A^{-1}x) + \alpha W_2(A^{-1}x)).$$

In order to obtain the Haar function $\mathbf{1}_U$, we can take $a = \alpha = 1$ here. Note also, that the construction of MRA-based wavelets proposed in [20, §5] can be generalized by the same method.

4. MRA-based wavelet frames on Vilenkin's group

A detailed and comprehensive description of MRA-based wavelet frame theory and its applications are provided in [67] (see also [57, Chapter 2]). In this section, we describe a method to construct MRA-based tight wavelet frames on the group $G = G_p$.

Given $\Psi := \{\psi^{(1)}, \dots, \psi^{(r)}\} \subset L^2(G)$ with $r \geq p$, we define the *wavelet system* (or *affine system*) as

$$X(\Psi) := \{\psi_{j,k}^{(\nu)} : 1 \leq \nu \leq r, j \in \mathbb{Z}, k \in \mathbb{Z}_+\}, \quad (4.1)$$

where

$$\psi_{j,k}^{(\nu)}(x) = p^{j/2} \psi^{(\nu)}(A^j x \ominus h_{[k]}), \quad x \in G.$$

The system $X(\Psi)$ is called a *Parseval frame* or (a *tight wavelet frame with bound one*) for $L^2(G)$ if

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_+} \sum_{\nu=1}^r |\langle f, \psi_{j,k}^{(\nu)} \rangle|^2 = \|f\|^2$$

holds for all $f \in L^2(G)$, where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(G)$ and $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$. This is equivalent to

$$f = \sum_{g \in X(\Psi)} \langle f, g \rangle g,$$

for all $f \in L^2(G)$. Using the unitary extension principle (see e.g. [67]), the Parseval frames for $L^2(G)$ can be constructed by the following algorithm:

Algorithm B.

- **Step 1.** Choose numbers b_s , $0 \leq s \leq p^n - 1$, so that

$$b_0 = 1, \quad |b_l|^2 + |b_{l+p^{n-1}}|^2 + \dots + |b_{l+(p-1)p^{n-1}}|^2 \leq 1, \quad 0 \leq l \leq p^{n-1} - 1. \quad (4.2)$$

- **Step 2.** Compute a_α , $0 \leq \alpha \leq p^n - 1$, by (3.3) and define

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W_\alpha(\omega)}. \quad (4.3)$$

- **Step 3.** Define $\varphi \in L^2(G)$ such that

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(A^{-j}\omega), \quad \omega \in G.$$

- **Step 4.** Find the Walsh polynomials

$$m_\nu(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha^{(\nu)} \overline{W_\alpha(\omega)}, \quad 1 \leq \nu \leq r,$$

such that, for each $\omega \in G$, the rows of the following matrix form an orthonormal system:

$$M(\omega) = \begin{pmatrix} m_0(\omega) & m_1(\omega) & \dots & m_r(\omega) \\ m_0(\omega \oplus \delta_1) & m_1(\omega \oplus \delta_1) & \dots & m_r(\omega \oplus \delta_1) \\ \dots & \dots & \dots & \dots \\ m_0(\omega \oplus \delta_{p-1}) & m_1(\omega \oplus \delta_{p-1}) & \dots & m_r(\omega \oplus \delta_{p-1}) \end{pmatrix}.$$

- **Step 5.** Define $\psi^{(1)}, \dots, \psi^{(r)}$ as follows:

$$\psi^{(\nu)}(x) = p \sum_{\alpha=0}^{p^n-1} a_\alpha^{(\nu)} \varphi(Ax \ominus h_{[\alpha]}), \quad 1 \leq \nu \leq r. \quad (4.4)$$

Remark 4.1. In comparison with Algorithm A we have more freedom in choosing values of parameters, since condition (b) and (c) from Theorem 3.3 are insignificant in Algorithm B (compare with Remark 3.7).

As above, let us set

$$b_s^{(\nu)} = m_\nu(A^{-n}h_{[s]}), \quad 0 \leq s \leq p^n - 1.$$

Then, by (4.2), we have

$$b_0^{(0)} = 1, \quad |b_l^{(0)}|^2 + |b_{l+p^{n-1}}^{(0)}|^2 + \dots + |b_{l+(p-1)p^{n-1}}^{(0)}|^2 \leq 1, \quad 0 \leq l \leq p^{n-1} - 1.$$

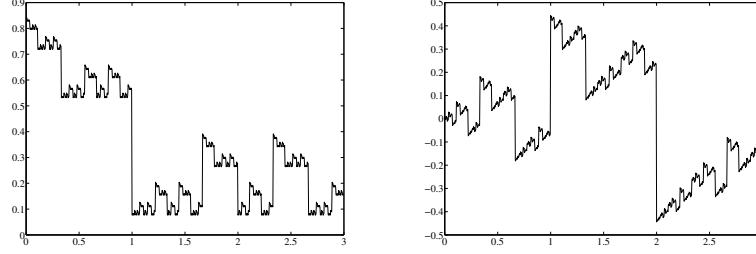
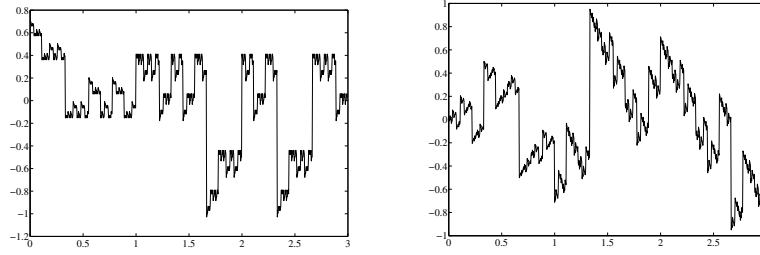
As usual, let $\mathcal{M}^*$ denote the adjoint matrix of a matrix $\mathcal{M}$. According to Step 4, for each l , we must find

$$b_l^{(1)}, b_{l+p^{n-1}}^{(1)}, \dots, b_{l+(p-1)p^{n-1}}^{(1)}, \quad \dots, \quad b_l^{(r)}, b_{l+p^{n-1}}^{(r)}, \dots, b_{l+(p-1)p^{n-1}}^{(r)},$$

such that the matrices

$$\mathcal{M}_l = \begin{pmatrix} b_l^{(0)} & b_{l+p^{n-1}}^{(0)} & \dots & b_{l+(p-1)p^{n-1}}^{(0)} \\ b_l^{(1)} & b_{l+p^{n-1}}^{(1)} & \dots & b_{l+(p-1)p^{n-1}}^{(1)} \\ \dots & \dots & \dots & \dots \\ b_l^{(r)} & b_{l+p^{n-1}}^{(r)} & \dots & b_{l+(p-1)p^{n-1}}^{(r)} \end{pmatrix}, \quad 0 \leq l \leq p^{n-1} - 1,$$

satisfy the condition $\mathcal{M}_l \mathcal{M}_l^* = I$, where I is the identity matrix. As in the orthogonal case, this problem is solved by using the Householder transform (see Proposition 3.5 and [32, § 3]). Notice that if instead of (4.2) we have condition (3.5) (in other words,


 FIGURE 4. $\text{Re } \Phi$ (left) and $\text{Im } \Phi$ (right) from Example 4.2.

 FIGURE 5. $\text{Re } \Psi^{(1)}$ (left) and $\text{Im } \Psi^{(1)}$ (right) from Example 4.2.

if all the inequalities in (4.2) become equalities), then Algorithm B can be applied to $r = p - 1$ (see also [31, Theorem 3.4] for the case $p = 2$).

Example 4.2. Let $p = 3$, $n = 2$, and let the notations be as in Example 5. Suppose that

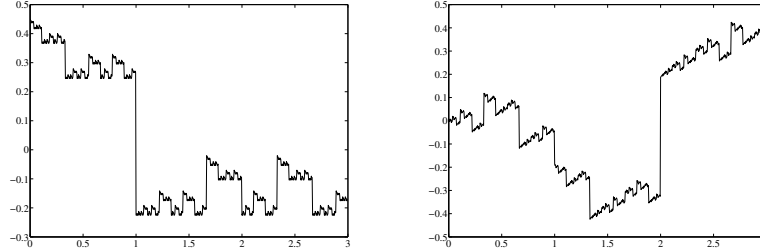
$$a = 0.909297, \quad \alpha = 0, \quad b = 0.411982, \quad \beta = 0, \quad c = -0.058727, \quad \gamma = 1.$$

Since $\alpha = \beta = 0$, the mask m_0 have the blocked set $U_{1,3}$. By Theorem 3.3, the refinable function φ does not generate an MRA. Moreover, in this case we have

$$\begin{aligned} b_0^{(1)} &= 0.000000, & b_1^{(1)} &= 0.411982, & b_2^{(1)} &= 0.000000, & b_3^{(1)} &= 1.000000, & b_4^{(1)} &= -0.871274, \\ b_5^{(1)} &= 1.000000, & b_6^{(1)} &= 0.000000, & b_7^{(1)} &= 0.266744, & b_8^{(1)} &= 0.000000, \end{aligned}$$

$$\begin{aligned} b_0^{(2)} &= 0.000000, & b_1^{(2)} &= -0.058727, & b_2^{(2)} &= 1.000000, & b_3^{(2)} &= 0.000000, & b_4^{(2)} &= 0.266744, \\ b_5^{(2)} &= 0.000000, & b_6^{(2)} &= 1.000000, & b_7^{(2)} &= 0.961977, & b_8^{(2)} &= 0.000000. \end{aligned}$$

Using Algorithm B, we find $\psi^{(1)}$ and $\psi^{(2)}$. The real and imaginary parts of Φ , $\Psi^{(1)}$ and $\Psi^{(2)}$ are shown in Figs 4, 5 and 6, respectively.

FIGURE 6. $\operatorname{Re} \Psi^{(2)}$ (left) and $\operatorname{Im} \Psi^{(2)}$ (right) from Example 4.2.

It is easy to check that (4.2) holds if and only if

$$m_0(\theta) = 1, \quad \sum_{l=0}^{p-1} |m_0(\omega \oplus \delta_l)|^2 \leq 1, \quad \omega \in G, \quad (4.5)$$

where m_0 is defined by (4.3). Further, from [32, Theorem 11] we have the following

Proposition 4.3. *Let g be the function given by*

$$g(\omega) := \prod_{j=1}^{\infty} m_0(A^{-j}\omega), \quad \omega \in G,$$

where m_0 is a Walsh polynomial satisfying (4.5). Then $g \in L^2(G)$ and a function φ defined by $\widehat{\varphi} = g$ is a compactly supported refinable function with mask m_0 .

Therefore, in the Fourier domain, (4.4) can be written as

$$\widehat{\psi}^{(\nu)}(\omega) = m_\nu(A^{-1}\omega) \widehat{\varphi}(A^{-1}\omega), \quad 1 \leq \nu \leq r,$$

where

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(A^{-j}\omega), \quad \omega \in G,$$

and, as a consequence of [32, Theorem 2] (cf. [33, Theorem 3.1]), we obtain the following

Theorem 4.4. *Suppose that $\Psi = \{\psi^{(1)}, \dots, \psi^{(r)}\}$ is defined by Algorithm B. Then $X(\Psi)$ is the Parseval frame for $L^2(G)$.*

Finally, we note that any tight wavelet frame decompositions in the Sobolev-type space $W_m^2(G)$ has an arbitrary approximation order (see [32, § 5]), which is in contrast to the real setting.

References

- [1] G. H. Agaev, N. Ya. Vilenkin, G. M. Dzhafarli, and A.I. Rubinstein, *Multiplicative Systems of Functions and Harmonic Analysis on 0-Dimensional Groups*, ELM, Baku, 1981 [in Russian].
- [2] F. Schipp, W. R. Wade, and P. Simon, *Walsh Series: An Introduction to Dyadic Harmonic Analysis*, Adam Hilger, New York, 1990.

- [3] B. I. Golubov, A. V. Efimov, and V. A. Skvortsov, *Walsh Series and Transforms*, URSS, Moscow, 2008. English transl. of 1st ed., Kluwer, Dordrecht, 1991.
- [4] R. E. Edwards. *Fourier Series. A Modern Introduction*, Vol. 2, Berlin: Springer Verlag, 1982.
- [5] T. P. Lukashenko, Wavelets on topological groups, *Izv. Ross. Akad. Nauk Ser. Mat.* **58**(3) (1994), 88-102. English transl., *Izvestiya: Mathematics* **44**(3) (1995), 515–529.
- [6] F. Schipp and W. R. Wade, *Transforms on Normed Fields*, Eötvös Loránd University, Budapest (1995).
- [7] W. C. Lang, Orthogonal wavelets on the Cantor dyadic group, *SIAM J. Math. Anal.* **27** (1996), 305-312.
- [8] W. C. Lang, Fractal multiwavelets related to the Cantor dyadic group, *Intern. J. Math. and Math. Sci.* **21** (1998), 307-317.
- [9] W. C. Lang, Wavelet analysis on the Cantor dyadic group, *Houston J. Math.* **24** (1998), 533-544.
- [10] Bl. Sendov, Multiresolution analysis of functions defined on the dyadic topological group, *East J. Approx.* **3**(2)(1997), 225–239.
- [11] N. Ya. Vilenkin, On a class of complete orthonormal systems, *Izv. Akad. Nauk SSSR, Ser. Mat.* **11** (1947), 363-400. English transl., *Amer. Math. Soc. Transl.* **28** (Ser. 2) (1963), 1-35.
- [12] N. J. Fine, On the Walsh functions, *Trans. Amer. Math. Soc.* **65**(3) (1949), 372-414.
- [13] Yu. A. Farkov, Orthogonal wavelets with compact support on locally compact abelian groups, *Izv. Ross. Akad. Nauk Ser. Mat.* **69**(3) (2005), 193-220. English transl., *Izvestiya: Mathematics* **69**(3) (2005), 623-650.
- [14] Yu. A. Farkov and V. Yu. Protasov, Dyadic wavelets and refinable functions on a half-line, *Mat. Sbornik* **197**(10) (2006), 129-160. English transl., *Sbornik: Mathematics* **197** (2006), 1529-1558.
- [15] Yu. A. Farkov, Orthogonal wavelets on direct products of cyclic groups, *Mat. Zametki* **82**(6) (2007), 934-952. English transl., *Math. Notes* **82**(6) (2007), 843-859.
- [16] Yu. A. Farkov, Multiresolution analysis and wavelets on Vilenkin groups, *Facta Universitatis (Niš)* **21**(3) (2008), 309-325.
- [17] Yu. A. Farkov, Biorthogonal wavelets on Vilenkin groups, *Tr. Mat. Inst. Steklova* **265**(1) (2009) 110-124. English transl., *Proc. Steklov Inst. Math.* **265**(1) (2009), 101-114.
- [18] Yu. A. Farkov, On wavelets related to Walsh series, *J. Approx. Theory* **161** (2009), 259-279.
- [19] Yu. A. Farkov and E. A. Rodionov, Algorithms for wavelet construction on Vilenkin groups, *p-Adic Numb. Ultr. Anal. Appl.* **3** (3) (2011), 181-195.
- [20] G. S. Berdnikov and S. F. Lukomskii, N -Valid trees in wavelet theory on Vilenkin groups, <http://arxiv.org/abs/1412.3096v1>.
- [21] S. F. Lukomskii, Step refinable functions and orthogonal MRA on p -adic Vilenkin groups, *J. Fourier Anal. Appl.* **20** (2014), 42-65.
- [22] S. F. Lukomskii, Multiresolution Riesz analysis on zero-dimensional groups, *Izv. Ross. Akad. Nauk Ser. Mat.* **79**(1) (2015), 153-184: English transl., *Izvestiya: Mathematics* **79**(1) (2015), 145-176.
- [23] Yu. A. Farkov and E. A. Rodionov, Estimates of the smoothness of dyadic orthogonal wavelets of Daubechies type, *Math. Notes* **86**(3) (2009), 407-421.
- [24] Yu. A. Farkov, U. Goginava, and T. Kopaliani, Unconditional convergence of wavelet expansion on the Cantor dyadic group, *Jaen J. Approx.* **3** (1) (2011), 117-133.
- [25] V. Yu. Protasov, Approximation by dyadic wavelets, *Mat. Sbornik* **198**(11) (2007), 135-152. English transl., *Sbornik: Mathematics* **198** (2007), 1665–1681.
- [26] Yu. A. Farkov, Wavelet expansions on the Cantor group, *Math. Notes* **96**(6) (2014), 996-1007.
- [27] Yu. A. Farkov, Periodic wavelets on the p -adic Vilenkin group, *p-Adic Numb. Ultr. Anal. Appl.* **3**(4)(2011), 281—287.

- [28] Yu. A. Farkov, Periodic wavelets in Walsh analysis, *Communic. Math. Appl.* **3**(3) (2012), 223-242.
- [29] Yu. A. Farkov, Discrete wavelets and the Vilenkin–Chrestenson transform, *Math. Notes* **89**(5) (2011), 871-884.
- [30] Yu. A. Farkov and E. A. Rodionov, On biorthogonal discrete wavelet bases, *Intern. J. Wavelets Multiresolut. Inf. Process.* **13**(1) (2015), 1550002, 18 p.
- [31] Yu. A. Farkov, Examples of frames on the Cantor dyadic group, *J. Math. Sc.* **187**(1) (2012), 22-34.
- [32] Yu. A. Farkov, E. A. Lebedeva, and M. A. Skopina, Wavelet frames on Vilenkin groups and their approximation properties, <http://arxiv.org/abs/1412.2699>
- [33] F. A. Shah, Tight wavelet frames generated by the Walsh polynomials, *Int. J. Wavelets Multiresolut. Inf. Process.* **11** (6) (2013), 1350042, 15 p.
- [34] Yu. A. Farkov, Orthogonal p -wavelets on $\mathbb{R}_+$, in *Proc. Intern. Conf. "Wavelets and splines" (July 3-8, 2003, St. Petersburg, Russia)*, St. Petersburg Univ. Press, St. Petersburg (2005), 4-26.
- [35] Yu. A. Farkov, Wavelets and frames in Walsh Analysis, in *Wavelets: Classification, Theory and Applications*, Chapter 11. Editors: Manel del Valle et al, Nova Science Publishers, New York (2012), 267-304.
- [36] Yu. A. Farkov and E. A. Rodionov, Nonstationary wavelets related to the Walsh functions, *American J. Comput. Math.* **2** (2012), 82-87.
- [37] Meenakshi, P. Manchanda and A. H. Siddiqi, Wavelets associated with nonuniform multiresolution analysis on positive half-line, *Int. J. Wavelets Multiresolut. Inf. Process.* **10**(2) (2012), 1250018, 27 p.
- [38] P. Manchanda and V. Sharma, Wavelet packets with nonuniform multiresolution analysis on positive half-line, *Asian-European J. Math.* **6**(1) (2013), 1350007, 16 p.
- [39] F. A. Shah, Biorthogonal p -wavelet packets related to the Walsh polynomials, *J. Classical Analysis* **1**(2) (2012), 135-146.
- [40] B. Behera and Q. Jahan, Wavelet packets and wavelet frame packets on local fields of positive characteristic, *J. Math. Anal. Appl.* **395**(2012), 1-14.
- [41] A. V. Krivoshein and E. A. Lebedeva, Uncertainty principle for the Cantor dyadic group, *J. Math. Anal. Appl.* **423** (2015), 1231-1242.
- [42] E. A. Lebedeva and M. A. Skopina, Walsh and wavelet methods for differential equations on the Cantor group, *J. Math. Anal. Appl.* (accepted, <http://arxiv.org/abs/1403.7349>)
- [43] Yu. A. Farkov, A. Yu. Maksimov and S. A. Stroganov, On biorthogonal wavelets related to the Walsh functions, *Int. J. Wavelets Multiresolut. Inf. Process.* **9**(3)(2011), 485-499.
- [44] L. Debnath and F. A. Shah, *Wavelet Transforms and Their Applications*, 2nd ed., Birkhäuser/Springer, New York, 2015.
- [45] Yu. A. Farkov and S. A. Stroganov, The use of discrete dyadic wavelets in image processing, *Russian Math. (Iz. VUZ)*, **55**(7)(2011), 47-55.
- [46] S. A. Stroganov, Dyadic wavelets for estimate of smoothness of low-frequency microseismic oscillations, *Geophysical Research* **13**(1) (2012), 60-65 [in Russian].
- [47] Yu. A. Farkov and M. E. Borisov, Periodic dyadic wavelets and coding of fractal functions, *Russian Math. (Iz. VUZ)*, **56**(9) (2012), 46-56.
- [48] J. J. Benedetto and R. L. Benedetto, A wavelet theory for local fields and related groups, *J. Geometric Analysis*, **14** (2004), 423-456.
- [49] J. J. Benedetto and R. L. Benedetto, The construction of wavelet sets, *Wavelets and multiscale analysis. Theory and applications. Selected papers based on the presentations at the international conference on wavelets: twenty years of wavelets*, Cohen, Jonathan (ed.) et al., DePaul University, Chicago, IL, USA, May 15-17, 2009. New York, NY: Springer (Applied and Numerical Harmonic Analysis) (2011), 17-56.

- [50] S. V. Kozyrev, Wavelet analysis as a p -adic spectral analysis, *Izv. Ross. Akad. Nauk Ser. Mat.* **66**(2) (2002), 149-158; English transl., *Izvestiya: Mathematics* **66**(2) (2002), 367-376.
- [51] S. F. Lukomskii, Multiresolution analysis on zero-dimensional abelian groups and wavelets bases, *Sb.: Math.* **201**(5) (2010), 669-691.
- [52] S. Albeverio, S. Evdokimov and M. Skopina, p -adic multiresolution analysis and wavelet frames, *J. Fourier Anal. Appl.* **16** (5) (2010), 693-714.
- [53] S. Evdokimov and M. Skopina, On orthogonal p -adic wavelet bases. *J. Math. Anal. Appl.*, **424** (2) (2015), 952-965.
- [54] I. Ya. Novikov and M. A. Skopina, Why are Haar bases in various structures the same? *Math. Notes* **91**(6) (2012), 895-898.
- [55] S. Evdokimov, Haar multiresolution analysis and Haar bases on the ring of rational adeles, *J. Math. Sci., New York*, **192** (2) (2013), 215-219 (2013).
- [56] B. Behera and Q. Jahan, Characterization of wavelets and MRA wavelets on local fields of positive characteristic, *Collect. Math.* **66**(1)(2015), 33-53.
- [57] I. Ya. Novikov, V. Yu. Protasov, and M. A. Skopina *Wavelet Theory*, Translations of Mathematical Monographs **239** (AMS, Providence), 2011.
- [58] F. Keinert, *Wavelets and Multiwavelets*, Studies in advanced mathematics, Chapman & Hall/CRC, 2004.
- [59] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Lecture Notes **61**, SIAM, Philadelphia, PA, 1992.
- [60] I. W. Selesnick, Multiwavelet bases with extra approximation properties, *IEEE Trans. signal process.*, **46**(11)(1998), 2898-2908.
- [61] N. Guglielmi and V. Yu. Protasov, Invariant polytopes of linear operators with applications to regularity of wavelets and of subdivisions, <http://arxiv.org/1502.01192v1>
- [62] V. N. Malozemov and S. M. Masharskii, Generalized wavelet bases related to the discrete Vilenkin–Chrestenson transform, *St. Petersburg Math. J.* **13** (2002), 75-106.
- [63] B. I. Golubov, On one class of complete orthogonal systems, *Siberian Math. J.* **9**(2) (1968), 297-314 (in Russian).
- [64] R. Q. Jia and Z. Shen, Multiresolution and wavelets, *Proc. Edinburgh Math. Soc.* **37** (1994), 271-300.
- [65] M. W. Frazier, *An Introduction to Wavelets Through Linear Algebra*, Springer, New York, 1999.
- [66] S. A. Broughton and K. M. Bryan, *Discrete Fourier Analysis and Wavelets. Applications to Signal and Image Processing*, Hoboken (NJ), Wiley, 2009.
- [67] B. Dong and Z. Shen, MRA-based wavelet frames and application, in *Mathematics in Image Processing* / Hogkai Zhao, editor. – (IAS/Park City mathematics series) **19** (2013), 7-158.

(Yu. A. Farkov) DEPARTMENT OF APPLIED INFORMATION TECHNOLOGY, RUSSIAN PRESIDENTIAL ACADEMY OF NATIONAL ECONOMY AND PUBLIC ADMINISTRATION, PROSPECT VERNADSKOGO, 82, MOSCOW, 119571, RUSSIA.

E-mail address: `farkov@list.ru`