

MULTIRESOLUTION ANALYSES ON QUASILATTICES

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Abstract. We derive relations between geometric means of the Fourier moduli of a refinable distribution and of a related polynomial. We use Pisot-Vijayaraghavan numbers to construct families of one dimension quasilattices and multiresolution analyses spanned by distributions that are refinable with respect to dilation by the PV numbers and translation by quasilattice points. We conjecture that scalar valued refinable distributions are never integrable, construct piecewise constant vector valued refinable functions, and discuss multidimensional extensions.

1. Refinable Distributions

In this paper $\mathbb{Z}, \mathbb{N} = \{1, 2, 3, \dots\}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the integer, natural, rational, real, and complex numbers and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the circle group. Let $\alpha_1, \alpha_2, \dots$ be a sequence of positive real numbers, let $\mu_1, \mu_2, \dots$ be a sequence of complex probability measures whose total variations are uniformly bounded and whose supports satisfy $\text{supp}(\mu_n) \subset [-\alpha_n, \alpha_n]$, and define the sequence of complex probability measures $f_n = \mu_1 * \dots * \mu_n$ where $*$ denotes convolution. Our result ([28], Theorem 2.1) implies that if $\alpha_j \rightarrow 0$ exponentially fast then f_n converges to a compactly supported distribution. Our result addresses general Lie groups and our proof avoids Fourier transforms. In [57] Uchida proved that f_n converges to a hyperfunction under the much weaker assumption that $\sum_n \alpha_n < \infty$. His result addresses commutative Lie groups and his proof uses the Fourier transform.

Remark 1.1. The complex Fourier transforms $\hat{f}_n(z) = \int_{-\infty}^{\infty} f_n(x) e^{2\pi i x z} dx = \prod_{j=1}^n \hat{\mu}_j(z)$ and if $\alpha_n \rightarrow 0$ exponentially fast then the argument that Daubechies credits to Deslauriers and Dubuc [13] and uses to prove ([10], Lemma 6.2.2) shows that $\hat{f}_n(z)$ converges to an entire function of exponential type which the Paley-Wiener Theorem ([15], Section 3.3), [39] shows is equal to the Fourier transform $\hat{f}$ of a compactly supported distribution f .

Let $\lambda \in \mathbb{R} \setminus [-1, 1]$, $a_1, \dots, a_m \in \mathbb{C} \setminus \{0\}$, $a_1 + \dots + a_m = |\lambda|$, $\tau_1 < \dots < \tau_m \in \mathbb{R}$, and define the sequence of complex probability measures by $\mu_n = |\lambda|^{-1} \sum_{j=1}^m a_j \delta_{\tau_j/\lambda^n}$

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where for $r \in \mathbb{R}$, δ_r denotes the Dirac point measure located at r . Since $\text{supp}(\mu_n) \subset [\tau_1/\lambda^n, \tau_m/\lambda^n]$ there exists a compactly supported distribution f such that $f_n = \mu_1 * \cdots * \mu_n \rightarrow f$. Since $|\lambda| f_n(\lambda x) = \mu_2 * \cdots * \mu_{n+1}(x)$, f is the unique compactly supported distribution that satisfies the refinement equation

$$f(x) = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \mu_1(y) |\lambda| f_n(\lambda(x-y)) dy = \sum_{j=1}^m a_j f(\lambda x - \tau_j). \quad (1.1)$$

and the condition $\int_{-\infty}^{\infty} f(x) dx = 1$. Define the function

$$A(y) = |\lambda|^{-1} \sum_{j=1}^m a_j e^{2\pi i \tau_j y}, \quad y \in \mathbb{R}. \quad (1.2)$$

The Fourier transform $\widehat{f}$ admits the expansion

$$\widehat{f}(y) = \prod_{k=1}^{\infty} A(y \lambda^{-k}) \quad (1.3)$$

and $\widehat{f}(0) = 1$. Therefore $\widehat{f}$ and A are related by the functional equation

$$\widehat{f}(y \lambda^k) = \widehat{f}(y) \prod_{j=0}^{k-1} A(y \lambda^j), \quad k \in \mathbb{N}. \quad (1.4)$$

The Mahler measure [33] (or height) $M(P)$ of a polynomial $P(z_1, \dots, z_d)$ is defined by

$$M(P) = \exp \int_0^1 \cdots \int_0^1 \ln |P(e^{2\pi i t_1}, \dots, e^{2\pi i t_d})| dt_1 \dots dt_d. \quad (1.5)$$

For a polynomial of a single variable $P(z) = c(z - \omega_1) \cdots (z - \omega_q)$ Jensen's theorem [21] gives $M(P) = |c| \prod_{j=1}^q \max\{1, |\omega_j|\}$.

Remark 1.2. Mahler measure arises in prediction theory of stationary random processes [25], in algebraic dynamics [17], and in Lehmer's open problem in number theory [32].

A is almost periodic in the (uniform) sense of Bohr [4]. Let d be the maximal number of linearly independent $\tau_1, \dots, \tau_m$ over $\mathbb{Q}$. A is periodic if and only if $d = 1$. Furthermore, there exists a polynomial $P(z_1, \dots, z_d)$ and real numbers $r_1, \dots, r_d$ such that

$$A(y) = P(e^{2\pi i r_1 y}, \dots, e^{2\pi i r_d y}), \quad y \in \mathbb{R}. \quad (1.6)$$

In the remainder of this section we derive the following two theorems.

Theorem 1.3. $\ln |A|$ is almost periodic, in the (mean squared) sense of Besicovitch [3], and

$$\lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L \ln |A(y)| dy = \ln M(P). \quad (1.7)$$

Definition 1.4. The Mahler measure of A is $M(A) = M(P)$.

Theorem 1.5. Define $\rho(f) = -\ln M(A)/\ln \lambda$. Then

$$\lim_{L \rightarrow \infty} \frac{1}{2L \ln L} \int_{-L}^L \ln |\widehat{f}(y)| dy = -\rho(f). \quad (1.8)$$

Lemma 1.6. (Titchmarsh) *If h is a nonzero compactly supported distribution and $[a, b]$ is the smallest interval containing the support of h then*

$$H(z) = \int_a^b h(t) e^{2\pi z t} dt \quad (1.9)$$

is an entire function of exponential type. Moreover the number $n(r)$ of zeros of $H(z)$ in the disk $|z| < r$ satisfies

$$\lim_{r \rightarrow \infty} \frac{n(r)}{r} = 2(b - a). \quad (1.10)$$

Titchmarsh's proof ([56], Theorem IV) assumed that h was a Lebesgue integrable function but a simple regularization argument implies that if h is a compactly supported distribution then (1.10) holds. Therefore there exists $\beta > 0$ such that

$$\text{card} \{ y : y \in [-L, L] \text{ and } H(y) = 0 \} \leq \beta L, \quad L \geq 1 \quad (1.11)$$

Lemma 1.7. *If $I = [a, b]$, $u > 0$, $v > 0$, $K : I \rightarrow \mathbb{C}$ is differentiable, $|K| \leq v$, the derivative K' satisfies $|K'| \geq u$, and $K'(I)$ is contained in a single quadrant of $\mathbb{C}$, then*

$$b - a \leq \frac{2\sqrt{2}v}{u}. \quad (1.12)$$

Proof We present the proof that we gave in ([27], Lemma 1). Since $u \leq |K'|$ the triangle inequality $|K'| \leq |\Re K'| + |\Im K'|$ gives

$$u(b - a) \leq \int_a^b |K'(y)| dy \leq \int_a^b (|\Re K'(y)| + |\Im K'(y)|) dy.$$

Since $K'(I)$ is contained in a single quadrant of $\mathbb{C}$ there exist $c \in \{1, -1\}$ and $d \in \{1, -1\}$ such that $|\Re K'| = c \Re K'$ and $|\Im K'| = d \Im K'$. Therefore

$$\int_a^b (|\Re K'(y)| + |\Im K'(y)|) dy = (c \Re K(b) + d \Im K(b)) - (c \Re K(a) + d \Im K(a)).$$

Since $|c| = |d| = 1$ and $|K| \leq v$ the quantity on right is bounded above by $2\sqrt{2}v$. Combining these inequalities completes the proof.

Lemma 1.8. *Let μ denote Lebesgue measure. There exists a sequence $c_2, c_3, \dots$ of positive real numbers such that for all monic polynomials $P(z)$ with $k \geq 2$ nonzero coefficients*

$$\mu(\{ y \in [0, 1] : |P(e^{2\pi i y})| \leq v \}) \leq c_k v^{1/(k-1)}, \quad v > 0. \quad (1.13)$$

Proof We proved this in ([27], Theorem 1) using induction on k , Lemma 1.7, and the fact that a polynomial of degree q can have at most q distinct roots.

Remark 1.9. In [27] we proved a conjecture of Boyd [7]. His conjecture expresses the Mahler measure of a multidimensional polynomial as a limit of Mahler measures of univariate polynomials and provides an alternative proof of the special case of Lehmer's conjecture proved in [14]. The conjecture is stated in http://en.wikipedia.org/wiki/Mahler_measure Our proof used Lemma 1.8 and a multidimensional extension of a result of Kronecker that we developed in [26] and Boyd independently developed in [7].

Lemma 1.10. *There exists a sequence $c_2, c_3, \dots$ of positive real numbers such that*

$$\mu(\{y \in [-L, L] : |A(y)| \leq v\}) \leq L c_m v^{1/(m-1)}, \quad L \geq 1, v > 0. \quad (1.14)$$

Proof We observe that Lemma 1.10 concerns (not necessarily periodic) trigonometric polynomials. The conclusion is obvious for $m = 2$ so we assume that $m \geq 3$ and proceed by induction on m using a technique similar to the one we used to prove Lemma 1.8. Since $|e^{-2\pi i \tau_1 y} A(y)| = |A(y)|$ we may assume without loss of generality that $0 = \tau_1 < \dots < \tau_m$. Then A' is a trigonometric polynomial with $m - 1$ terms so by induction there exists $c_{m-1} > 0$ such that

$$\mu(\{y \in [-L, L] : |A'(y)| \leq u\}) \leq L c_{m-1} u^{1/(m-2)}, \quad L \geq 1, u > 0. \quad (1.15)$$

We observe that A and A' , their real and imaginary parts, and their squared moduli are restrictions of functions of exponential type so they satisfy the hypothesis in Lemma 1.6. Therefore (1.11) implies that there exists $\beta > 0$ such that for all $u > 0, v > 0$ and $L \geq 1$ the subset of $[-L, L]$ where $|A'(y)| \geq u$ and $|A(y)| \leq v$ can be expressed as the union of not more than βL closed intervals I such that $A'(I)$ is contained in a single quadrant of $\mathbb{C}$. Therefore Lemma 1.7 implies that there exists $\beta > 0$ such that

$$\mu(\{y \in [-L, L] : |A(y)| \leq v \text{ and } |A'(y)| \geq u\}) \leq \frac{L \beta v}{u}, \quad L \geq 1, u > 0, v > 0. \quad (1.16)$$

Combining (1.15) and (1.16) gives for every $L \geq 1, v > 0$

$$\mu(\{y \in [-L, L] : |A(y)| \leq v\}) \leq L \min_{u>0} \left[c_{m-1} u^{1/(m-2)} + \frac{\beta v}{u} \right] = L c_m v^{1/(m-1)} \quad (1.17)$$

where $c_m = (1 + \beta(m-2)) \left[\frac{c_{m-1}}{\beta(m-2)} \right]^{\frac{m-2}{m-1}}$.

For $n \in \mathbb{N}$, $\mathbb{R}^n$ is n -dimensional Euclidean vector space with norm $\|\cdot\|$, $\mathbb{Z}^n$ is a rank n lattice subgroup of $\mathbb{R}^n$, $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ is the n -dimensional torus group equipped with Haar measure, and $p_n : \mathbb{R}^n \rightarrow \mathbb{T}^n$ is the canonical epimorphism. Vectors are represented as column vectors and T denotes transpose. We define $\|\cdot\|_{\mathbb{T}^n} : \mathbb{T}^n \rightarrow \mathbb{R}$ by

$$\|g\|_{\mathbb{T}^n} = \min\{\|x\| : x \in \mathbb{R}^n \text{ and } p_n(x) = g\}, \quad g \in \mathbb{T}^n. \quad (1.18)$$

For $d \in \mathbb{N}$ and $r = [r_1, \dots, r_d]^T \in \mathbb{R}^d$ we define the homomorphism $\Psi_r : \mathbb{R} \rightarrow \mathbb{T}^d$ by

$$\Psi_r(y) = p_d(yr), \quad y \in \mathbb{R} \quad (1.19)$$

where $p_d : \mathbb{R}^d \rightarrow \mathbb{T}^d$ is the canonical epimorphism.

Lemma 1.11. (Bohl–Sierpinski–Weyl) *The image of Ψ_r is a dense subgroup of $\mathbb{T}^d$ if and only if the components $r_1, \dots, r_d$ of r are linearly independent over $\mathbb{Q}$. If they are independent then for every continuous function $S : \mathbb{T}^d \rightarrow \mathbb{C}$*

$$\lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L S \circ \Psi_r(y) dy = \int_{\mathbb{T}^d} S(g) dg \quad (1.20)$$

where $S \circ \Psi_r : \mathbb{R} \rightarrow \mathbb{C}$ is the composition of S with Ψ_r and dg is Haar measure on $\mathbb{T}^d$.

Proof Arnold ([1], p. 285–289) discusses the significance and gives Weyl’s proof of this classic result called the *Theorem on Averages* and asserts it “may be found implicitly in the work of Laplace, Lagrange, and Gauss on celestial mechanics” and “A rigorous

proof was given only in 1909 by P. Bohl, V. Sierpinski, and H. Weyl in connection with a problem of Lagrange on the mean motion of the earth's perihelion."

For every nonzero polynomial $P(z_1, \dots, z_d)$ we define the trigonometric polynomial $P_{\text{trig}} : \mathbb{T}^d \rightarrow \mathbb{C}$ by

$$P_{\text{trig}}(p_d([t_1, \dots, t_d]^T)) = P(e^{2\pi i t_1}, \dots, e^{2\pi i t_d}) \quad (1.21)$$

and observe through a simple induction based computation that $\ln |P_{\text{trig}}| \in L^2(\mathbb{T}^d)$. We now assume that $r = [r_1, \dots, r_d]^T \in \mathbb{R}^d$ has linearly independent entries over $\mathbb{Q}$ and that $P(z_1, \dots, z_d)$ is a polynomial that satisfies (1.6). Therefore

$$A(y) = P_{\text{trig}} \circ \Psi_r(y), \quad y \in \mathbb{R}. \quad (1.22)$$

Proof of Theorem 1.3 For all $v > 0$ define $S_v : \mathbb{T}^d \rightarrow \mathbb{R}$ by

$$S_v(g) = \ln(\max\{v, |P_{\text{trig}}(g)|\}).$$

Lebesgue's dominated convergence theorem implies that

$$\lim_{v \rightarrow 0} \int_{\mathbb{T}^d} |S_v(g) - \ln |P_{\text{trig}}(g)||^2 dg = 0. \quad (1.23)$$

Lemma 1.10 implies that

$$\lim_{v \rightarrow 0} \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L |\ln |A(y)| - S_v \circ \Psi_r(y)|^2 dy = 0 \quad (1.24)$$

and hence $\ln |A(y)|$ is almost periodic in the sense of Besicovitch. We combine (1.23) and (1.24) with Lemma 1.11 to complete the proof by computing

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L \ln |A(y)| dy &= \lim_{v \rightarrow 0} \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L S_v \circ \Psi_r(y) dy = \\ \lim_{v \rightarrow 0} \int_{\mathbb{T}^d} S_v(g) dg &= \int_{\mathbb{T}^d} \ln |P_{\text{trig}}(g)| dg = M(P). \end{aligned}$$

Remark 1.12. Since the zero set of P_{trig} is a real analytic set, an alternative proof based on Lojasiewicz's structure theorem [23], [30] for real analytic sets may be possible using methods that we developed in [29] to prove the Lagarias-Wang Conjecture.

Proof of Theorem 1.5 Let $L = b|\lambda|^k$ with $k \in \mathbb{N}$ and $b \in [|\lambda^{-1}|, 1)$ then use (1.4) and Theorem 1.3 to compute

$$\begin{aligned}
& \lim_{L \rightarrow \infty} \frac{1}{2L \ln L} \int_{-L}^L \ln |\widehat{f}(y)| dy = \\
& \lim_{k \rightarrow \infty} \frac{1}{2kb |\lambda|^k \ln |\lambda|} \sum_{j=1}^{k-1} \int_{b|\lambda|^j}^{b|\lambda|^{j+1}} \ln \left(|\widehat{f}(y) \widehat{f}(-y)| \right) dy = \\
& \lim_{k \rightarrow \infty} \frac{1}{2k |\lambda|^k \ln |\lambda|} \sum_{j=1}^{k-1} |\lambda|^j \int_1^{|\lambda|} \ln \left(|\widehat{f}(ub|\lambda|^j) \widehat{f}(-ub|\lambda|^j)| \right) du = \\
& \lim_{k \rightarrow \infty} \frac{1}{2k |\lambda|^k \ln |\lambda|} \sum_{j=1}^{k-1} |\lambda|^j \int_1^{|\lambda|} \ln \left(|\widehat{f}(ub\lambda^j) \widehat{f}(-ub\lambda^j)| \right) du = \\
& \lim_{k \rightarrow \infty} \frac{1}{2k |\lambda|^k \ln |\lambda|} \sum_{j=1}^{k-1} |\lambda|^j \int_1^{|\lambda|} \left(\ln |\widehat{f}(bu) \widehat{f}(-bu)| + \sum_{i=1}^j \ln |A(ub\lambda^i) A(-ub\lambda^i)| \right) du = \\
& \lim_{k \rightarrow \infty} \frac{1}{2k |\lambda|^k \ln |\lambda|} \sum_{j=1}^{k-1} |\lambda|^j \sum_{i=1}^j \int_1^{|\lambda|} (\ln |A(ub\lambda^i)| + \ln |A(-ub\lambda^i)|) du = \\
& \lim_{k \rightarrow \infty} \frac{(|\lambda|-1) \ln M(A)}{k |\lambda|^k \ln |\lambda|} \sum_{j=1}^{k-1} j |\lambda|^j = \\
& \lim_{k \rightarrow \infty} \frac{(|\lambda|-1) \ln M(A)}{k |\lambda|^k \ln |\lambda|} \left(\frac{|\lambda| - |\lambda|^{k+1}}{(|\lambda|-1)^{-2}} + \frac{k|\lambda|^{k-1}-1}{|\lambda|-1} \right) = \\
& \frac{\ln M(A)}{\ln \lambda}.
\end{aligned}$$

2. Quasilattices

If λ is an algebraic integer with minimal polynomial $\Lambda(z) = z^n + c_{n-1}z^{n-1} + \dots + c_0$, $c_j \in \mathbb{Z}$ that has roots $\lambda = \lambda_1, \lambda_2, \dots, \lambda_n$, then the Frobenius companion matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -c_0 & -c_1 & \dots & \dots & -c_{n-1} \end{bmatrix}, \quad (2.1)$$

Vandermonde matrix

$$V = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \vdots & \vdots & \dots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix}, \quad (2.2)$$

and diagonal matrix $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ satisfy

$$C^k V = V D^k, \quad k \in \mathbb{N}. \quad (2.3)$$

A Pisot—Vijayaraghavan (PV) number is an algebraic integer λ whose Galois conjugates $\lambda = \lambda_1, \dots, \lambda_n$ satisfy $|\lambda_j| < 1$ whenever $j > 1$. Then $|\lambda| > 1$. They were discovered by Thue [55], rediscovered by Hardy [19] who with Vijayaraghavan studied their

Diophantine approximation [8] properties, and studied by Pisot [44] in his dissertation. Examples are:

$\lambda = \lambda_1$	λ_2	λ_3	λ_4	minimal polynomial
1.6180	-0.6180			$z^2 - z - 1$
-1.6180	0.6180			$z^2 + z - 1$
3.4142	0.5858			$z^2 - 4z + 2$
2.2470	0.5550	-0.8019		$z^3 - 2z^2 - z + 1$
1.3247	$-0.6624 + 0.5623i$	$-0.6624 - 0.5623i$		$z^3 - z - 1$
1.3803	-0.8192	$0.2194 + 0.9145i$	$0.2194 - 0.9145i$	$z^4 - z^3 - 1$

For $\lambda \in \mathbb{C}$ let $\mathbb{Z}[\lambda]$ be the $\mathbb{Z}$ -module generated by powers of λ .

Lemma 2.1. *If λ is a PV number and $\alpha \in \mathbb{Z}[\lambda]$ then there exists a sequence $n_k \in \mathbb{Z}$ such that $\lambda^k \alpha - n_k \rightarrow 0$ exponentially fast.*

Proof Define $n_k = \sum_{j=1}^n \lambda_j^k \alpha_j$ where $\lambda = \lambda_1, \dots, \lambda_n$ are the Galois conjugates of λ and $\alpha = \alpha_1, \dots, \alpha_n$ are the Galois conjugates of α . Each n_k is a sum of the Galois conjugates of $\lambda^k \alpha$ and hence is an integer. The assertion follows since $\sum_{j=2}^n \lambda_j^k \alpha_j \rightarrow 0$ exponentially fast.

Let λ be a PV number and let $C, D, V, \lambda_1, \dots, \lambda_n$ be as above. We call a vector $\sigma \in \mathbb{R}^n$ admissible if $\sigma_1 = 0$, $\sigma_j > 0$ for $j \geq 2$, and whenever $2 \leq j < k \leq n$ and λ_j and λ_k are nonreal complex conjugate pairs then $\sigma_j = \sigma_k$. We denote the set of admissible vectors by $\mathbb{R}_a^n$. It is an open cone and admits the partial order $\sigma \leq \xi$ if and only if $\sigma_j \leq \xi_j$, $1 \leq j \leq n$.

Definition 2.2. The quasilattice corresponding to $\sigma \in \mathbb{R}_a^n$ is

$$\mathfrak{L}(\sigma) = \{ (V^T \ell)_1 : \ell \in \mathbb{Z}^n \text{ and } |(V^T \ell)_j| < \sigma_j \text{ for all } 2 \leq j \leq n \}.$$

Remark 2.3. Quasilattices are a subset of cut-and-project sets studied by Meyer [35]. de Bruijn [11], [2] showed that quasicrystals associated to Penrose tilings [40, 41, 42] are higher dimensional cut-and-project sets. Bombieri and Taylor [5, 6] discuss the diffraction and algebraic properties of quasicrystals. Hof [20] proved that every cut-and-project set has a pure point Fourier spectrum. The book by Senechal [53] and references therein provide a basic introduction to quasicrystals.

Lemma 2.4. *If $\sigma, \xi \in \mathbb{R}_a^n$ then*

$$0 \in \mathfrak{L}(\sigma) \text{ and } \mathfrak{L}(\sigma) = -\mathfrak{L}(\sigma),$$

$$\mathfrak{L}(\sigma) \subseteq \mathfrak{L}(\xi) \text{ if and only if } \sigma \leq \xi,$$

$$\mathfrak{L}(\sigma) + \mathfrak{L}(\xi) = \mathfrak{L}(\sigma + \xi).$$

Proof Last two assertions follow from density assertion in Lemma 1.11, others are obvious.

Lemma 2.5. *Let $\sigma \in \mathbb{R}_a^n$. If $\ell \in \mathfrak{L}(\sigma) \setminus \{0\}$ then $|(V^T \ell)_1| \geq \prod_{j=2}^n \sigma_j^{-1}$. The minimal distance between the points in $\mathfrak{L}(\sigma)$ is $2^{1-n} \prod_{j=2}^n \sigma_j^{-1}$.*

Proof If $\ell \in \mathbb{Z}^n \setminus \{0\}$ then $\prod_{j=1}^n (V^T \ell)_j \in \mathbb{Z} \setminus \{0\}$ is nonzero and a symmetric polynomial in $\mathbb{Z}[\lambda_1, \dots, \lambda_n]$ so is a nonzero integer, implying the first assertion. The second assertion follows since Lemma 2.4 implies that the differences between points in $\mathfrak{L}(\sigma)$ are in $\mathfrak{L}(2\sigma)$.

The following result, proved by Minkowsky in 1896 [37], is the foundation of the geometry of numbers ([18], II.7.2, Theorem 1), ([34], Theorem 4.7)

Lemma 2.6. (Minkowski) *If $X \subset \mathbb{R}^n$ is convex, $X = -X$, and the volume of X exceeds 2^n then X contains a point in $\mathbb{Z}^n \setminus \{0\}$.*

Lemma 2.7. *If $\sigma \in \mathbb{R}_a^n$ and $L > |\det V| \prod_{j=2}^n \sigma_j^{-1}$ then $\mathfrak{L}(\sigma)$ contains a nonzero point in the interval $(-L, L)$.*

Proof Let $X = (V^T)^{-1} [(-L, L) \times (-\sigma_1, \sigma_1) \cdots (-\sigma_n, \sigma_n)]^T$. Then $X = -X$ and

$$\text{volume } X = 2^n |\det V|^{-1} L \prod_{j=2}^n \sigma_j > 2^n.$$

The result follows from Minkowski's Lemma 2.6 since if $\ell \in \mathbb{Z}^n$ then $(V^T \ell)_1 \in (-L, L) \cap \mathfrak{L}(\sigma)$ if and only if $\ell \in X$.

Lemma 2.8. *If $\sigma \in \mathbb{R}_a^n$ then there exists $M(\sigma) > 0$ such that every open interval of length $M(\sigma)$ contains a point in $\mathfrak{L}(\sigma)$.*

Proof Clearly matrix VV^T is invertible and it has integer entries since

$$(VV^T)_{i,j} = \sum_{\ell=1}^n \lambda_\ell^{i+j-2} = \text{trace } C^{i+j-2}, \quad i, j \in \{1, \dots, n\}. \quad (2.4)$$

Therefore $VV^T \mathbb{Z}^n$ is a rank n subgroup of $\mathbb{Z}^n$ so the quotient group $G = \mathbb{R}^n / (VV^T \mathbb{Z}^n)$ is isomorphic to the n -dimensional torus group $\mathbb{T}^n$. Let $q_n : \mathbb{R}^n \rightarrow G$ and $\varphi_n : G \rightarrow \mathbb{T}^n$ be the canonical epimorphisms and observe that

$$p_n((VV^T)^{-1}x) = \varphi_n \circ q_n(x), \quad x \in \mathbb{R}^n. \quad (2.5)$$

Since the entries of $v = [1, \lambda, \dots, \lambda^{n-1}]^T$ are linearly independent over $\mathbb{Q}$ the density assertion in Lemma 1.11 implies that

$$G = q_n \left((V [\mathbb{R} \times (-\sigma_1, \sigma_1) \times (-\sigma_n, \sigma_n)]^T) \right). \quad (2.6)$$

Since G is compact there exists $\kappa > 0$ such that

$$G = q_n \left((V [(-\kappa, \kappa) \times (-\sigma_1, \sigma_1) \times (-\sigma_n, \sigma_n)]^T) \right). \quad (2.7)$$

Therefore (2.5) implies that

$$\mathbb{T}^n = \varphi_n(G) = p_n \left((V^T)^{-1} [(-\kappa, \kappa) \times (-\sigma_1, \sigma_1) \times (-\sigma_n, \sigma_n)]^T \right) \quad (2.8)$$

and the result follows by choosing $M(\sigma) = 2\kappa$.

Remark 2.9. Lemma 2.5 shows that quasilattices are uniformly discrete, Lemma 2.8 shows that they are relatively dense. Sets having both of these properties are called Delone sets, named after Boris Nikolaevich Delone (aka Delaunay) [12].

An inflation symmetry of $\mathfrak{L}(\sigma)$ is a real number $\eta > 1$ such that $\eta \mathfrak{L}(\sigma) \subseteq \mathfrak{L}(\sigma)$.

Lemma 2.10. *If $|c_0| = 1$ then $\lambda \mathfrak{L}(\sigma) = \mathfrak{L}([0, |\lambda_2| \sigma_2, \dots, |\lambda_n| \sigma_n]^T)$. Therefore λ is an inflation symmetry of $\mathfrak{L}(\sigma)$.*

Proof Equation (2.3) gives $\lambda_j(V^T \ell)_j = (V^T C^T \ell)_j$, $j = 1, \dots, n$ and hence

$$\lambda \mathfrak{L}(\sigma) = \{ (V^T C^T \ell)_1 : \ell \in \mathbb{Z}^n \text{ and } |(V^T C^T \ell)_j| < |\lambda_j| \sigma_j \text{ for } j = 2, \dots, n \}.$$

Since $\det C = (-1)^n c_0 = \pm 1$, $C^T \mathbb{Z}^n = \mathbb{Z}^n$ and hence $\lambda \mathfrak{L}(\sigma) =$

$$\{ (V^T \ell)_1 : \ell \in \mathbb{Z}^n \text{ and } |(V^T \ell)_j| < |\lambda_j| \sigma_j \text{ for } j = 2, \dots, n \} = \mathfrak{L}([0, |\lambda_2| \sigma_2, \dots, |\lambda_n| \sigma_n]^T).$$

The second assertion then follows from Lemma 2.4 since $|\lambda_j| < 1$ for $j = 2, \dots, n$.

Points $x < y$ in $\mathfrak{L}(\sigma)$ are consecutive if $x \neq y$ and $(x, y) \cap \mathfrak{L}(\sigma) = \emptyset$. Define

$$\mathfrak{D}(\sigma) = \{ y - x : x < y \text{ and } x \text{ and } y \text{ are consecutive points in } \mathfrak{L}(\sigma) \}.$$

Lemma 2.5 implies that $\mathfrak{D}(\sigma)$ is bounded below by a positive number and Lemma 2.8 implies that $\mathfrak{D}(\sigma)$ is bounded above. The following result is stronger.

Lemma 2.11. *$\mathfrak{D}(\sigma)$ is finite.*

Proof Let $x < y$ be consecutive points in $\mathfrak{L}(\sigma)$. Lemma 2.4 implies that $y - x \in \mathfrak{L}(2\sigma)$ so Definition 2.2 implies that there exists $\ell \in \mathbb{Z}^n$ such that $y - x = (V^T \ell)_1$ and $|(V^T \ell)_j| < 2\sigma_j$ for $j > 1$. Lemma 2.8 implies that $|(V^T \ell)_1| = |y - x| < M(2\sigma)$. Therefore $V^T \ell$ and hence ℓ must belong to a bounded subset of $\mathbb{R}^n$ and therefore ℓ must belong to a finite subset of $\mathbb{Z}^n$ so $\mathfrak{D}(\sigma)$ is finite.

Lemma 2.12. *There exists a finite subset $F \subset \mathbb{R}$ such that $\mathfrak{L}(\sigma) + \mathfrak{L}(\sigma) \subseteq \mathfrak{L}(\sigma) + F$.*

Proof Let $x, y \in \mathfrak{L}(\sigma)$ and $z = x + y$. Lemma 2.8 implies there exists $w \in \mathfrak{L}(\sigma)$ such that $|z - w| < M(\sigma)$. Lemma 2.4 implies that z and w are in the Delone set $\mathfrak{L}(2\sigma)$ so there exists an integer m bounded by a function of σ and sequence of consecutive points $z_0 = w, z_1, \dots, z_m = z$ in $\mathfrak{L}(2\sigma)$. Lemma 2.11 implies that $\mathfrak{D}(2\sigma)$ is finite and hence the set of possible values for $z - w = (z_m - z_{m-1}) + \dots + (z_1 - z_0)$ lies in the finite m -fold sum set $F = \mathfrak{D}(2\sigma) + \dots + \mathfrak{D}(2\sigma)$.

A Salem number is an algebraic integer λ whose Galois conjugates $\lambda = \lambda_1, \dots, \lambda_n$ satisfy $|\lambda_j| \leq 1$ whenever $j > 1$ and $|\lambda| > 1$. Like PV numbers, Salem numbers appear in Diophantine approximation and harmonic analysis [35, 50, 51]. The smallest 1.17628 is the Mahler measure of Lehmer's polynomial $z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$ and is conjectured to be the smallest known Mahler measure > 1 of any polynomial.

Remark 2.13. Lemma 2.12 shows that $\mathfrak{L}(\sigma)$ is a Meyer set [36] and Lagarias [24] showed that every inflation symmetry of a Meyer set is a Salem number.

Let $\mathfrak{D}(\sigma) = \{c_1, \dots, c_m\}$ and $I_j = [0, c_j)$, $j = 1, \dots, m$ where without loss of generality we assume that $c_1 \in \mathfrak{L}(\sigma)$, and define $\mathfrak{I} = \{I_j : j = 1, \dots, m\}$. Then for every $j = 1, \dots, m$ there exists $T_j \subset \mathfrak{L}(\sigma)$ and a function $\Upsilon : T_j \rightarrow \mathfrak{I}$ such that $0 \in T_1$ and $\Upsilon(0) = I_1$ and

$$\lambda I_j = \bigcup_{\tau \in T_j} (\Upsilon(\tau) + \tau). \quad (2.9)$$

It follows that for every $k \in \mathbb{N}$, the interval $[0, \lambda^k c_1)$ can be expressed as a disjoint union of translates of intervals in the set $\mathfrak{I}$ by elements in $\mathfrak{L}(\sigma)$ and that every positive element in $\mathfrak{L}(\sigma)$ arises as a translate for some k . The negative elements arise similarly. This gives a way of constructing the quasilattice by a substitution dynamical system. Schlottmann [52] and Sirvent and Wang [54] discuss the relationship between quasicrystals, tilings and substitution dynamical systems.

3. Multiresolution Analyses

We refer to the reader to the textbook by Resnikoff and Wells [47] for background on multiresolution analyses and its relation to refinement equations (called scaling equations) and to multiwavelets. In this paper we only address the scaling functions and do not attempt to construct wavelets from them.

Theorem 3.1. *If $|c_0| = 1$ and $\sigma \in \mathbb{R}_a^n$ then $\xi = [0, (1 - |\lambda_2|)\sigma_2, \dots, (1 - |\lambda_n|)\sigma_n]^T \in \mathbb{R}_a^n$ and if f is a refinable distribution satisfying (1.1) and $\tau_j \in \mathfrak{L}(\xi)$ then the spaces*

$$W_k = \text{span} \{ f(\lambda^k x - \tau) : \tau \in \mathbb{L}(\sigma) \}, \quad k \in \mathbb{Z}, \quad (3.1)$$

satisfy $W_k \subset W_{k+1}$.

Proof If $\tau \in \mathbb{L}(\sigma)$ then (1.1) implies that

$$f(\lambda^k x - \tau) = \sum_{j=1}^m a_j f(\lambda^{k+1} x - \lambda\tau - \tau_j).$$

Since $|c_0| = 1$, Lemma 2.10 implies that

$$\lambda\tau \in \mathfrak{L}([0, |\lambda_2|\sigma_2, \dots, |\lambda_n|\sigma_n]^T),$$

and hence Lemma 2.4 implies that

$$\lambda\tau + \tau_j \in \mathfrak{L}([0, |\lambda_2|\sigma + \xi_2, \dots, |\lambda_n|\sigma_n + \xi_n]^T) = \mathfrak{L}(\sigma).$$

Theorem 3.1 provides a multiresolution analysis for spaces of distributions. We have been unable to construct the refinable distribution f that is an integrable function. In the next section we conjecture that such a construction is impossible.

We now construct a vector valued integrable function f that is refinable meaning that it satisfies (1.1) when the coefficients a_j are suitable matrices. f is called a vector valued scaling function in the multiwavelet discussion in ([47], p. 137–138). We choose the entries of f to be the characteristic functions χ_{I_j} , $j = 1, \dots, m$ of the intervals in $\mathfrak{I}$. Then (2.9) gives

$$\chi_{I_j}(x) = \sum_{\tau \in T_j} \chi_{\Upsilon(\tau)}(\lambda x - \tau), \quad j = 1, \dots, m \quad (3.2)$$

which determines the matrices a_j . Suitable translates of entries of f can be used to define the multiresolution spaces W_k so that W_k equals the set of functions that are constant on intervals of the form $[x, y)$ where x and y are consecutive points in $\lambda^{-k}\mathfrak{L}(\sigma)$. Refinable vectors with smoother entries are constructed from convolutions of entries in f .

4. Conjectures

The Hausdorff dimension of a Borel measure is defined in ([43], p. 6) as

$$\dim(\nu) = \inf\{\dim(B) : B \text{ is a Borel subset of } \mathbb{R} \text{ and } \nu(\mathbb{R} \setminus B) = 0\} \quad (4.1)$$

where $\dim(B)$ is the Hausdorff dimension of B .

Conjecture 4.1. If a refinable distribution f is a Borel measure then $\rho(f) = \dim(f)$.

If Conjecture 4.1 is validated we propose to call $\rho(f)$ defined in Theorem 1.5 the Hausdorff dimension of f . The following observations support this conjecture.

The characteristic function f of $[0, 1]$ satisfies the refinement equation $f(x) = f(2x) + f(2x - 1)$. Therefore $A(y) = (1 + e^{2\pi i y})/2$ so $\rho(f) = 1$ which equals the Hausdorff dimension of f . We observe that $|\hat{f}(y)| = |\sin y/y|$ decays like $|y|^{-1}$.

The uniform measure f on Cantor's ternary set satisfies the refinement equation $f(x) = \frac{3}{2}f(3x) + \frac{3}{2}f(3x - 2)$. Therefore $A(y) = (1 + e^{6\pi i y})/2$ so $\rho(f) = \ln 2 / \ln 3 \approx 0.6309$ which equals the Hausdorff dimension of f . This suggests that $\hat{f}(y)$ decays like $|y|^{-\ln 2 / \ln 3}$.

The measure f defined by a Bernoulli convolution and PV number λ satisfies the refinement equation $f(x) = \frac{|\lambda|}{2}f(\lambda x) + \frac{|\lambda|}{2}f(\lambda x - 1)$. Therefore $A(y) = (1 + e^{2\pi i \lambda y})/2$ so $\rho(f) = \ln 2 / \ln |\lambda|$ which equals the Hausdorff dimension of f computed by Peres, Schlag and Solomnyak [43]. This suggests that $\hat{f}(y)$ decays like $|y|^{-\ln 2 / \ln |\lambda|}$.

Let f be a nonzero distribution that satisfies the refinement equation $f(2x) = 2f(2x) + 2f(2x - 1) - 2f(2x - 3)$. Therefore $A(y) = 1 + e^{2\pi i y} - e^{4\pi i y}$ so $\rho(f) = -\ln((1 + \sqrt{5})/2) / \ln 2 \approx 0.6942$. This suggests that $|\hat{f}(y)|$ grows like $|y|^{-\rho(f)}$.

If λ is a PV number, $m = 2$, $a_1 = a_2 = |\lambda|/2$, $\tau_1 = 0$, and $\tau_2 = 1$ then f is a measure defined by a Bernoulli convolution and Erdős [16] used Lemma 2.1 to prove that there exists $\alpha > 0$ and $\gamma \in \mathbb{C} \setminus \{0\}$ such that

$$\lim_{k \rightarrow \infty} \hat{f}(\alpha \lambda^k) = \gamma. \quad (4.2)$$

This proves that f is not integrable since otherwise the Riemann-Lebesgue lemma would imply that the limit in (4.2) equals 0.

Kahane [22] extended Erdős' result for λ a Salem number and Dai, Feng and Wang [9] extended Kahane's result for $m > 2$ and $\tau_j \in \mathbb{Z}$.

Conjecture 4.2. If f is a nonzero refinable distribution with respect to dilation by a PV number λ and translations in $\mathbb{Z}[1, \lambda, \dots, \lambda^{n-1}]$ then there exists $\alpha \in \mathbb{Q}[\lambda] \setminus \{0\}$ such that

$$\lim_{k \rightarrow \infty} \hat{f}(\lambda^k \alpha) = \gamma$$

where $\gamma \in \mathbb{C} \setminus \{0\}$. This implies, by the Riemann-Lebesgue lemma, that f is not integrable thus extending the results in [9] to certain noninteger values of τ_j .

A proof of this conjecture eludes us but we present some observations that support it. Henceforth let λ be a PV number of degree n , let $C : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be the endomorphism induced by its Frobenius matrix defined in (2.1), let f be a refinable distribution that satisfies (1.1) with $\tau_j \in \mathbb{Z}[\lambda]$, let $p_n : \mathbb{R}^n \rightarrow \mathbb{T}^n$ be the canonical epimorphism, and let $P_{trig} : \mathbb{T}^d \rightarrow \mathbb{C}$ be the trigonometric polynomial that satisfies

$$A(y) = P_{trig}(p_n([y, \lambda y, \dots, \lambda^{n-1}y]^T)), \quad y \in \mathbb{R}. \quad (4.3)$$

Therefore (2.3) implies that

$$A(\lambda y) = P_{trig}(C(p_n([y, \lambda y, \dots, \lambda^{n-1}y]^T))), \quad y \in \mathbb{R}. \quad (4.4)$$

Define the real algebraic set $V(P_{trig}) = \{x \in \mathbb{T}^n : P_{trig}(x) = 0\}$.

Lemma 4.3. *If f is integrable and $\alpha \in \mathbb{Q}[\lambda] \setminus \{0\}$ then there exists an integer k such that $A(\lambda^k \alpha) = 0$. For $k \geq 0$ this is equivalent to*

$$C^k(p_n([\alpha, \lambda \alpha, \dots, \lambda^{n-1} \alpha]^T)) \in V(P_{trig}), \quad (4.5)$$

and for $k < 0$ this is equivalent to

$$p_n([\alpha, \lambda \alpha, \dots, \lambda^{n-1} \alpha]^T) \in C^{-k}(V(P_{trig})). \quad (4.6)$$

If $|\det C| = |c_0| = 1$ then $C : \mathbb{T}^n \rightarrow \mathbb{T}^n$ is invertible and (4.6) is equivalent to

$$C^k(p_n([\alpha, \lambda \alpha, \dots, \lambda^{n-1} \alpha]^T)) \in V(P_{trig}). \quad (4.7)$$

Proof Lemma 2.1 implies that $\lim_{j \rightarrow \infty} \prod_{m=j}^{\infty} A(\lambda^m \alpha) = 1$. Since f is integrable the Riemann–Lebesgue lemma implies that $\hat{f}(\lambda^j \alpha) \rightarrow 0$ so the first assertion follows from (1.3). The equivalences follow easily from (4.4) and the fact that $|\det C| = 1$ implies that the entries of C^{-1} are integers.

Remark 4.4. Lemma 4.3 shows that if f is integrable then the real algebraic set $V(P_{trig})$ contains a rich set of points which suggests that it is not a proper subset of $\mathbb{T}^n$ and hence that Conjecture 4.2 holds.

The following discussion shows that $V(P_{trig})$ may contain an even richer set of points than Lemma 4.3 implies. Let $A(\lambda)$ denote the $\mathbb{Q}$ –subspace of $\mathbb{C}^n$ consisting of vectors $[\alpha_1, \dots, \alpha_n]^T$ whose components are Galois conjugates and satisfy $\alpha_j \in \mathbb{Q}(\lambda_j)$, $j = 1, \dots, n$. Then the map $\Gamma_V : \mathbb{Q}^n \rightarrow A(\lambda)$ defined by $\Gamma_V(q) = V^T(VV^T)^{-1}q$, $q \in \mathbb{Q}^n$ is a $\mathbb{Q}$ –linear bijection of $\mathbb{Q}^n$ onto $A(\lambda)$ and

$$q = V\Gamma_V q. \quad (4.8)$$

Now assume that λ is a PV number and let v be the first column vector of V and α_1 be the first entry of $\Gamma_V q$. Then (2.3) and (2.4) give

$$\|C^k q - \alpha_1 \lambda^k v\| \rightarrow 0 \quad (4.9)$$

where the convergence is exponentially fast. Then (4.9) implies that

$$\|p_n(C^k q - \alpha \lambda^k v)\|_{\mathbb{T}^n} = \|C^k p_n(q) - p_n(\alpha \lambda^k v)\|_{\mathbb{T}^n} \rightarrow 0. \quad (4.10)$$

Since $p_n(\mathbb{Q}^n)$ is the set of preperiodic points of the endomorphism $C : \mathbb{T}^n \rightarrow \mathbb{T}^n$, if $\alpha \in \mathbb{Q}(\lambda)$ then the sequence $p_n(\alpha \lambda^k v) \in \mathbb{T}^n$ converges exponentially fast to a periodic orbit of C . Conversely every periodic orbit is the limit of such a sequence.

Lemma 4.5. *If λ is a PV number with degree n , $\alpha \in \mathbb{Q}(\lambda)$, $\{\tau_1, \dots, \tau_m\} \subset \mathbb{Z}[1, \lambda, \dots, \lambda^{n-1}]$, and $\widehat{f}(\alpha \lambda^k) \neq 0$ for all $k \in \mathbb{N}$ then the following limit exists*

$$\lim_{k \rightarrow \infty} \frac{\ln |\widehat{f}(\alpha \lambda^k)|}{k \ln |\lambda|} \quad (4.11)$$

and equals the mean value of $\ln |P_{trig}|$ over a finite orbit of C in the n -dimensional torus group.

Proof Follows from the same calculation used to prove Theorem 1.5.

Remark 4.6. Lemmas 4.3 and 4.5 ensure that $V(P_{trig})$ contains a rich set of points whenever f is integrable. We think that the techniques that we used to prove the Lagarias–Wang conjecture in [29], suitably modified, can be applied to prove Conjecture 4.2 by showing that there do not exist proper real algebraic subsets of $\mathbb{T}^n$ that contains these rich set of points. To this end we suggest studying the preperiodic points of C using results from integral matrices discussed by Newman [38], the open dynamical version of the Manin-Mumford Conjecture solved by Raynaud [45], [46], and connections between Mahler measure and torsion points developed by Le [31].

5. Multidimensional Extensions

Constructing quasilattices in dimension $d > 1$ is simple. Choose any irreducible monic integral polynomial $\Lambda(x)$ of degree $n > 2$ and constant term ± 1 having roots $\lambda_1, \dots, \lambda_d$ with modulus > 1 and whose remaining roots have modulus < 1 . The quasilattice $\mathfrak{L}(\sigma)$ consists of all vectors

$$[(V^T \ell)_1, \dots, (V^T \ell)_d]^T \in \mathbb{R}^d$$

where $\ell \in \mathbb{Z}^n$ are chosen to satisfy constraints of the form $|(V^T \ell)_j| < \sigma_j$ for $j = d+1, \dots, n$ where σ is an appropriate admissible vector with $\sigma_1 = \dots = \sigma_d = 0$ and the remaining entries positive. These multidimensional quasilattices have similar properties to the one dimensional quasilattices except that multiplication by λ is replaced by multiplication by the diagonal matrix $\text{diag}[\lambda_1, \dots, \lambda_d]$. The construction of refinable distributions is similar but deriving their analytic properties is more difficult. It is easy to show that the number of Voronoi cells for these quasilattices is finite and hence a finite number of dual Delaunay tiles, each a polytope whose vertices are in the quasilattice, can be constructed which tile $\mathbb{R}^d$. However, although the quasilattice satisfies an inflation related to the Frobenius matrix for Λ , in general the Delaunay tiling will not admit this inflation. One challenge is to determine possible symmetry conditions on C that will allow this.

Remark 5.1. We suggest research to extend the constructions in this short paper. Multiresolution analysis based on a single dilation can be replaced a more general nonstationary multiresolution analysis based on a sequence of dilations that gives spaces of hyperfunctions studied by Uchida [57]. Another extension is to stratified nilpotent groups with rational structure constants. We discussed in [28] that these are the only Lie groups that admit discrete uniform subgroups that are invariant under a dilation automorphism and we showed how to construct multiresolution analyses on these subgroups. This suggests that it may be possible to construct analogues of the quasicrystals on these Lie groups using a suitable modification of the algebraic number theory approach discussed in this paper. A far more ambitious extension

would be to construct quasilattices that give tilings by a single tile. Resnikoff [48, 49] recently constructed an algebraic realization of the Taylor–Socolar aperiodic monotilings. However, these mono tilings are not invariant under a dilation.

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