

p -ADIC WAVELETS

M. Skopina

Date of Receiving : 21. 02. 2015
 Date of Revision : 08. 05. 2015
 Date of Acceptance : 09. 05. 2015

Abstract. This paper is a short survey on wavelet and MRA theory for functions of p -adic argument.

1. Introduction

A variety of different orthogonal wavelet bases has been found for $L_2(\mathbb{R})$ for the last three decades. Such bases are not only interesting by themselves, a number of theoretical and applied problems were solved with using them. A general scheme for construction of wavelets of real argument was developed by Meyer and Mallat [19], [20] (see also, [7], [21]). This scheme is based on the concept of multiresolution analysis (MRA in the sequel). The theory allowed to construct wavelet bases and frames with desired properties, in particular, the Daubechies wavelets [7]

It appeared that similar constructions also exist for functions defined on some other algebraic structures, such as the Cantor and Vilenkin groups, local fields of positive characteristic (see, e.g., [17], [22], [10], [4], [5], [6]). Orthogonal wavelet bases essentially different from the Haar basis were found on these structures.

There were several attempts to develop MRA theory with natural translations in a more general setting, e.g., for locally compact groups or zero-dimensional groups (see [3], [18]). Analyzing these papers, one can see that actually nothing except for the Haar basis and its trivial modifications was constructed. p -Adic wavelet theory gives an explanation for these failures. From the wavelet theory point of view, the additive group of p -adic numbers looks very similar to the Cantor/Vilenkin groups. In both structures, Schwartz-Bruhat functions (i.e., band-limited and compactly supported ones) are the most convenient for MRA constructions. A variety of different orthogonal wavelet bases generated by Schwartz-Bruhat functions has been found for the Cantor/Vilenkin groups in [17], [22], [10]. However the situation is quite different in p -adics. Namely, any

2010 *Mathematics Subject Classification.* 42C40, 11E95.

Key words and phrases. p -adic field, orthogonal wavelet basis, wavelet frames, multiresolution analysis.

This research was supported by the RFBR-grant # 15-01-05796 and the SPbGU-grant # 9.38.198.2015.

Communicated by. S. Lukomskii

orthogonal wavelet basis consisting of band-limited functions is a modification of Haar basis. Note that all p -adic wavelet bases and even frames appeared in the literature consist of Schwartz-Bruhat functions.

The development of the p -adic wavelet theory is of both theoretical and applied interest. According to the well-known Ostrovsky theorem, *any nontrivial valuation on the field $\mathbb{Q}$ is equivalent either to the real valuation $|\cdot|$ or to one of the p -adic valuations $|\cdot|_p$* . Thus there are two equal in rights universes: the “real universe” and the “ p -adic one”, what involves us to study of p -adic analysis in different aspects. On the other hand, it appeared that each element of p -adic Haar basis is an eigenfunction of some p -adic pseudo-differential operators [16]. This property may be used to solve p -adic pseudo-differential equations. A. Khrennikov and V. Shelkovich [12] developed a method (an analog of the classical Fourier method) for finding solutions to the following Cauchy problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + D_x^\alpha u(x,t) &= f(x,t), \\ u(x,0) &= u^0(x). \end{cases}$$

where D_x^α is the fractional operator (introduced and studied by M.H. Taibleson [25]), t is a real variable and x is a p -adic variable. Such p -adic pseudo-differential equations are needed for some physical problems. In particular, the simplest p -adic pseudo-differential heat type equation

$$\frac{\partial u(x,t)}{\partial t} + D_x^\alpha u(x,t) = 0$$

arises in modeling interbasin kinetics of macromolecules (here t is time, while the p -adic parameter x describes the hierarchy of basins).

2. Notations and basic facts

In this section we give basic facts of p -adic analysis which can be found, e.g., in the monographs [23], [26] in more details.

As usual, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ we denote the ring of rational integers and the fields of rational, real and complex numbers, respectively.

The field $\mathbb{Q}_p$ of p -adic numbers is the completion of the field $\mathbb{Q}$ with respect to the p -adic norm $|\cdot|_p$ defined as follows:

$$|x|_p = \begin{cases} p^{-\gamma}, & \text{if } x = p^\gamma \frac{m}{n} \neq 0; \\ 0, & \text{if } x = 0, \end{cases}$$

where $\gamma \in \mathbb{Z}$ and m, n are integers not divisible by p . The extension of this norm to $\mathbb{Q}_p$ is also denoted by $|\cdot|_p$.

The norm $|\cdot|_p$ is non-Archimedean, i.e. satisfies the strong triangle inequality

$$|x + y|_p \leq \max(|x|_p, |y|_p), \quad x, y \in \mathbb{Q}_p.$$

Any p -adic number $x \neq 0$ can uniquely be written in the form

$$x = \sum_{j=\gamma}^{\infty} x_j p^j \tag{2.1}$$

where $\gamma \in \mathbb{Z}$ and $x_j \in \{0, 1, \dots, p-1\}$ with $x_\gamma \neq 0$. The fractional part $\{x\}_p$ of the number x equals by definition $\sum_{j=\gamma}^{-1} x_j p^j$. We also set $\{0\}_p = 0$. The ring of p -adic integers is

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p : \{x\}_p = 0\} = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

Denote by $B_N(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^{-N}\}$ the disc of radius p^N with the center at a point $a \in \mathbb{Q}_p$, $N \in \mathbb{Z}$. In particular, we have $B_0(0) = \mathbb{Z}_p$. Note that any two balls in $\mathbb{Q}_p$ either are disjoint or one of them contains the other.

Set

$$I_p := \{x \in \mathbb{Q}_p : \{x\}_p = x\} = \left\{ \frac{k}{p^n} \in \mathbb{Q}_p : k = 0, \dots, p^n - 1, n = 0, 1, \dots \right\}.$$

We observe that the translations of $\mathbb{Z}_p$ by the elements of I_p are mutually disjoint and the union of them equals $\mathbb{Q}_p$. Thus, I_p is an analog of the set of integers on the real line, and it is natural to consider I_p as the basic set of translations for wavelets. Of course I_p is not a unique set with such properties, any coset representatives of $\mathbb{Q}_p/\mathbb{Z}_p$ can be taken instead. The advantage of I_p is in its convexity. On the other hand, in contrast to the integers on the real line, I_p is not a group. This drawback cannot be fixed because $\mathbb{Q}_p$ does not contain discrete subgroups.

The additive character χ_p of the field $\mathbb{Q}_p$ is defined by

$$\chi_p(x) = e^{2\pi i \{x\}_p}, \quad x \in \mathbb{Q}_p.$$

The field $\mathbb{Q}_p$ is locally compact. Denote by dx the normalized Haar measure on it. By definition this measure is positive, invariant under translations, i.e., $d(x+a) = dx$ for all $a \in \mathbb{Q}_p$, and satisfies the condition $\int_{\mathbb{Z}_p} dx = 1$. Moreover,

$$d(ax) = |a|_p dx, \quad a \in \mathbb{Q}_p \setminus \{0\}.$$

The Hilbert space of all complex-valued functions on $\mathbb{Q}_p$ summing with square with respect to the measure dx , is denoted by $L_2(\mathbb{Q}_p)$.

Denote by $\mathcal{D}$ the linear space of locally-constant compactly supported functions defined on $\mathbb{Q}_p$ (so-called test functions). This space is a p -adic analog of the Schwartz space in the real analysis.

The Fourier transform of a function $f \in \mathcal{D}$ is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{Q}_p} \chi_p(\xi x) f(x) dx, \quad \xi \in \mathbb{Q}_p,$$

This yields a linear isomorphism taking $\mathcal{D}$ onto $\mathcal{D}$. It can uniquely be extended to a linear isomorphism of $L_2(\mathbb{Q}_p)$. Moreover, the Plancherel equality holds

$$\int_{\mathbb{Q}_p} f(x) \overline{g(x)} dx = \int_{\mathbb{Q}_p} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi, \quad f, g \in L^2(\mathbb{Q}_p).$$

A function $f \in L_2(\mathbb{Q}_p)$ is said to be band-limited if its Fourier transform $\widehat{f}$ is compactly supported. Given $m \in \mathbb{Z}$, the function $\widehat{f}$ is supported on $p^{-m}\mathbb{Z}_p$ if and only if f is p^m -periodic, i.e.

$$f(x + p^m) = f(x), \quad x \in \mathbb{Q}_p.$$

Thus the spaces of band-limited and periodic functions coincide. Moreover, $\mathcal{D}$ equals the space consisting of all both band-limited (periodic) and compactly supported functions in $L_2(\mathbb{Q}_p)$.

Below for $f \in L_2(\mathbb{Q}_p)$, $j \in \mathbb{Z}$ and $a \in \mathbb{Q}_p$, we set

$$f_{j,a}(x) = p^{j/2} f\left(\frac{x}{p^j} - a\right), \quad x \in \mathbb{Q}_p. \quad (2.2)$$

3. Multiresolution analysis

Definition 3.1. A collection of closed spaces $V_j \subset L_2(\mathbb{Q}_p)$, $j \in \mathbb{Z}$, is called a *multiresolution analysis (MRA)* in $L_2(\mathbb{Q}_p)$ if the following axioms hold

- (a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
- (b) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L_2(\mathbb{Q}_p)$;
- (c) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;
- (d) $f(\cdot) \in V_j \iff f(p^{-1}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
- (e) there exists a function $\varphi \in V_0$ such that

$$V_0 := \overline{\text{span}\{\varphi(x-a) : a \in I_p\}}.$$

The function φ from axiom (e) is called a *scaling function*, the spaces V_j are called sampling spaces. One also says that an MRA is generated by its scaling function φ (or φ generates the MRA). It follows immediately from axioms (d) and (e) that

$$V_j := \overline{\text{span}\{\varphi(p^{-j}x - a) : a \in I_p\}}, \quad j \in \mathbb{Z}. \quad (3.1)$$

An important class of MRAs consists of those generated by so-called *orthogonal scaling functions*. A scaling function φ is said to be orthogonal if $\{\varphi(x-a), a \in I_p\}$ is an orthonormal basis for V_0 . Consider such an MRA. Evidently, the functions $p^{j/2}\varphi(p^{-j}x - a)$, $a \in I_p$, form an orthonormal basis for V_j , $j \in \mathbb{Z}$. According to the standard scheme (see, e.g., [21, §1.3]) for the construction of MRA-based wavelets, for each j we define a space W_j (*wavelet space*) as the orthogonal complement of V_j in V_{j+1} , i.e., $V_{j+1} = V_j \oplus W_j$, $j \in \mathbb{Z}$, where $W_j \perp V_j$, $j \in \mathbb{Z}$. It is not difficult to see that

$$f(\cdot) \in W_j \iff f(p^{-1}\cdot) \in W_{j+1}, \quad \text{for all } j \in \mathbb{Z} \quad (3.2)$$

and $W_j \perp W_k$, $j \neq k$. Taking into account axioms (b) and (c), we obtain

$$\bigoplus_{j \in \mathbb{Z}} W_j = L_2(\mathbb{Q}_p) \quad (\text{orthogonal direct sum}). \quad (3.3)$$

If we now find functions $\psi^{(\nu)} \in W_0$, $\nu \in A$, such that the functions $\psi^{(\nu)}(x-a)$, $a \in I_p$, $\nu \in A$, form an orthonormal basis for W_0 , then, due to (3.2) and (3.3), the system $\{p^{j/2}\psi^{(\nu)}(p^{-j}x - a), a \in I_p, j \in \mathbb{Z}, \nu \in A\}$ is an orthonormal basis for $L_2(\mathbb{Q}_p)$. Such functions $\psi^{(\nu)}$ are called *wavelet functions* and the basis is a *wavelet basis*.

Theorem 3.2. [24] *Let φ be the characteristic function of the set $\mathbb{Z}_p$,*

$$V_j = \overline{\text{span}\{\varphi_{j,a} : a \in I_p\}}, \quad j \in \mathbb{Z}.$$

Then $\{V_j\}_{j \in \mathbb{Z}}$ is an MRA and φ is its orthogonal scaling function.

The MRA described in Theorem 3.2 is called the *Haar MRA*. The functions

$$\theta^{(\nu)}(x) = \varphi(x) \chi_p \left(\frac{\nu x}{p} \right), \quad \nu = 1, \dots, p-1, \quad (3.4)$$

are wavelet functions (see [24]), and the corresponding wavelet system

$$\{p^{j/2} \theta^{(\nu)}(p^{-j}x - a), a \in I_p, j \in \mathbb{Z}, \nu = 1, \dots, p-1\}$$

is an orthonormal basis for $L_2(\mathbb{Q}_p)$ (*Haar basis*). For the first time this basis was found by S. Kozyrev in [15] without using the MRA approach.

Next question is: how to construct another MRA?

Let φ be a scaling function for an MRA. It follows from axiom (a) that $V_0 \subset V_1$. In the real setting, the relation $V_0 \subset V_1$ holds if and only if φ is refinable, i.e. φ satisfies a refinement equations: $\varphi(x) = \sum_{n \in \mathbb{Z}} h_n \varphi(2x - n)$.

Let us look what happens in p -adics. If $\varphi \in L_2(\mathbb{Q}_p)$ is a scaling function, then φ is refinable, i.e. satisfies a p -adic refinement equations $\varphi(x) = \sum_{a \in I_p} \beta_a \varphi(p^{-1}x - a)$, moreover, if $\text{supp } \varphi \subset B_N(0)$, $N \geq 0$, then the refinement equation takes the form

$$\varphi(x) = \sum_{k=0}^{p^{N+1}-1} h_k \varphi \left(\frac{x}{p} - \frac{k}{p^{N+1}} \right).$$

However the converse is not true. Let $\varphi \in \mathcal{D}$ be refinable. For axiom (a), every $\varphi(x-b)$, $b \in I_p$, should be decomposed with respect to the system

$$\{p^{1/2} \varphi(p^{-1}x - a), a \in I_p\}.$$

Generally speaking, we cannot state that axiom (a) is fulfilled because I_p is not a group. For example, $a = \frac{3}{4} \in I_2$, $b = \frac{1}{2} \in I_2$, but $a + b \notin I_2$. The trouble is missing only for 1-periodic functions, i.e. for φ such that $\text{supp } \widehat{\varphi} \subset \mathbb{Z}_p$. The characteristic function of $\mathbb{Z}_p$ generating the Haar MRA just is such a function. An attempt to find new 1-periodic scaling function was successfully solved in the following statement.

Theorem 3.3. [13] Let $m_0(\xi) = \frac{1}{p} \sum_{k=0}^{p^{N+1}-1} \beta_k \chi_p(k\xi)$, $m_0(0) = 1$, $\widehat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{p^{N-j+1}}\right)$.

If $m_0\left(\frac{k}{p^{N+1}}\right) = 0$ for all $k = 1, \dots, p^{N+1} - 1$ not divisible by p , then $\text{supp } \widehat{\varphi} \subset \mathbb{Z}_p$. If, furthermore, $|m_0\left(\frac{k}{p^{N+1}}\right)| = 1$ for all $k = 1, \dots, p^{N+1} - 1$ divisible by p , then $\{\varphi(x-a) : a \in I_p\}$ is an orthonormal system and φ generates an MRA.

Thus, a variety of different orthogonal scaling functions was found. Unfortunately it appeared that these functions do not lead to new wavelet bases.

Theorem 3.4. [1] There exists a unique MRA generated by an orthogonal scaling function $\varphi \in \mathcal{D}$ such that $\text{supp } \widehat{\varphi} \subset \mathbb{Z}_p$. It is the Haar MRA.

Next question is whether a function φ with $\text{supp } \widehat{\varphi} \not\subset \mathbb{Z}_p$ can be an orthogonal scaling function. It was much more complicated to solve this problem. A negative answer was given in [1].

Theorem 3.5. [1] Let $\varphi \in \mathcal{D}$ be an orthogonal scaling function and $\widehat{\varphi}(0) \neq 0$. Then $\text{supp } \widehat{\varphi} \subset \mathbb{Z}_p$.

Corollary 3.6. [1] There exists a unique MRA generated by an orthogonal test scaling function. It is the Haar MRA.

4. Wavelet equivalence

So, it is not possible to construct orthogonal wavelets based on an MRA generated by a scaling function $\varphi \in \mathcal{D}$ with $\text{supp } \widehat{\varphi} \not\subset \mathbb{Z}_p$, and all scaling functions φ with $\text{supp } \widehat{\varphi} \subset \mathbb{Z}_p$ generate the Haar MRA. Hence it's worth to consider a wider class of orthogonal wavelet bases, not necessary MRA-based, and try to find a basis which is essentially different from the Haar basis. First of all we have to explain what means "essentially different from the Haar basis".

Definition 4.1. Let r be a positive integer and $\Psi = (\psi^{(1)}, \dots, \psi^{(r)})^T$, $\psi^{(\nu)} \in L_2(\mathbb{Q}_p)$, $\nu = 1, \dots, r$. The system of functions

$$\{\psi_{m,a}^{(\nu)} : m \in \mathbb{Z}, a \in I_p, \nu = 1, \dots, r\}$$

is called *wavelet system* generated by the vector-function Ψ (or by its components $\psi^{(\nu)}$). If this system is an orthonormal basis for $L_2(\mathbb{Q}_p)$, one says that Ψ generates an *orthonormal wavelet basis* (ONWB in the sequel).

Two vector-functions Ψ and Ψ' are said to be *unitary equivalent* if there exists a unitary matrix U such that $\Psi = U\Psi'$. Evidently, if Ψ generates an ONWB and Ψ' is unitary equivalent to Ψ , then Ψ' generates an ONWB.

The number of components of a vector-function Ψ is called its *rank* and denoted by $r = \text{rk}(\Psi)$.

Let φ be the characteristic function of the set $\mathbb{Z}_p$. As was said above, φ is an orthogonal scaling function for the Haar MRA. We call the corresponding wavelet spaces W_j by the *Haar wavelet spaces*.

Definition 4.2. A vector-function generating an ONWB is called *standard Haar* if it is of rank $p - 1$ and all its components are in the Haar wavelet space W_0 . The ONWB is called *standard Haar basis* in this case.

The vector-function $\Theta = (\theta^{(1)}, \dots, \theta^{(p-1)})^T$, where $\theta^{(\nu)}$ is defined by (3.4), is standard Haar, we call it by the *basic Haar vector-function*. The following statement gives a complete description of all compactly supported standard Haar vector-functions.

Theorem 4.3. [14] *A vector-function $\Psi = (\psi^{(1)}, \dots, \psi^{(p-1)})^T$, where $\psi^{(\nu)} \in L_2(\mathbb{Q}_p)$ is compactly supported for every $\nu = 1, \dots, p - 1$, is standard Haar if and only if*

$$\psi^{(\mu)}(x) = \sum_{\nu=1}^{p-1} \sum_{k=0}^{p^s-1} \alpha_{\nu;k}^{\mu} \theta^{(\nu)}\left(x - \frac{k}{p^s}\right), \quad \mu = 1, 2, \dots, p - 1, \quad (4.1)$$

where $s = 0, 1, 2, \dots$, and

$$\alpha_{\nu;k}^{\mu} = \begin{cases} -p^{-s} \sum_{m=0}^{p^s-1} e^{-2\pi i \frac{-\frac{\nu}{p}+m}{p^s} k} \sigma_{\mu m} z_{\mu\mu}, & \text{if } \mu = \nu, \\ p^{-2s} \sum_{m=0}^{p^s-1} \sum_{n=0}^{p^s-1} e^{-2\pi i \frac{-\frac{\nu}{p}+m}{p^s} k} \frac{1 - e^{\frac{2\pi i \frac{\mu-\nu}{p}}{p^s}}}{e^{\frac{2\pi i \frac{\nu-\nu}{p}+m-n}{p^s}} - 1} \sigma_{\nu m} z_{\nu\mu}, & \text{if } \mu \neq \nu, \end{cases} \quad (4.2)$$

$|\sigma_{\mu m}| = 1$, $z_{\mu\nu}$ are entries of a unitary $(p - 1) \times (p - 1)$ matrix Z .

Given a standard Haar vector-function, one can easily "damage" it to obtain another vector-function (which is non-standard) generating the same ONWB. Using such a

technique together with replacing some vector-functions by unitary equivalent ones, new orthogonal wavelet bases can be obtained. This is illustrated as follows.

Let $p = 2$, $\psi = \theta^{(1)}$. Then ψ can be treated as a standard Haar vector-function consisting of one function ψ . Set

$$\psi^{(1)}(x) = \sqrt{2}\psi(x/2), \quad \psi^{(2)}(x) = \sqrt{2}\psi((x-1)/2).$$

Evidently, the sets $\{\psi_{m,a}^{(\nu)} : m \in \mathbb{Z}, a \in I_p, \nu = 1, 2\}$ and $\{\psi_{m,a} : m \in \mathbb{Z}, a \in I_p\}$ coincide. So the wavelet system generated by $\psi^{(1)}, \psi^{(2)}$ is a standard Haar basis, but $(\psi^{(1)}, \psi^{(2)})^T$ is not a standard Haar vector-function. Similarly, the functions $\psi^{(1)}, \psi^{(2,1)}, \psi^{(2,2)}$, where

$$\psi^{(2,1)}(x) = \sqrt{2}\psi^{(2)}(x/2), \quad \psi^{(2,2)}(x) = \sqrt{2}\psi^{(2)}((x-1)/2),$$

generate the same standard Haar basis. Set

$$\tilde{\psi}^{(1)} = \frac{1}{\sqrt{2}}(\psi^{(1)} + \psi^{(2,1)}), \quad \tilde{\psi}^{(2)} = \frac{1}{\sqrt{2}}(\psi^{(1)} - \psi^{(2,1)}), \quad \tilde{\psi}^{(3)} = \psi^{(2,2)}.$$

These functions generate an ONWB which is not a standard Haar basis, but the vector-function $\tilde{\Psi} = (\tilde{\psi}^{(1)}, \tilde{\psi}^{(2)}, \tilde{\psi}^{(3)})^T$ is unitary equivalent to a vector-function generating a standard Haar basis. Next set

$$\tilde{\psi}^{(1,1)}(x) = \sqrt{2}\tilde{\psi}^{(1)}(x/2), \quad \tilde{\psi}^{(1,2)}(x) = \sqrt{2}\tilde{\psi}^{(1)}((x-1)/2).$$

It is not difficult to see that the vector-function $\tilde{\Psi}' = (\tilde{\psi}^{(1,1)}, \tilde{\psi}^{(1,2)}, \tilde{\psi}^{(2)}, \tilde{\psi}^{(3)})^T$ generates an ONWB, but $\tilde{\Psi}'$ does not generate a standard Haar basis, and $\tilde{\Psi}'$ is not unitary equivalent to a vector-function generating a standard Haar basis.

Definition 4.4. Two vector-functions Ψ, Ψ' are said to be *wavelet equivalent* if there exist vector-functions $\Psi_0, \dots, \Psi_N$ such that $\Psi_0 = \Psi$, $\Psi_N = \Psi'$, and for every $j > 0$ either Ψ_j is unitary equivalent to Ψ_{j-1} or Ψ_j and Ψ_{j-1} generate the same wavelet system (as a set).

Evidently, the property of vector-functions to be wavelet equivalent is preserved under changing the order of components in one of them. So, we can use the term "wavelet equivalent" for two sets of functions, or to say that a vector-function is wavelet equivalent to a set of functions (which can be considered as the components of another vector-function).

We note also that the wavelet equivalence relation is reflexive, symmetric and transitive. Moreover, if Ψ generates an ONWB and Ψ' is wavelet equivalent to Ψ , then Ψ' generates an ONWB.

It is easy to see that any standard Haar basis consists of periodic functions. So, if a vector-function is wavelet equivalent to a standard Haar vector-function, then all its components are periodic. It appears that the converse statement is also true.

Theorem 4.5. [8] *Any periodic vector-function Ψ generating an ONWB is wavelet equivalent to a standard Haar vector-function which can be taken eigen. If, moreover, Ψ is compactly supported, then it is wavelet equivalent to the basic Haar vector-function Θ .*

The following statement is a byproduct of Theorem 4.5.

Theorem 4.6. [8] *The rank r of a periodic vector-function generating an ONWB is divisible by $p-1$, in particular, $r \geq p-1$.*

It was illustrated above how to find ONWBs which are not standard Haar bases. Every vector-function constructed in this way is wavelet equivalent to a standard Haar one. However, really the situation is more specific. Namely, at each step, where the vector-functions Ψ_j and Ψ_{j-1} generate the same wavelet basis, in fact Ψ_j is obtained from Ψ_{j-1} by changing one of its components for two functions. To describe all bases that can be constructed in such a way for an arbitrary p , we say that Ψ' is *reducible* to Ψ if there exist vector-functions $\Psi_0, \dots, \Psi_N$ such that $\Psi_0 = \Psi$, $\Psi_N = \Psi'$, and for every $j > 0$, either Ψ_j is unitary equivalent to Ψ_{j-1} or Ψ_{j-1} is obtained from Ψ_j by changing one of its components, say $\psi_j^{(\nu)}(x)$, for p functions $\psi_j^{(\nu)}\left(\frac{x-k}{p}\right)$, $k = 0, \dots, p-1$.

It would be attractive to replace the words "wavelet equivalent" by "reducible" in Theorem 4.5. However this is impossible due to the following statement.

Theorem 4.7. [8] *There exists a vector-function generating an ONWB that cannot be reduced to a standard Haar vector-function.*

5. Riesz wavelet bases

As was shown above, a function $\varphi \in \mathcal{D}$ cannot be an orthogonal scaling function for an MRA different from the Haar MRA. A natural next question is whether a function $\varphi \in \mathcal{D}$ with $\text{supp } \varphi \not\subset \mathbb{Z}_p$ can be a non-orthogonal scaling function for an MRA. Recall that, in contrast to the real setting, for axiom (a) of Definition 4.2 it is not sufficient that φ be refinable because I_p is not a group. The following statement gives a complete description of functions $\varphi \in \mathcal{D}$ generating MRAs.

Theorem 5.1. [1] *A function $\varphi \in \mathcal{D}$, $\text{supp } \varphi \subset B_N(0)$, $\text{supp } \widehat{\varphi} \subset B_M(0)$, $N, M \geq 0$, $\widehat{\varphi}(0) \neq 0$, generates an MRA if and only if*

- (1) *φ is refinable;*
- (2) *there exist at least $p^{M+N} - p^N$ integers n such that $0 \leq n < p^{M+N}$ and $\widehat{\varphi}\left(\frac{n}{p^M}\right) = 0$.*

Next question is whether MRA-based non-orthogonal wavelets generated by a scaling function φ such that $\text{supp } \widehat{\varphi} \not\subset \mathbb{Z}_p$ can be constructed. A positive answer was given in [9] for $p = 2$ and in [2] for arbitrary p .

Since wavelet systems are numerated by double index, a natural order does not exist. Thus, construction of unconditional wavelet bases is of interest.

Definition 5.2. A system $\{f_n\}_{n=1}^\infty$ of elements of a Hilbert space H is called a *Riesz system with constants* $A, B > 0$ if the series $\sum_{n=1}^\infty c_n f_n$ converges in H for any $c = \{c_n\}_{n=1}^\infty \in \ell^2$ and

$$A\|c\|_{\ell^2}^2 \leq \left\| \sum_{n=1}^\infty c_n f_n \right\|_H^2 \leq B\|c\|_{\ell^2}^2. \quad (5.1)$$

If a Riesz system forms a basis, it is called a *Riesz basis*. It is well known that the unconditional bases and only they are Riesz bases in Hilbert spaces.

The following construction leads to MRA-based wavelet Riesz bases for $L_2(\mathbb{Q}_p)$.

For an integer $K \geq 0$, we set

$$A_K = \left\{ \frac{ap^j - b}{p^{j+1}}, \frac{p^{K+1} - b}{p^{K+1}} : j = 1, \dots, K, a, b = 1, \dots, p-1 \right\},$$

$$B_K = \left\{ 0, \frac{p^j - b}{p^j} : j = 1, \dots, K, b = 1, \dots, p-1 \right\}.$$

For instance, we have for $p = 2$:

$$A_0 = \left\{ \frac{1}{2} \right\}, B_0 = \{0\}; A_1 = \left\{ \frac{1}{4}, \frac{3}{4} \right\}, B_1 = \left\{ 0, \frac{1}{2} \right\}; A_2 = \left\{ \frac{1}{4}, \frac{3}{8}, \frac{7}{8} \right\}, B_2 = \left\{ 0, \frac{1}{2}, \frac{3}{4} \right\};$$

for $p = 3$:

$$A_0 = \left\{ \frac{1}{3}, \frac{2}{3} \right\}, B_0 = \{0\}; A_1 = \left\{ \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9} \right\}, B_1 = \left\{ 0, \frac{1}{3}, \frac{2}{3} \right\};$$

$$A_2 = \left\{ \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{27}, \frac{8}{27}, \frac{16}{27}, \frac{17}{27}, \frac{25}{27}, \frac{26}{27} \right\}, B_2 = \left\{ 0, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9} \right\}.$$

It is easily seen that

$$\sharp A_K = (p-1)\sharp B_K, \quad \sharp B_K = 1 + K(p-1) \quad (5.2)$$

for all K . Moreover, $A_K, B_K \subset I_p$ and

$$A_K \cap B_K = \emptyset, \quad (5.3)$$

$$A_K \cup B_K = \bigcup_{j=0}^{p-1} \frac{1}{p}(j + B_K). \quad (5.4)$$

Let us define trigonometric polynomials $m_0 = m_{0,K}$ and $n_0 = n_{0,K}$, respectively of degrees $(p-1)(1 + (p-1)K)$ and $1 + (p-1)K$, by

$$m_{0,K}(\xi) = \frac{1}{p} \prod_{r \in A_K} (\chi_p(\xi) - \chi_p(r)), \quad n_{0,K}(\xi) = \frac{1}{p} \prod_{r \in B_K} (\chi_p(\xi) - \chi_p(r)). \quad (5.5)$$

One can easily verify that $m_{0,K}(0) = 1$ for all $K \geq 0$.

Given integers $M \geq 0$ and $N \geq 0$, we define the Fourier transform of functions $\varphi = \varphi_{M,N}$ and $\psi = \psi_{M,N}$ by

$$\widehat{\varphi}_{M,N}(\xi) = \prod_{j=0}^{\infty} m_{0,K} \left(\frac{\xi}{p^{N-j}} \right), \quad \xi \in \mathbb{Q}_p, \quad (5.6)$$

$$\widehat{\psi}_{M,N}(\xi) = n_{0,K} \left(\frac{\xi}{p^N} \right) \widehat{\varphi}_{M,N}(p\xi), \quad \xi \in \mathbb{Q}_p, \quad (5.7)$$

where $K = M + N$.

Theorem 5.3. [2] *For integers $M, N \geq 0$, the function $\varphi_{M,N}$ generates an MRA if and only if $M \leq \frac{p^N - 1}{p-1} - N$. Moreover, in this case, the functions*

$$p^{-j/2} \psi_{M,N}^{(\nu)}(p^j x - a), \quad j \in \mathbb{Z}, a \in I_p, \nu = 1, \dots, p-1,$$

form a Riesz basis for $L_2(\mathbb{Q}_p)$ if and only if $M = \frac{p^N - 1}{p-1} - N$.

References

- [1] S. Albeverio, S. Evdokimov and M. Skopina, p -adic multiresolution analysis and wavelet frames, *J. Fourier Anal. Appl.*, Vol. 16, Number 5, (2010), 693–714.
- [2] S. Albeverio, S. Evdokimov, and M. Skopina, p -Adic Nonorthogonal Wavelet Bases, *Proceedings of the Steklov Institute of Mathematics*, 265 (2009), 1–12.
- [3] J.J. Benedetto, and R.L. Benedetto, A wavelet theory for local fields and related groups, *The Journal of Geometric Analysis* **3** (2004) 423–456.
- [4] Behera, B., Jahan, Q., Multiresolution analysis on local fields and characterization of scaling functions. *Adv. Pure Appl. Math.* 3 (2012), no. 2, 181–202.
- [5] Behera, B., Jahan, Q., Wavelet packets and wavelet frame packets on local fields of positive characteristic. *J. Math. Anal. Appl.* 395 (2012), no. 1, 1–14.
- [6] Golubov, B. I., A certain class of complete orthogonal systems. (Russian) *Sibirsk. Mat. Zh.* 9 (1968), 297–314.
- [7] Daubechies I. *Ten Lectures on wavelets*, CBMS-NSR Series in Appl. Math., SIAM, 1992.
- [8] S. Evdokimov and M. Skopina, On orthogonal p -adic wavelet bases, *J. Math. Anal. Appl.*, 424 (2015), N 2, 952–965.
- [9] S.A. Evdokimov and M.A. Skopina, 2-Adic wavelet bases, *Proceedings of Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences*, 15 (2009), N 1, 135–146 (Russian).
- [10] Farkov, Yu. A., Orthogonal wavelets on direct products of cyclic groups. *Mat. Zametki* 82 (2007), no. 6, 934–952 (Russian); translation in *Math. Notes* 82 (2007), no. 5-6, 843–859
- [11] Khrennikov, A. Yu. and Shelkovich V. M. (2009), Non-Haar p -adic wavelets and their application to pseudo-differential operators and equations, *Appl. Comput. Harmon. Anal.* , **28** (1), 1–23.
- [12] Khrennikov, A. Yu. Shelkovich, V. M. (2009), An infinite family of p -adic non-Haar wavelet bases and pseudo-differential operators, *P-Adic Numbers, Ultrametric Analysis, and Applications*, **1** (3), 204–216.
- [13] A.Yu. Khrennikov, V.M. Shelkovich, M. Skopina, p -Adic refinable functions and MRA-based wavelets. *J. Approx. Theory* 161 (2009) 226–238
- [14] A.Yu. Khrennikov, V.M. Shelkovich, M. Skopina, p -Adic Orthogonal Wavelet Bases, p -Adic Numbers, Ultrametric Analysis and Applications, 1 (2009), No 2, 145–156.
- [15] S.V. Kozyrev, Wavelet analysis as a p -adic spectral analysis, *Izvestia Akademii Nauk, Seria Math.* **66** no. 2 (2002) 149–158.
- [16] Kozyrev, S.V. (2004), p -Adic pseudodifferential operators and p -adic wavelets, *Theor. Math. Physics*, **138** (3), 1–42.
- [17] Lang, W. C., Orthogonal wavelets on the Cantor dyadic group. *SIAM J. Math. Anal.* 27 (1996), no. 1, 305–312.
- [18] Lukomskii S.F., Multiresolution analysis on product of zero-dimensional groups, *J. Math. Anal. Appl.*, 385 (2012), n. 2, 1162–1178.
- [19] Mallat, S. (1988). Multiresolution representation and wavelets, Ph. D. Thesis, University of Pennsylvania, Philadelphia, PA.
- [20] Meyer, Y. (Décembre 1986). Ondelettes et fonctions splines, Séminaire EDP. Paris.
- [21] Novikov, I. Ya., Protasov, V. Yu., Skopina, M. A., Wavelet Theory. AMS, Translations Mathematical Monographs, V. 239 (2011).
- [22] Protasov, V. Yu.; Farkov, Yu. A. Dyadic wavelets and scaling functions on a half-line. *Mat. Sb.* 197 (2006), no. 10, 129–160 (Russian); translation in *Sb. Math.* 197 (2006), no. 9-10, 1529–1558
- [23] J.-P. Serre, Linear representations of finite groups. Graduate Texts in Mathematics, vol. 42, Springer-Verlag, New York, Heidelberg, Berlin, 1977

- [24] V. M. Shelkovich and M. Skopina, p -Adic Haar multiresolution analysis and pseudo-differential operators, *J. Fourier Analysis and Appl.*, 15 (2009), N 3, 366–393.
- [25] Taibleson, M.H. (1975), *Fourier analysis on local fields*. Princeton University Press, Princeton.
- [26] V.S. Vladimirov, I.V. Volovich and E.I. Zelenov, p -Adic analysis and mathematical physics. World Scientific, Singapore, 1994.

(M. Skopina) ST.PETERSBURG STATE UNIVERSITY, UNIVERSITETSKII PR.-35, 198504
ST.PETERSBURG, RUSSIA

E-mail address: **skopina@MS1167.spb.edu**