

O-FRAMES FOR OPERATORS IN BANACH SPACES

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Abstract. A new notion of O-frame for an operator $T : X \rightarrow Y$ in Banach spaces is introduced: we say that a system $\mathcal{F} := ((x'_k)_{k=1}^\infty, (y_k)_{k=1}^\infty)$ is O-frame (operator frame) for T , if for every $x \in X$ the series $\sum_{k=1}^\infty \langle x'_k, x \rangle y_k$ converges in Y and

$$Tx = \sum_{k=1}^\infty \langle x'_k, x \rangle y_k, \quad x \in X.$$

This is an evident, but, as we hope, interesting generalization of the well known notion of a Banach (Schauder) frame. We used the classical tools as well as (and mainly - also now classical) the technique, developed by Aleksander Pełczyński, to get first general properties of O-frames.

We present a small survey on the topic. Let us mention only that, in particular, as in the case of the identity operators, it can be proved that

- (i) an operator has an operator frame if and only if
- (ii) it factors through a Banach space with a basis, and, when it is defined on a separable Banach space, iff
- (iii) it has the bounded approximation property.

Also, we consider a new notion of an unconditional operator frame and present some similar results.

1. O-frames

We use the standard terminology and notations of the theory of operators in Banach spaces (see, e.g., [6]). All definition, results and details, concerning the notions of frames in Hilbert and Banach spaces, can be found in [1], [2], [5]. Recall only a definition of a Schauder frame for a Banach space (cf. [2] or [5]): a system $(x'_k)_{k=1}^\infty \subset X^*, (x_k)_{k=1}^\infty \subset X$ is said to be a Schauder frame for X , if for every $x \in X$ the series $\sum_{k=1}^\infty \langle x'_k, x \rangle x_k$ converges in X to x . An evident generalization is given by the following

Definition 1.1. Let $T \in L(X, W)$, $(x'_k)_{k=1}^\infty \subset X^*, (w_k)_{k=1}^\infty \subset W$. We say that $\mathcal{F} := ((x'_k)_{k=1}^\infty, (w_k)_{k=1}^\infty)$ is an O-frame (operator frame) for T , if for every $x \in X$

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the series $\sum_{k=1}^{\infty} \langle x'_k, x \rangle w_k$ converges in W and

$$Tx = \sum_{k=1}^{\infty} \langle x'_k, x \rangle w_k, \quad x \in X.$$

If there exists an O-frame for T , then we say that T has an O-frame.

Example 1.2. Let $\Delta : l_{\infty} \rightarrow l_1$ be a diagonal operator with a diagonal $(\delta_k) \in l_1$. Then $\Delta x = \sum \langle e_k, x \rangle e_k$, where (e_k) is the sequence of the unit vectors in l_p (in our case, in l_1), $x = (x_k) \in l_{\infty}$.

Example 1.3. If X has a Schauder basis $(f_k)_{k=1}^{\infty}$, then, for every $x \in X$, we have: $x = \sum_{k=1}^{\infty} \langle f'_k, x \rangle f_k$, where (f'_k) is a biorthogonal system for (f_k) . Hence, for every Banach space W and for each operator $T : X \rightarrow W$

$$Tx = \sum_{k=1}^{\infty} \langle f'_k, x \rangle T f_k, \quad x \in X.$$

Example 1.4. If W has a Schauder basis, for instance, (w_k) with a biorthogonal system (w'_k) , then every element $w \in W$ has an expansion $w = \sum_{k=1}^{\infty} \langle w'_k, w \rangle w_k$ and we get for $x \in X$ $\hat{=} T : X \rightarrow W$: $\langle Tx, w'_k \rangle = \langle x, T^* w'_k \rangle$; therefore, $Tx = \sum_{k=1}^{\infty} \langle T^* w'_k, x \rangle w_k$. Let us mention that in this example the space W is separable, but the space X need not be separable.

Example 1.5. If the space X is separable and has the bounded approximation property, then it can be imbedded, as a complemented subspace, in a space X_0 with a basis [7]. Let $P : X_0 \rightarrow X$ be a continuous projector and $((x'_k), (x_k))$ be a basis for X_0 (i.e., the corresponding biorthogonal system). For $x_0 \in X_0$, we have an expansion $x_0 = \sum_{k=1}^{\infty} \langle x'_k, x_0 \rangle x_k$ and if $x \in X$ $\hat{=} x = Px_0$, then $x = \sum_{k=1}^{\infty} \langle x'_k, x_0 \rangle Px_k$. But $x \in X \subset X_0$; therefore, $x = Px$ and

$$x = \sum_{k=1}^{\infty} \langle x'_k, x \rangle x_k = P(x) = \sum_{k=1}^{\infty} \langle x'_k, x \rangle Px_k.$$

Now, any operator $T \in L(X, W)$ has an O-frame with an expansion of the kind

$$Tx = \sum_{k=1}^{\infty} \langle x'_k|_X, x \rangle TPx_k.$$

An analogous example can be considered for the case where W is separable and has the bounded approximation property.

Lemma 1.6. Let X, V, Z, W be Banach spaces, $T \in L(X, W)$, $A \in L(W, V)$, $B \in L(Z, X)$. If an operator T has an O-frame, Then the operator $ATB : Z \rightarrow V$ has an O-frame too.

Proof. Let $((x'_k), (w_k))$ be an O-frame for T . Fix $z \in Z$, and let $x = Bz$. Then $Tx = \sum \langle x'_k, x \rangle w_k$, $ATx = \sum \langle x'_k, x \rangle Aw_k$ and $ATBz = ATx = \sum \langle x'_k, Bz \rangle Aw_k = \sum \langle B^* x'_k, z \rangle Aw_k$. Thus, $((B^* x'_k), (Aw_k))$ is an O-frame for the operator ATB . $\square$

Corollary 1.7. If an operator $T \in L(X, W)$ can be factored through a Banach space with a Schauder basis, then it has an O-frame.

Proof. Apply Lemma 1.1 and Examples 2-3. $\square$

Corollary 1.8. *If an operator $T \in L(X, W)$ can be factored through a Banach space with the bounded approximation property, then it has an O-frame.*

One more property of O-frames:

Proposition 1.9. *Let $\mathcal{F} := ((x'_k), (w_k))$ be an O-frame for $T \in L(X, W)$. Then the dual system $\mathcal{F}^d := ((w_k), (x'_k))$ is a $*$ -weak O-frame for T^* , i.e.,*

$$T^*w' = w^* - \lim_N \sum_{k=1}^N \langle w', w_k \rangle x'_k, \quad w' \in W^*.$$

Proof. For $w' \in W^*$ and $x \in X$,

$$\langle Tx, w' \rangle = \left\langle \sum_{k=1}^{\infty} \langle x'_k, x \rangle w_k, w' \right\rangle = \left\langle \sum_{k=1}^{\infty} \langle w', w_k \rangle x'_k, x \right\rangle;$$

hence, $T^*w' = w^* - \lim_N \sum_{k=1}^N x'_k \langle w', w_k \rangle$ (the limit in the topology $\sigma(X^*, X)$). $\square$

We have considered above only the simplest properties of O-frames. More serious results are to appear below.

In the theory of bases and frames, we can find many theorems in which such terms as "shrinking", "boundedly complete" appear (see., for example, [1], [5], [6]). We are introducing the corresponding notions for the operator frames.

Definition 1.10. An O-frame $((x'_k), (w_k))$ for T is said to be boundedly complete if it has the following property: For every $x'' \in X^{**}$, it follows from the boundedness of the sequence of partial sums $(\sum_{k=1}^N \langle x'', x'_k \rangle w_k)_{N=1}^{\infty}$ that the series $\sum_{k=1}^{\infty} \langle x'', x'_k \rangle w_k$ converges in the space W .

Definition 1.11. An O-frame $((x'_k), (w_k))$ for T is called shrinking, if for every $w' \in W^*$ the norm $\|\sum_{k=n+1}^{\infty} x'_k \langle w', w_k \rangle\| \rightarrow 0$ as $n \rightarrow \infty$.

Next proposition gives us conditions for the dual $*$ -weak frames to be O-frames.

Proposition 1.12. *Let $\mathcal{F} := ((x'_k), (w_k))$ be an O-frame for $T \in L(X, W)$. The dual system $\mathcal{F}^d := ((w_k), (x'_k))$ is an O-frame for T^* if and only if the O-frame $\mathcal{F}$ is shrinking.*

Proof. If an O-frame $\mathcal{F}$ is shrinking, then, firstly (Proposition 1.1), for $w' \in W^*$

$$T^*w' = w^* - \lim_N \sum_{k=1}^N \langle w', w_k \rangle x'_k$$

and, secondly, ,

$$\left\| \sum_{k=n+1}^{\infty} x'_k \langle w', w_k \rangle \right\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Therefore, the series $\sum_{k=1}^{\infty} \langle w', w_k \rangle x'_k$ converges in X^* , evidently, to T^*w' .

Conversely, if the series $\sum_{k=1}^{\infty} \langle w', w_k \rangle x'_k$ converges to $\hat{x}_0 \in X^*$, then, by the same Proposition 1.1, it is convergent to T^*w' . Hence, the remainder of this series tends to zero in X^* , that is, the O-frame $\mathcal{F}$ is shrinking. $\square$

Proposition 1.13. *Let $\mathcal{F} := ((x'_k), (w_k))$ be an O-frame for $T \in L(X, W)$. The following assertions are equivalent:*

- 1) *O-frame $\mathcal{F}$ is boundedly complete;*
- 2) *for each $x'' \in X^{**}$ the series $\sum_{k=1}^{\infty} \langle x'', x'_k \rangle w_k$ converges in the space W .*

Proof. The implication 2) $\Rightarrow$ 1) is evident.

1) $\Rightarrow$ 2). Let $x'' \in X^{**}$. It is to be shown that the series $\sum_{k=1}^{\infty} \langle x'', x'_k \rangle w_k$ converges in W .

For any $w' \in W^*$ and $x \in X$, we have:

$$\lim_{N \rightarrow \infty} \left\langle \sum_{k=1}^N \langle w', w_k \rangle x'_k, x \right\rangle = \lim_{N \rightarrow \infty} \langle w', \sum_{k=1}^N w_k \langle x'_k, x \rangle \rangle = \langle w', Tx \rangle.$$

In particular, the sequence $(\sum_{k=1}^N \langle w', w_k \rangle x'_k)_{N=1}^{\infty}$ is bounded in X^* . Therefore, for $w' \in W^*$ and $x'' \in X^{**}$, the scalar sequence

$$\left(\sum_{k=1}^N \langle w', w_k \rangle \langle x'', x'_k \rangle \right)_{N=1}^{\infty}$$

is bounded, i.e., the sequence $(\sum_{k=1}^N w_k \langle x'', x'_k \rangle)_{N=1}^{\infty}$ is $\sigma(W, W^*)$ -bounded. Hence, it is strongly bounded in W . By condition 1), the series $\sum_{k=1}^{\infty} \langle x'', x'_k \rangle w_k$ converges in the space W . $\square$

Applying both propositions, we get one of the main facts of the section:

Theorem 1.14. *Let $\mathcal{F} := ((x'_k), (w_k))$ be an O-frame for $T \in L(X, W)$. If this O-frame $\mathcal{F}$ is boundedly complete and shrinking, then the operator T is weakly compact.*

Proof. Since $\mathcal{F}$ is shrinking, then by Proposition 1.2, $\mathcal{F}^d$ is an O-frame for T^* , i.e., for every $w' \in W^*$

$$T^* w' = \sum_{k=1}^{\infty} \langle w', w_k \rangle x'_k.$$

Therefore, for $x'' \in X^{**}$

$$\langle T^* w', x'' \rangle = \sum_{k=1}^{\infty} \langle w', w_k \rangle \langle x'', x'_k \rangle = \lim_N \langle w', \sum_{k=1}^N \langle x'', x'_k \rangle w_k \rangle,$$

or

$$\langle w', T^{**} x'' \rangle = \lim_N \langle w', \sum_{k=1}^N \langle x'', x'_k \rangle w_k \rangle.$$

Since $\mathcal{F}$ is a boundedly complete O-frame, then the series $\sum_{k=1}^{\infty} \langle x'', x'_k \rangle w_k$ converges in W , e.g., to an element $w_0 \in W$. Therefore,

$$\langle w', T^{**} x'' \rangle = \langle w', \sum_{k=1}^{\infty} \langle x'', x'_k \rangle w_k \rangle. \quad w' \in W^*.$$

Or: $\langle T^{**} x'', w' \rangle = \langle w', w_0 \rangle$ for every $w' \in W^*$. Thus, $T^{**} x'' = w_0$ and, in particular, $T^{**} x'' \in W$, that is, if $x'' \in X^{**}$, then $T^{**} x'' \in W$ and, consequently, the operator T is weakly compact. $\square$

Next theorem is the first of the theorems, which connect the introduced here notion of O-frame and the notion of the BAP (bounded approximation property) for operators (see below for some details on BAP). The idea of the proof is well known and can be found, for example, in [14] or [6]; see also [5] and [2]

Theorem 1.15. *Let $T \in L(X, W)$. The following assertions are equivalent:*

- 1) *T has an O-frame;*
- 2) *the operator T factors through a Banach space with a Schauder basis;*
- 3) *the operator T factors through a Banach sequence space with a Schauder basis.*

Proof. At first, we show that 1) implies 2). Let the operator T have an O-frame $\mathcal{F} := ((x'_k), (w_k))$. We may (and do) assume that $w_k \neq 0$ for every k . Since the series $\sum_{k=1}^{\infty} \langle x'_k, x \rangle w_k$ converges (to Tx) for every $x \in X$, the norms of operators $\sum_{k=1}^N x'_k \otimes w_k$ (uniformly) bounded (e.g., by a constant $K > 0$).

Consider the following space of scalar sequences:

$$V := \{a = (a_k)_{k=1}^{\infty} : \text{the series } \sum_{k=1}^{\infty} a_k w_k \text{ converges in } W\},$$

and let e_k be the k 's unit vector in V (i.e., $(e_k)_s = 0$ for $k \neq s$ and $(e_k)_k = 1$).

Put

$$|||a|||_V := \sup_N \left\| \sum_{k=1}^N a_k w_k \right\| \quad (\geq \lim_N \left\| \sum_{k=1}^N a_k w_k \right\|).$$

Since for a finite sequence $a = (a_1, a_2, \dots, a_{N+s}, 0, 0, \dots)$

$$||| \sum_{k=1}^N a_k e_k ||| \leq ||| \sum_{k=1}^{N+s} a_k e_k |||$$

and the linear span of the vectors $(e_k)_{k=1}^{\infty}$ is dense in V , it follows that (e_k) is a monotone basis (cf. [6], [14, Prop. 3.1]) in the Banach space V . Moreover, if $j : V \rightarrow W$ is a natural mapping $a \mapsto \sum_{k=1}^{\infty} a_k w_k$, then $\|j\| \leq 1$ (by definition of the norm $|||\cdot|||$ in V). Set $Ax := (\langle x'_k, x \rangle)_{k=1}^{\infty}$; since the series $\sum_{k=1}^{\infty} \langle x'_k, x \rangle w_k$ is convergent, $Ax \in V$. Furthermore,

$$|||Ax|||_V = \sup_N \left\| \sum_{k=1}^N \langle x'_k, x \rangle w_k \right\| \leq K \|x\|, \quad \forall x \in X.$$

Thus, the linear mapping A is continuous from X in V and $T = jA : X \rightarrow V \rightarrow W$.

That 3) implies 2) is evident, and the assertion 1) follows from the assertion 2) (by Corollary 1.1). $\square$

Equally with the notion of an O-frame, we can consider (and consider shortly) a notion of an unconditional O-frame (by changing a little the definition).

Definition 1.16. Let $T \in L(X, W)$, $(x'_k)_{k=1}^{\infty} \subset X^*$, $(w_k)_{k=1}^{\infty} \subset W$. We say that $\mathcal{F} := ((x'_k)_{k=1}^{\infty}, (w_k)_{k=1}^{\infty})$ is an UO-frame for T , if for every $x \in X$ the series $\sum_{k=1}^{\infty} \langle x'_k, x \rangle w_k$ converges unconditionally in W and

$$Tx = \sum_{k=1}^{\infty} \langle x'_k, x \rangle w_k, \quad x \in X.$$

If there exists an UO-frame for T , we say that T has an UO-frame.

The following theorem is just an analogue of Theorem 1.2 with almost the same standard proof (cf. [2, Proposition 2.4] and [5]).

Theorem 1.17. *Let $T \in L(X, W)$. The following assertions are equivalent:*

- 1) T has an UO-frame;
- 2) the operator T can be factored through a Banach space with an unconditional basis;
- 3) the operator T can be factored through a Banach sequence space with an unconditional basis.

Proof. At first, we show that 1) implies 3). Let the operator T has an UO-frame $\mathcal{F} := ((x'_k), (w_k))$. We may (and do) assume that $w_k \neq 0$ for every k . Since the series $\sum_{k=1}^{\infty} \langle x'_k, x \rangle w_k$ is unconditionally convergent (to Tx) for each $x \in X$, the family $\{T_b : b \in l_{\infty}; \|b\| \leq 1\}$ of operators, which are defined by $T_b := \sum_{k=1}^{\infty} b_k x'_k \otimes w_k$, is uniformly bounded, e.g., $\|T_b\| \leq K$ if $\|b\| \leq 1$.

Consider the following space of scalar sequences:

$$U := \{a = (a_k)_{k=1}^{\infty} : \text{the series } \sum_{k=1}^{\infty} a_k w_k \text{ converges unconditionally in } W\},$$

and let e_k be the k 's unit vector in U (i.e., $(e_k)_s = 0$ for $k \neq s$ and $(e_k)_k = 1$).

Put

$$\|a\|_U := \sup_{\|(b_k)\|_{l_{\infty}} \leq 1} \left\| \sum_{k=1}^{\infty} b_k a_k w_k \right\|.$$

Then (e_k) is an unconditional basis in the Banach space U . Moreover, if $j : U \rightarrow W$ is the natural mapping $a \mapsto \sum_{k=1}^{\infty} a_k w_k$, then $\|j\| \leq 1$ (by definition of the norm $\|\cdot\|$ in U). Set $Ax := (\langle x'_k, x \rangle)_{k=1}^{\infty}$; since the series $\sum_{k=1}^{\infty} \langle x'_k, x \rangle w_k$ is unconditionally convergent, $Ax \in U$. Furthermore,

$$\|Ax\|_U = \sup_{\|(b_k)\|_{l_{\infty}} \leq 1} \left\| \sum_{k=1}^{\infty} b_k \langle x'_k, x \rangle w_k \right\| \leq K \|x\| \quad \forall x \in X.$$

Thus, the linear mapping A is continuous from X to U , and $T = jA : X \rightarrow U \rightarrow W$.

That 3) implies 2) and 2) implies 1) is evident. $\square$

Remark 1.18. Let $\mathcal{F} := ((x'_k)_{k=1}^{\infty}, (w_k)_{k=1}^{\infty})$ be an UO-frame for $T, T \in L(X, W)$. Then, for all $x \in X$ and $w' \in W$, the series $\sum \langle x'_k, x \rangle \langle w', w_k \rangle$ converges unconditionally, or (what is the same) absolutely. Since, for a constant $K > 0$, one has $\sup\{\|\sum_{k=1}^{\infty} b_k x'_k \otimes w_k\| : \|b\|_{l_{\infty}} \leq 1\} \leq K$, we get that

$$\sup_{\|x\| \leq 1, \|w'\| \leq 1} \sum_{k=1}^{\infty} |\langle x'_k, x \rangle \langle w', w_k \rangle| \leq K < \infty.$$

Therefore, the operator T is σ -nuclear in the sense of [8; 23.2]. Evidently, converse is also true; so, the class of operators with UO-frames is exactly the class of all σ -nuclear operator and we have shown in Theorem 1.3 that every σ -nuclear operator can be factored through a Banach space with an unconditional basis. But, as was noted in [8, 23.2.3], every Banach space with an unconditional basis can be renormed such that the basis becomes hyperorthogonal (see again Section 22.2.3 in [8] for a definition). Hence, we present here a different proof of Theorem 23.2.5 from [8].

Next our aim is to connect the notion of an O-frame with the bounded approximation property (as it was done, e.g., in the paper [2])

2. On operators with the bounded approximation property

Definition 2.1. Let $T \in L(X, W)$, $C \geq 1$. We say that T has the property C-BAP (C-bounded approximation property), if for every compact subset K of X , for each $\varepsilon > 0$ there exists a finite rank operator $R : X \rightarrow W$ such that $\|R\| \leq C\|T\|$ and $\sup_{x \in K} \|Rx - Tx\| \leq \varepsilon$. The operator T has (the property) BAP, if it has the property C-BAP for some constant $C \in [1, \infty)$.

Lemma 2.2. T has C-BAP iff for every finite family $(x_k)_{k=1}^N \subset X$, for each $\varepsilon > 0$ there exists a finite rank operator $R : X \rightarrow W$ such that $\|R\| \leq C\|T\|$ and $\sup_{1 \leq k \leq N} \|Rx_k - Tx_k\| \leq \varepsilon$.

Proof. We may (and do) assume that $\|T\| = 1$. Fix a compact subset $K \subset X$ and $\varepsilon > 0$. Let $\varepsilon_0 := \varepsilon/(2 + C)$, $(x_k)_{k=1}^M$ be an ε_0 -net for K in X , $R \in X^* \otimes W$, $\|R\| \leq C$ and $\sup_{1 \leq k \leq M} \|Rx_k - Tx_k\| \leq \varepsilon_0$. Take any $x \in K$, and let x_k be such that $\|x - x_k\| \leq \varepsilon_0$. Then $\|Tx - Rx\| \leq \|x - x_k\| + \|Tx_k - Rx_k\| + \|Rx_k - Rx\| \leq \varepsilon_0 + \varepsilon_0 + C\varepsilon_0 = \varepsilon$. $\square$

Lemma 2.3. Let X, W be Banach spaces with X separable and let $T \in L(X, W)$. T has the C-BAP iff there exists a sequence $(Q_l)_{l=1}^\infty$ of finite rank operators from X to W , for which

1) for any $x \in X$ the series $\sum_{l=1}^\infty Q_l x$ is convergent and

$$Tx = \sum_{l=1}^\infty Q_l x, \quad x \in X;$$

2) $\sup_N \|\sum_{l=1}^N Q_l\| \leq C\|T\|$.

Proof. Since X is separable, there exists a sequence $(x_k)_1^\infty$, which is dense in the closed unit ball $\bar{B}_1(0)$ of the space X . Suppose, as above, that $\|T\| = 1$ and T has the C-BAP, that is for every finite set $F \subset X$, for each $\varepsilon > 0$ there is a finite rank operator $R : X \rightarrow W$ such that $\|R\| \leq C$ and $\sup_{f \in F} \|Rf - Tf\| \leq \varepsilon$. For each N , let R_N be a finite rank operator from X to W with the following properties:

- (i) $\|R_N\| \leq C$ and
- (ii) $\sup_{1 \leq n \leq N} \|R_N x_n - T x_n\| \leq 1/2^{N+1}$.

If $n \in \mathbb{N}$, then for every $N \geq n$ we have:

$$(iii) \quad \|R_N x_n - T x_n\| \leq \frac{1}{2^{N+1}}$$

and, therefore, for a fixed x_n

$$R_N x_n \rightarrow T x_n$$

as N tends to ∞ .

Now, fix $\varepsilon > 0$ and let $\delta > 0$ be such that $C\delta + \delta < \varepsilon$. For $x \in \bar{B}_1(0)$, take an x_n such that $\|x_n - x\| < \delta$. Then one can find a number N_0 such that if $N \geq N_0$, then

$$\|R_N x - T x\| \leq \|R_N\| \|x_n - x\| + \|R_N x_n - T x_n\| + \|T x_n - T x\| \leq C\delta + \|T\|\delta < \varepsilon.$$

Thus, if $x \in X$, then $R_N x \rightarrow T x$ as $N \rightarrow +\infty$.

To finish the proof of the "only it" part, we apply a lemma: $\square$

Lemma 2.4. *Let X, W be any Banach spaces, $C \geq 1$ and $T \in L(X, W)$. Suppose that (*) there exists a sequence $(S_N)_{N=1}^\infty$ of finite rank operators from X to W such that if $x \in X$, then $S_N x \rightarrow Tx$ as $N \rightarrow +\infty$ and $\|S_N\| \leq C\|T\|$ for each N .*

Then there exists a sequence $(Q_l)_{l=1}^\infty$ of finite rank operators from X to W such that

1) for every $x \in X$ the series $\sum_{l=1}^\infty Q_l x$ is convergent and

$$Tx = \sum_{l=1}^\infty Q_l x, \quad x \in X;$$

2) $\sup_N \|\sum_{l=1}^N Q_l\| \leq C\|T\|$.

Proof. Again, we assume that $\|T\| = 1$. Set $Q_1 := S_1, Q_l := S_l - S_{l-1}$ for $l > 1$. Then

$$\begin{aligned} S_N &= S_1 + (S_2 - S_1) + \cdots + (S_{N-1} - S_{N-2}) + (S_N - S_{N-1}) \\ &= Q_1 + Q_2 + \cdots + Q_N. \end{aligned}$$

It follows that

$$(1) \quad Tx = \sum_{l=1}^\infty Q_l x \quad \forall x \in X$$

and

$$(2) \quad \sup_N \left\| \sum_{l=1}^N Q_l \right\| = \sup_N \|S_N\| \leq C.$$

The "if" part of the proof of Lemma 2.2 follows from the following assertion. □

Lemma 2.5. *Let X, W be any Banach spaces, $C \geq 1$ and $T \in L(X, W)$. If there exists a sequence $(R_N)_1^\infty$ of finite rank operators from X to W , converging pointwise to T and such that $\|R_N\| \leq C\|T\|$ for all N , then T has the C -BAP.*

Proof. Indeed, let $R_N x \rightarrow Tx$ for each $x \in X$ ($\|R_N\| \leq C\|T\|$). Fix an $\varepsilon > 0$ and a compact subset $K \subset X$. Put $\varepsilon_0 := \varepsilon(\|T\| + 1 + C\|T\|)^{-1}$. Take a finite ε_0 -net $F \subset X$ for K and consider a finite rank operator R_{N_0} such that $\sup_{f \in F} \|R_{N_0} f - Tf\| \leq \varepsilon_0$. Then, for every $x \in K$, one can find an element $f_0 \in F$ such that $\|f_0 - x\| \leq \varepsilon_0$, and we get:

$$\|Tx - R_{N_0} x\| \leq \|T\| \varepsilon_0 + \varepsilon_0 + \|R_{N_0}\| \varepsilon_0 \leq \varepsilon_0(\|T\| + 1 + C\|T\|) = \varepsilon.$$

□

Corollary 2.6. *If X is separable and $T \in L(X, W)$, then T has the C -BAP iff there exists a sequence $(R_N)_1^\infty$ of finite rank operators from X to W , which converges pointwise to the operator T and such that $\|R_N\| \leq C\|T\|$ for all N .*

Corollary 2.7. *If X is separable and $T \in L(X, W)$, then T has the BAP iff there exists a sequence of finite rank operators from X to W , converging pointwise to the operator T .*

3. Bounded approximation property and O-frames.

In this section, we show, in particular, that every operator in separable Banach spaces, which has the bounded approximation property, has an O-frame.

We begin with a theorem which, essentially, has been proved in A. Pełczyński's paper [7]. Since the proof of Theorem 3.1 below repeats *almost word by word* a part of the corresponding proof from [7], we omit it. To check that this is the case, one can find the full proof of Theorem 3.1 in preprints [11], [12] and [13].

Theorem 3.1. *If $T : X \rightarrow W$ has the property (*), that is, if there exists a sequence $(S_N)_{N=1}^\infty$ of finite rank operators from X to W such that for every $x \in X$ $S_N x \rightarrow Tx$ as $N \rightarrow +\infty$ and $\|S_N\| \leq C\|T\|$ for each N , then the operator T has an O-frame..*

Theorem 3.1 gives us two immediate consequences.

Corollary 3.2. *For any Banach spaces X and W and for every operator $T \in L(X, W)$, the following assertions are equivalent:*

- (1) *T has an O-frame;*
- (2) *the operator T factors through a Banach sequence space with a basis;*
- (3) *there exists a sequence of finite rank operators from X to W , which converges to T pointwise.*

Corollary 3.3. *Let X, W be Banach spaces with X being separable. For $T \in L(X, W)$, the following assertions are equivalent:*

- (1) *T has an O-frame;*
- (2) *T has the bounded approximation property;*
- (3) *the operator T can be factored through a Banach space with Schauder basis.*

4. Comparing the usual Schauder frames with O-frames

I. It is known that if X has an unconditional Schauder frame, then:

1. The frame is shrinking iff X does not contain l_1 iff X is almost reflexive (i.e., every bounded sequence in X contains a weak Cauchy subsequence).
2. X is reflexive iff it does not contain both l_1 and c_0 .

This follows, in particular, from the properties of subspaces of the spaces with unconditional bases (see, e.g., [L-Tz])

For UO-frames, the situation is different. We have

Example 4.1. There exists an operator $T : l_1 \rightarrow C[0, 1]$ such that

1. T is conditionally weakly compact and, thus, does not contain l_1 . T has no shrinking O-frame.
2. T does not contain also c_0 , but is not weakly compact.

Indeed, let V be an integral Volterra operator, $Vf(s) = \int_0^s f(t) dt$ for $f \in L_1[0, 1]$. Then V is conditionally weakly compact and, thus, does not contain l_1 (see, e.g., [3]). Evidently, the image of V^* is not separable; therefore, V cannot have a shrinking O-frame (by Proposition 2). Now, take any linear operator P from l_1 onto $L_1[0, 1]$ and consider the composition $T := VP$. This operator is not weakly compact (otherwise, $Im T^*$ would be separable).

II. Trivially, If X does not have the approximation property, then it can not have a Schauder frame.

For O-frames, the situation is different. We have

Example 4.2. There exist two separable reflexive Banach spaces X, Y and an operator $T : X \rightarrow Y$ so that: Both X and Y do not have the approximation property, but T has an unconditional O-frame.

Indeed, take any $p \in (2, \infty), q \in [1, 2)$ and consider subspaces $X \subset l_p$ and $Y \subset l_q$ without approximation property [15]. By Maurey-Kwapień theorem [9, Theorem 3.4, p. 33], any operator from X to Y factors through a Hilbert space.

III. It is known that if a reflexive space has the approximation property, then it has a Schauder frame. (Indeed, by Grothendieck [4], the space has the MAP (= 1-BAP), by Pełczyński [7], it has a Schauder frame).

For O-frames, the situation is different. We have

Example 4.3. There exists a weakly compact operator, which has the approximation property, but has no operator frame.

Indeed, one can find in [10] an example of a weakly compact operator (in separable Banach spaces) with the approximation property, but without the property BAP. Now, it is enough to apply Corollary 3.2.

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