

MATRIX APPROACH TO THE GLOBAL AND LOCAL REGULARITY OF WAVELETS

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Abstract. We apply recent methods of computing the joint spectral characteristics of matrices to find precise values of the Hölder exponents of Daubechies wavelets of order $N \leq 20$. The local regularity of these wavelets is also analyzed. In particular, we find the points of the worst local regularity. In all cases they are rational, but not always dyadic.

1. Introduction

The regularity of wavelets and of refinement equations is one of key characteristics, important for their approximation properties, for applications in PDE, in functional analysis etc. This problem was studied in detail, see [8, 9, 7, 12, 11, 19, 32, 35, 36] and references therein. Among several methods of estimating the regularity of wavelets elaborated in the literature the matrix approach is the only one that gives (at least theoretically) precise values of the Hölder exponent. Moreover, in some cases it allows us to make a refine analysis of local regularity of wavelet functions. The main disadvantage is that the matrix approach reduces the problem of finding the Hölder exponent to computing of the joint spectral characteristics of matrices. This problem is notoriously hard even for matrices of low dimensions. That is why the exponents of regularity was computed only for wavelets with small support. Recently several efficient methods of computing the joint spectral characteristics was elaborated. In this paper we apply one of them, the invariant polytope algorithm, to find precise values of the Hölder exponent of compactly supported wavelets. In particular, we find the regularity of Daubechies

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wavelets of order $N \leq 20$. We disprove the conjecture that the Hölder exponent is always attained at dyadic points and find the points of the worst local regularity. We also demonstrate our technique to analyse the local regularity of Daubechies wavelets of low orders.

2. Notation and basic facts on the global regularity

For a given set of linear operators $\mathcal{A} = \{A_1, \dots, A_m\}$ acting in $\mathbb{R}^d$ and for $p \in [-\infty, +\infty]$ the p -radius of $\mathcal{A}$ is

$$\rho_p(\mathcal{A}) = \lim_{k \rightarrow \infty} \left(m^{-k} \sum_{d_1, \dots, d_k \in \{1, \dots, m\}} \|A_{d_k} \cdots A_{d_1}\|^p \right)^{1/pk} \quad (2.1)$$

The existence of this limit for every p follows from the Fekete subadditive lemma [13]. This limit does not depend on an operator norm in $\mathbb{R}^d$, because all norms in a finite-dimensional space are equivalent. Thus, $\rho_p(\mathcal{A})$ are well defined for all p . These numbers are alternatively called the joint spectral characteristics of the family $\mathcal{A}$. If $m = 1$, i.e., the family consists of one operator, all these p -radii are equal to the (usual) spectral radius $\rho(A_1)$, which is the maximal modulus of eigenvalues of A_1 . If $m \geq 2$, then these characteristics are usually different for different parameters p . For $p = +\infty$, the L_p -mean becomes the maximum and the corresponding $+\infty$ -radius becomes the *joint spectral radius* denoted usually by $\rho(\mathcal{A})$ (the subscript $+\infty$ is dropped):

$$\rho(\mathcal{A}) = \lim_{k \rightarrow \infty} \max_{d_1, \dots, d_k \in \{1, \dots, m\}} \|A_{d_k} \cdots A_{d_1}\|^{1/k}. \quad (2.2)$$

For $p = -\infty$ we obtain the *lower spectral radius* $\check{\rho}(\mathcal{A})$ defined by the same formula (2.2) with the maximum replaced by minimum. For $p = 0$ the L_p -mean in (2.1) turns to the geometric mean and we obtain the Lyapunov exponent ρ_0 .

The pioneering work of 1991 by Daubechies and Lagarias [11] revealed the relation of the smoothness of wavelets and of the joint spectral radius. The latter notion was almost forgotten by that time since its invention by Rota and Strang in 1960 [33]. Now there is an extensive literature studying the joint spectral radius. See [1, 4, 15, 16, 17, 23, 26, 27, 31] for methods to estimate and to compute the joint spectral radius. This problem is algorithmically hard [5, 6].

Let us recall that the Daubechies wavelets is a system of functions $2^{j/2}\psi(2^jx - n)$, $j, n \in \mathbb{Z}$, that constitutes an orthonormal basis in $L_2(\mathbb{R})$. All functions of this system are generated by double dilates and integer translates of the compactly supported *wavelet function* ψ . I. Daubechies in [10] constructed a countable family of wavelet functions $\psi = \psi_N$, $N \geq 1$, each generates its own wavelet system. For every N , we consider a *refinement equation* for the *scaling function* $\varphi = \varphi_N$:

$$\varphi(x) = \sum_{k=0}^M c_k \varphi(2x - k), \quad (2.3)$$

where $M = 2N - 1$, and $c_0, \dots, c_M$ are *Daubechies filter coefficients*. They possess some special properties, in particular, $\sum_{i=0}^M c_i = 2$ and the polynomial (the *symbol* of the

equation) $m(z) = \frac{1}{2} \sum_{n=0}^M c_n z^n$ has zero of order N at the point $z = -1$. We have

$$b(z) = \frac{m(z)}{((1+z)/2)^N} = \frac{1}{2} \sum_{n=0}^{N-1} b_n z^n, \quad (2.4)$$

where the coefficients b_n are explicitly constructed, see [10, 25]. The wavelet function ψ_N is obtained from the scaling function φ_N by an explicit formula and has the same regularity [10]. The main idea of the matrix approach to study refinement equations of the type (2.3) is to consider the vector function $v(x) = (\varphi(x), \dots, \varphi(x-M+1)) \in \mathbb{R}^M$. Then equation (2.3) becomes the *self-similarity* equation on the function $v : [0, 1] \rightarrow \mathbb{R}^M$.

$$v(x) = \begin{cases} T_0 v(2x), & x \in [0, \frac{1}{2}); \\ T_1 v(2x-1), & x \in [\frac{1}{2}, 1], \end{cases} \quad (2.5)$$

where T_0, T_1 are $M \times M$ -matrices (*transition matrices*) defined by the formulas

$$\begin{aligned} (T_0)_{ij} &= c_{2i-j-1} \\ (T_1)_{ij} &= c_{2i-j}. \end{aligned} \quad i, j = 1, \dots, M \quad (2.6)$$

(we set $c_k = 0$ for $k > M$ and for $k < 0$). Since $\sum_i c_i = 2$, the hyperplane $W = \{x \in \mathbb{R}^M \mid \sum_i x_i = 0\}$ is invariant under both T_0 and T_1 . It was first shown in [11] that if $\rho(T_0|_W, T_1|_W) < 1$, then equation (2.5) and hence equation (2.3) possesses a continuous solution. Moreover, $\alpha_\varphi = \alpha_v \geq -\log \rho(T_0|_W, T_1|_W)$. This condition is not necessary, as it was shown by Collela and Heil in [9]. It was proved that actually $\alpha_\varphi = \alpha_v = -\log \rho(T_0|_V, T_1|_V)$, where $V \subset W$ is the smallest common invariant subspace of the transition operators T_0, T_1 containing the vector $v(1) - v(0)$. That subspace and the action of the operators T_0, T_1 on it was the subject of several papers [2, 3, 9, 7, 14, 24, 34]. Eventually this problem was solved in terms of *clean symbol* of the equation [30]. For refinable functions generating wavelets, that result can be formulated simpler (Theorem 2.1 below). To this end, let us introduce some further notation. For a given $p \in [1, +\infty]$ and $f \in L_p[a, b]$ (in case $p = +\infty$ we replace $L_\infty[a, b]$ by $C[a, b]$) denote

$$\omega_{f,p}(h) = \sup_{|t| \leq h} \|f(\cdot+t) - f(\cdot)\|_p; \quad \alpha_{f,p}(h) = l + \sup \{\alpha \geq 0 \mid \omega_{f^{(l)},p}(h) \leq C h^\alpha\}, \quad (2.7)$$

where l is the largest integer such that $f \in W_p^l$ (let us recall that $W_p^0 = L_p$; for $l \geq 1$, the Sobolev space $W_p^l[a, b]$ consists of absolutely continuous functions such that $f^{(l)} \in L_p$). For $p = +\infty$ we replace W_p^l by C^l . For refinable functions, $\varphi \in W_p^l$ if and only if $\alpha_{\varphi,p} > l$ [9, 20, 29]. The following theorem is a special case of general Theorem 10 from [30] for Daubechies wavelets. Let the sequence $\{b_n\}_{n=0}^{N-1}$ be given by (2.4). We set $b_n = 0$ for $n < 0$ and for $n \geq N$. The $(N-1) \times (N-1)$ -matrices B_0, B_1 are defined by the sequence $\{b_n\}_{n=0}^{N-1}$ by formulas analogous to (2.6):

$$\begin{aligned} (B_0)_{ij} &= b_{2i-j-1} \\ (B_1)_{ij} &= b_{2i-j}. \end{aligned} \quad i, j = 1, \dots, N-1 \quad (2.8)$$

The operators T_0, T_1 on the space V in a special basis are written by matrices B_0 and B_1 respectively.

Theorem 2.1. [9, 21, 30]. For every $N \in \mathbb{N}$ and $p \in [1, +\infty]$ for the Daubechies wavelet function $\psi = \psi_N$, we have

$$\alpha_{\psi,p} = N - \log \rho_p(B_0, B_1). \quad (2.9)$$

Moreover,

$$C_1 h^\alpha \leq \omega_{\psi^{(l)},p}(h) \leq C_2 h^\alpha,$$

where $l = [\alpha_{\psi,p}]$, $\alpha = \alpha_{\psi,p} - l$, and C_1, C_2 are positive constants.

We denote $\omega_{f,+\infty} = \omega_f$ and $\alpha_{f,+\infty} = \alpha_f$.

3. The local regularity of wavelets

Local regularity of wavelets is important for applications in functional analysis and in the signal processing (see [22, 18] and references therein) and was studied in the literature [11, 29]. For an arbitrary x we denote

$$\omega_f(x, h) = \sup_{|t| \leq h} \|f(x+t) - f(x)\|; \quad \alpha_f(h) = \sup \{\alpha \geq 0 \mid \omega_f(x, h) \leq C h^\alpha\}. \quad (3.1)$$

Definition 3.1. A number $x = 0.d_1 d_2 \dots$ is *normal* if for any $\varepsilon > 0$ there is $N(\varepsilon)$ such that for every $m > N(\varepsilon)$ there are two different digits among those d_k with numbers $k \in [m, m(1 + \varepsilon)]$.

So, normal numbers cannot be approximated “too close” by dyadic numbers. Almost all (in Lebesgue measure) numbers of the segment $[0, 1]$ are normal. All rational, but not dyadic, numbers are normal.

Definition 3.2. For given linear operators B_0, B_1 and for a given binary sequence $(x) = d_1, d_2, \dots$ the value

$$\rho_x = \rho_x(B_0, B_1) = \limsup_{k \rightarrow \infty} \|B_{d_1} \cdots B_{d_k}\|^{1/k}$$

is the *joint spectral radius of B_0, B_1 along the sequence (x)* .

Theorem 3.3. [29] Let $v : [0, 1] \rightarrow \mathbb{R}^N$ be the vector-function corresponding to the Daubechies wavelet function $\psi = \psi_N$. The exponent of local regularity $\alpha_{v^{(l)}}(x)$ does not exceed $N - \log_2 \rho_x(B_0, B_1)$ for all points $x \in [0, 1]$, and is equal to this value for all normal points x . For rational points x , we have

$$\alpha_{v^{(l)}}(x) = N - \frac{1}{r} \log_2 \rho(\Pi),$$

where Π is the product of matrices B_0, B_1 corresponding to the period of dyadic expansion of x and r is its length. For dyadic points x , when $r = 1$, we have

$$\alpha_{v^{(l)}}(x) = N - \log_2 \max \{ \rho(B_0), \rho(B_1) \}$$

For an arbitrary point $x \in [0, 1]$, one has

$$\alpha_{v^{(l)}}(x) \in [\alpha_{\min}, \alpha_{\max}]$$

where $\alpha_{\min} = \alpha_{v^{(l)}} = N - \log_2 \rho(B_0, B_1)$ and $\alpha_{\max} = N - \log_2 \check{\rho}(B_0, B_1)$. For each value $\nu \in [\alpha_{\min}, \alpha_{\max}]$ the set $\Omega_\nu = \{x \in [0, 1] \mid \alpha_{v^{(l)}}(x) = \nu\}$ is everywhere dense on $[0, 1]$. If $\nu = N - \log_2 \rho_0(B_0, B_1)$, then Ω_ν has a full measure on $[0, 1]$, for all other $\nu \in [\alpha_{\min}, \alpha_{\max}]$ it has zero measure.

Theorems 2.1 and 3.3 make it possible to analyse the local behaviour of Daubechies wavelets ψ_N . Let us recall that the function ψ_1 is the Haar function $\chi_{[0,1/2]} - \chi_{[1/2,1]}$, where χ_A is the characteristic function of the set A . For all $N \geq 2$, the functions ψ_N are supported on the segments $\text{supp } \psi_N = [0, 2N - 1]$, are continuous, their smoothness increases in N and $\alpha_{\psi_N} > 0.2N$ [10].

The Daubechies function ψ_2 . The wavelet-function of Daubechies $\psi_2(x)$ (from now on we omit the subscript 2) is defined by a refinable function φ satisfying the equation:

$$\varphi(x) = \frac{1 + \sqrt{3}}{4} \varphi(2x) + \frac{3 + \sqrt{3}}{4} \varphi(2x - 1) + \frac{3 - \sqrt{3}}{4} \varphi(2x - 2) + \frac{1 - \sqrt{3}}{4} \varphi(2x - 3). \quad (3.2)$$

The operators T_0, T_1 act in $\mathbb{R}^3$, the operators $B_i = T_i|_V$, $i = 0, 1$, are one-dimensional, they are operators of multiplication by numbers $b_0 = 1 + \sqrt{3}$ and $b_1 = 1 - \sqrt{3}$ respectively. Clearly, $\alpha_\psi = 2 - \log_2(1 + \sqrt{3}) = 0.550\dots$. Consequently, ψ is continuous and (Theorem 2.1) we have $\omega_\psi(h) \asymp h^{0.550\dots}$. The derivative ψ' is summable, but not from $L_2(\mathbb{R})$. Indeed, $\alpha_{\psi',1} = 1 - \log_2 \sqrt{3} = 0.2075\dots$. On the other hand, $\alpha_{\psi',2} = 1$, and so $\psi' \notin L_2(\mathbb{R})$.

The local regularity at every normal point x is $\alpha_v(x) = 2 - \log_2 \rho_x$, where $\rho_x = \max\{b_1^{r(x)} b_0^{1-r(x)}, \frac{1}{2}\}$, $r(x)$ is the upper density of ones in the dyadic expansion of x . Furthermore, at dyadic points we have different local regularity exponents from the left and from the right:

$$\begin{aligned} \alpha_v^-(x) &= \min\{2 - \log_2(\sqrt{3} - 1), 1\} = \min\{2.449\dots, 1\} = 1, \\ \alpha_v^+(x) &= \min\{2 - \log_2(\sqrt{3} + 1), 1\} = \min\{0.550\dots, 1\} = 0.550\dots \end{aligned}$$

Thus, at all dyadic points the derivative from the right is infinite, while the derivative from the left is finite. For all possible $x \in [0, 1]$ the value $\alpha_v(x)$ fills out the segment $[1 - \log_2 \frac{1+\sqrt{3}}{2}, 1] = [0.550, 1]$. Finally, since $\psi \in W_1^1$, we see that the average regularity is 1.

The third Daubechies function ψ_3 . This function is defined by a refinable function φ , satisfying equation (2.3) with coefficients

$$\begin{aligned} c_0 &= \frac{1}{16} \left[1 + \sqrt{10} + \sqrt{5 + 2\sqrt{10}} \right], \\ c_1 &= \frac{1}{16} \left[5 + \sqrt{10} + 3\sqrt{5 + 2\sqrt{10}} \right], \\ c_2 &= \frac{1}{16} \left[10 - 2\sqrt{10} + 2\sqrt{5 + 2\sqrt{10}} \right], \\ c_3 &= \frac{1}{16} \left[10 - 2\sqrt{10} - 2\sqrt{5 + 2\sqrt{10}} \right], \\ c_4 &= \frac{1}{16} \left[5 + \sqrt{10} - 3\sqrt{5 + 2\sqrt{10}} \right], \\ c_5 &= \frac{1}{16} \left[1 + \sqrt{10} - \sqrt{5 + 2\sqrt{10}} \right]. \end{aligned} \quad (3.3)$$

In this case we have

$$B_0 = \begin{pmatrix} 3.764 & 0 \\ 0.399 & -2.162 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -2.162 & 3.764 \\ 0 & 0.399 \end{pmatrix}$$

(the coefficients are given approximately). In this case $\rho(B_0, B_1) = 3.760 \dots$ [11]. Since $N - \log_2 \rho > 1$, from Theorem 2.1 it follows that $\psi \in C^1$ and $\alpha_{\psi'} = 2 - \log_2 3.760 = 0.088$. Moreover, $\omega_{\psi'}(h) \asymp h^{0.088}$. The smallest local regularity of the function ψ is achieved at all dyadic points. Indeed, $\alpha_{\psi'}^+(x) = 2 - \log_2 \rho(B_0) = 0.088$. On the other hand, $\alpha_{\psi'}^-(x) = 2 - \log_2 \rho(B_1) = 0.887$. We see that at dyadic points the regularities from two sides are different. The regularity from the left equals to the minimal local regularity of ψ' . It turns out that the regularity from the right equals to the maximal one. In other words $\check{\rho}(B_0, B_1) = \rho(B_0) = 3.760 \dots$. This is shown in [29, Example 12].

We see that the local regularity of the function $\psi'(x)$ at any point does not exceed $0.887 \dots$. It follows that $\psi'(x)$ is nowhere differentiable. The values of the local regularity fills out the segment $[0.088, 0.887]$.

The Daubechies function ψ_4 . The corresponding refinement equation is of degree $N = 7$ and has the following coefficients:

$$\begin{aligned} c_0 &= 0.325, & c_4 &= -0.264, \\ c_1 &= 1.010, & c_5 &= 0.043, \\ c_2 &= 0.892, & c_6 &= -0.046, \\ c_3 &= -0.039, & c_7 &= -0.014. \end{aligned} \tag{3.4}$$

We have two 3×3 -matrices (all coefficients are rounded to the third digit after the decimal point):

$$\begin{aligned} B_0 &= \begin{pmatrix} 5.212 & 0 & 0 \\ 1.703 & -4.676 & 5.212 \\ 0 & -0.239 & 1.703 \end{pmatrix}, \\ B_1 &= \begin{pmatrix} -4.676 & 5.212 & 0 \\ -0.239 & 1.703 & -4.676 \\ 0 & 0 & -0.239 \end{pmatrix}, \end{aligned}$$

It turns out that $\rho(B_0, B_1) = \rho(B_0) = 5.212 \dots$. This was shown in [11], then by a different method in [15]. We give one more proof in Section 5 by constructing an invariant polytope norm with Algorithm 1. We have $\alpha_{\psi'} = 4 - \log_2 5.212 = 0.618 \dots$. Therefore $\psi \in C^1(\mathbb{R})$, but $\psi \notin C^2(\mathbb{R})$, and $\omega_{\psi'}(h) \asymp h^{0.618}$. The smallest local regularity of $\psi'(x)$ is attained at dyadic points from the right. The regularity at the same points from the left is $\alpha_{\psi'}^+(x) = 4 - \log_2 \rho(B_1) = 0.851 \dots$. As for the lower spectral radius, we have, for instance, the following bound:

$$\check{\rho} \leq \min_{d_1, \dots, d_6 \in \{0, 1\}^6} \|B_{d_1} \cdots B_{d_6}\|^{1/6} = \|B_0^3 B_1 B_0 B_1\|^{1/6} = 3.904 \dots$$

Since $4 - \log_2 3.904 = 2.034 \dots$, we see that the biggest local regularity of the function ψ' is 1. This yields that ψ is twice differentiable on an everywhere dense set. In particular, this is the case for all rational numbers x with a period (000101). However, the function ψ does not possess a summable second derivative. According to Theorem 2.1 this means that $4 - \log_2 \rho_1(B_0, B_1) \leq 2$. To show this we observe first

that $\rho_1(B_0, B_1) = \rho_1(B_0, -B_1) \geq \rho(\frac{1}{2}(B_0 - B_1))$. By a direct calculation we verify that $\rho(\frac{1}{2}(B_0 - B_1)) = 4.944\dots > 4$. Thus, $\psi \notin W_1^2$ and $\alpha_{\psi', 1} = 4 - \log_2 4.944 = 0.694\dots$.

Points with the worst local regularity. In those three examples for $N = 2, 3, 4$ we see that the smallest local regularity $\alpha_{\min}$ is attained at dyadic points. In view of Theorem 3.3, this means that $\rho(B_0, B_1) = \max\{\rho(B_0), \rho(B_1)\}$.

Conjecture A. *For all $N \geq 2$, the worst local regularity of the Daubechies wavelets is always attained at dyadic points.*

To the best of our knowledge, this conjecture has not been explicitly formulated in the literature, although it was a general belief among the specialists that this is true. In fact, Gripenberg [15] proved it for all $N \leq 8$. In Section 6 we compute the exact values of α_{ψ_N} for all $N \leq 20$ and, in particular, show that Conjecture A fails for $N = 10, 15$, and 16. See Theorem 7.1 for details.

4. Computation of the joint spectral radius

To find the Hölder exponents of Daubechies wavelets of higher order we apply the Invariant polytope algorithm from [16]. We give a brief description of this algorithm here, the details and the theoretical base can be found in [16].

For a given set $G \subset \mathbb{R}^d$ we denote by $\text{co}(G)$ the convex hull of G and by $\text{absco}(G) = \text{co}\{G, -G\}$ the symmetrized convex hull. The sign $\asymp$ denotes as usual the asymptotic equivalence of two values (i.e., equivalence up to multiplication by a constant).

Let $\mathcal{A} = \{A_1, \dots, A_m\}$ be a set of $d \times d$ matrices. A product $\Pi = A_{d_n} \cdots A_{d_1}$ is called a *spectral maximizing product* (s.m.p) if $\rho(\mathcal{A}) = [\rho(\Pi)]^{1/n}$. So, if one finds a spectrum maximizing product, then he finds the joint spectral radius. The idea of the algorithm is to prove that a chosen product Π is spectrum maximizing by constructing a convex polytope $P \subset \mathbb{R}^d$ such that $A_i P \subset \rho_n P$, $i = 1, \dots, m$, where $\rho_n = [\rho(\Pi)]^{1/n}$. This means that in the Minkowski norm $\|\cdot\|_P$ defined by the polytope P we have $\|A_i\|_P \leq \rho_n$, and therefore $\rho(\mathcal{A}) = \rho_n$.

Algorithm 1 (the Invariant polytope algorithm) (see [16]).

Initialization. First, we fix some number $\bar{n}$ and find a product $\Pi = A_{d_n} \cdots A_{d_1}$ with the maximal value $[\rho(\Pi)]^{1/n}$ among all products of lengths $n \leq \bar{n}$. We call this product a *candidate s.m.p.* and try to prove that it is actually an s.m.p. Denote $\rho_n = [\rho(\Pi)]^{1/n}$ and normalize all the matrices A_i as $\tilde{A}_i = \rho_n^{-1} A_i$. Thus we obtain the family $\tilde{\mathcal{A}}$ and the product $\tilde{\Pi} = \tilde{A}_{d_n} \cdots \tilde{A}_{d_1}$ such that $\rho(\tilde{\Pi}) = 1$. For the sake of simplicity we assume that the largest by modulo eigenvalue of $\tilde{\Pi}$ is real, in which case it is ± 1 . We assume it is 1, the case of -1 is considered in the same way. The eigenvector $v^{(1)}$ corresponding to this eigenvalue is called *leading eigenvector*. The vectors $v^{(j)} = \tilde{A}_{d_{j-1}} \cdots \tilde{A}_{d_1} v^{(1)}$, $j = 2, \dots, n$, are leading eigenvectors of cyclic permutations of $\tilde{\Pi}$. Then we construct a sequence of finite sets $\mathcal{V}_i \subset \mathbb{R}^d$ and their subsets $\mathcal{R}_i \subset \mathcal{V}_i$ as follows:

Zero iteration. We set $\mathcal{V}_0 = \mathcal{R}_0 = \{v^{(1)}, \dots, v^{(n)}\}$.

k th iteration, $k \geq 1$. We have finite set $\mathcal{V}_{k-1}$ and its subset $\mathcal{R}_{k-1}$. We set $\mathcal{V}_k = \mathcal{V}_{k-1}$, $\mathcal{R}_k = \emptyset$ and for every $v \in \mathcal{R}_{k-1}$, $\tilde{A} \in \tilde{\mathcal{A}}$, check whether $\tilde{A}v$ is an *interior point* of $\text{absco}(\mathcal{V}_k)$ (this is an LP problem). If so, we omit this point and take the next

pair $(v, \tilde{A}) \in \mathcal{R}_{k-1} \times \tilde{\mathcal{A}}$, otherwise we add $\tilde{A}v$ to $\mathcal{V}_k$ and to $\mathcal{R}_k$. When all pairs $(v, \tilde{A})$ are exhausted, both $\mathcal{V}_k$ and $\mathcal{R}_k$ are constructed. Let $P_k = \text{absco}(\mathcal{V}_k)$. We have

$$\mathcal{V}_k = \mathcal{V}_{k-1} \cup \mathcal{R}_k, \quad P_k = \text{co} \{ \tilde{A}_1 P_{k-1}, \dots, \tilde{A}_m P_{k-1} \}.$$

Termination. The algorithm halts when $\mathcal{V}_k = \mathcal{V}_{k-1}$, i.e., $\mathcal{R}_k = \emptyset$ (no new vertices are added in the k th iteration). In this case $P_{k-1} = P_k$, and hence P_{k-1} is an invariant polytope, Π is an s.m.p., and $\rho(\mathcal{A}) = [\rho(\Pi)]^{1/n}$. **End of the algorithm.**

Actually, the algorithm works with the sets $\mathcal{V}_k$ only, the polytopes P_k are needed to illustrate the idea. Thus, in each iteration of the algorithm, we construct a polytope $P_k \subset \mathbb{R}^d$, store all its vertices in the set $\mathcal{V}_k$ and spot the set $\mathcal{R}_k \subset \mathcal{V}_k$ of newly appeared (after the previous iteration) vertices. Every time we check whether $\tilde{A}P_k \subset P_k$. If so, then P_k is an invariant polytope, $\|\tilde{A}_i\|_{P_k} \leq 1$ for all i , and Π is an s.m.p. Otherwise, we update the sets $\mathcal{V}_k$ and $\mathcal{R}_k$ and continue.

If the algorithm terminates within finite time, then it proves that the chosen candidate is indeed an s.m.p. and gives the corresponding polytope norm. Although there are simple examples of matrix families for which the algorithm does not terminate, we believe that such cases are rare in practice. In fact, in all numerical experiments made with randomly generated matrices and with matrices from applications, the algorithm did terminate in finite time providing an invariant polytope.

Remark 4.1. *We can introduce an arbitrary number of extra initial vertices to the vertices $v^{(1)}, \dots, v^{(n)}$ sometimes it helps to speed up the algorithm. We use extra vertices in the next section when computing the joint spectral radius of Daubechies matrices.*

5. The regularity of Daubechies wavelets of higher order

We apply Algorithm 1 to compute the joint spectral radius of the transition matrices B_0, B_1 of the Daubechies refinement equation. For $N = 2, 3$, and 4, the value α_{ψ_N} were found by Daubechies and Lagarias in [11]; for $N = 5, 6, 7$, and 8, they were computed by G.Gripenberg [15]. Every time a delicate analysis of special properties of transition matrices was involved. In all the cases the s.m.p. of the family $\{B_0, B_1\}$ was one of those two matrices, and it was a general belief that this is the case for all N (Conjecture A). We apply the standard routine of Algorithm 1 to find the precise values of α_{ψ_N} for all $N \leq 20$. In particular, we shall see that for $N = 10$, the conjecture of one matrix s.m.p. is violated and the s.m.p. is $B_0^2 B_1^2$.

There is a trouble applying Algorithm 1 for the matrices B_0, B_1 : with growing N they become very ill-conditioned. All vertices of the constructed polytope have very small last components, which corresponds to the property of quasi-invariance of a certain subspace and determines a polytope strongly flattened along certain directions. For $N = 10$, the last components are about $10^{-12} - 10^{-19}$ of the values of the first components. This creates enormous numerical difficulties in the running of Algorithm 1, in particular, in the linear programming routines. That is why we add extra initial vertices to the polytope P .

We present three illustrative cases ($N = 4, 10, 12$) and report the computed Hölder regularity of Daubechies wavelets in Table 5.

The case $N = 4$.

We have two 3×3 -matrices B_0, B_1 given in (3.5). The candidate s.m.p. is B_0 with $\rho(B_0) = 5.212\dots$. The leading eigenvector $v_1 = (1.000, 0.015, 0.000)$ lies on the subspace E_2 of $\mathbb{R}^3$. Applying the norm to the resulting vectors also shows the flatness of the invariant polytope. We observe that v_1 almost lies on the canonical basis e_1 .

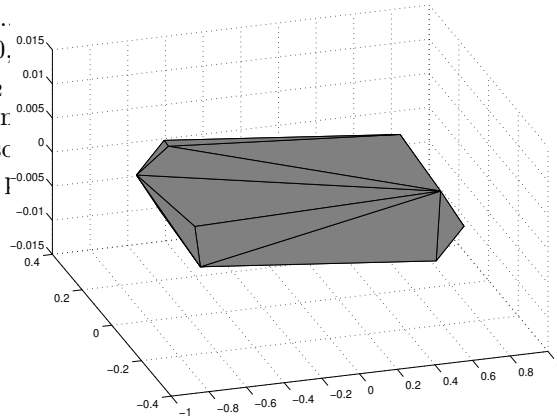


FIGURE 1. The invariant polytope for the Daubechies matrices for $N = 4$ computed by the standard algorithm.

In its basic implementation the algorithm converges and generates a centrally symmetric polytope in $\mathbb{R}^3$ of $6 \cdot 2$ vertices, in 5 iterations. The partial polytope norms of the family $\tilde{\mathcal{A}} = \{\tilde{B}_0, \tilde{B}_1\}$ are reported in Table 1. The vertices (beyond v_1) of the polytope follow

Iteration k	2	3	4	5
$\ \cdot\ _{\mathcal{P}_i}$	1.7845	1.1288	1.0571	1

TABLE 1. The partial polytope norms computed by the standard algorithm for $N = 4$.

(we report only half of them):

$$v_2 = \begin{pmatrix} -0.730 \\ 0.018 \\ 5.224 \cdot 10^{-4} \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0.730 \\ 0.254 \\ 6.806 \cdot 10^{-4} \end{pmatrix}, \quad v_4 = \begin{pmatrix} -0.730 \\ -0.010 \\ 0.011 \end{pmatrix},$$

$$v_5 = \begin{pmatrix} -0.730 \\ -0.217 \\ 0.004 \end{pmatrix}, \quad v_6 = \begin{pmatrix} -0.730 \\ -0.039 \\ 0.011 \end{pmatrix}$$

and the corresponding unit polytope $P = \text{absco}\{v_i, i = 1, \dots, 6\}$ is shown in Figure 1. We see that P appears to be very flat.

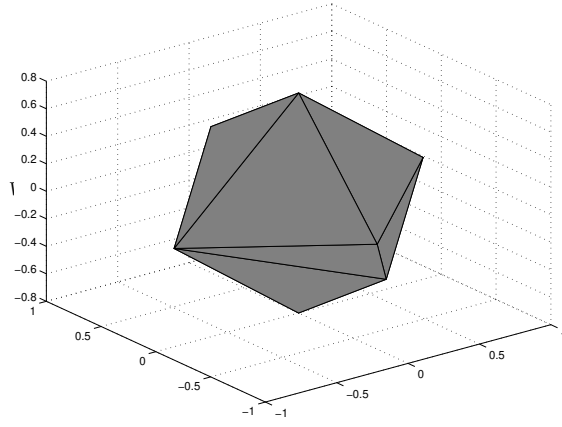


FIGURE 2. Polytope extremal norm for the Daubechies matrices for $N = 4$ computed with an extra initial vertex.

In fact, the largest singular value¹ of the matrix of its vertices $V = \{\pm v_i, i = 1, \dots, 6\}$ is $\sigma_1 = 1.9365$, while the smallest one is $\sigma_3 = 1.0923 \cdot 10^{-2}$ (note that if σ_3 would be zero then V would not span the whole space and the polytope would be contained in a subspace). This almost 200 times difference gives a numerical evidence of the flattening phenomenon. To avoid it we add an extra initial vertex $x_1 = \frac{4}{5}e_3$ and obtain a better behaviour of Algorithm 1 and a more balanced polytope (see Figure 2). The polytope has now $4 \cdot 2$ vertices which are computed in only one (!) iteration. Denote $v_2 = x_1$. The vertices beyond v_1 and v_2 of the polytope follow (we report only half of them):

$$v'_3 = \begin{pmatrix} 0 \\ 0.800 \\ 0.261 \end{pmatrix}, \quad v'_4 = \begin{pmatrix} 0 \\ -0.717 \\ -0.036 \end{pmatrix}.$$

The largest singular value is now $\sigma_1 = 1.159 \cdot 10^0$ and the smallest singular value is now $\sigma_3 = 7.740 \cdot 10^{-1}$, which demonstrate a much more balanced shape of the unit ball of the polytope extremal norm. The computed Hölder exponent of ψ_4 is $\alpha_{\varphi_4} = 4 - \log_2 \rho(B_0, B_1) = 4 - \log_2 \rho(B_0) = 1.617 \dots$

The case $N = 10$.

We have the 9×9 matrices B_0, B_1 defined by formulas (2.8) with the polynomial $b(z)$ of degree 9. We are going to prove that the s.m.p. is

$$\Pi = B_0^2 B_1^2, \quad \rho_n = \rho(\Pi)^{1/4} = 99.636 \dots$$

which already contradicts to Conjecture A.

Let $\tilde{B}_0 = B_0/\rho_n$, $\tilde{B}_1 = B_1/\rho_n$. Applying Algorithm 1 we compute the starting set of vectors $\mathcal{V}_0 = \{v_1, v_2, v_3, v_4\}$, where v_1 is the leading eigenvector of Π , $v_2 = \tilde{B}_1 v_1$, $v_3 =$

¹ Recall that for a matrix $B \in \mathbb{R}^{p,q}$ (or $B \in \mathbb{C}^{p,q}$), the *reduced* singular value decomposition is given by $B = U \Sigma W^*$ where $U \in \mathbb{C}^{p,q}$ and $W \in \mathbb{C}^{q,q}$ are unitary matrices and $\Sigma \in \mathbb{R}^{q,q}$ is a diagonal matrix with nonnegative diagonal elements $\{\sigma_i\}_{i=1}^q$ (the singular values, usually ordered in decreasing way) that are the square roots of the eigenvalues of the Hermitian (semi)-positive definite matrix $B^* B$.

$\tilde{B}_1 v_2, v_4 = \tilde{B}_0 v_3$. Clearly, $\tilde{B}_0 v_4 = v_1$. Thus, v_2, v_3 and v_4 are the leading eigenvectors of cyclic permutations of the product Π . We have

$$v_1 = \begin{pmatrix} 1.712 \cdot 10^{-3} \\ 1.000 \cdot 10^0 \\ 3.034 \cdot 10^{-1} \\ -3.051 \cdot 10^{-1} \\ -5.921 \cdot 10^{-2} \\ 5.951 \cdot 10^{-4} \\ 9.097 \cdot 10^{-5} \\ -2.522 \cdot 10^{-7} \\ -6.628 \cdot 10^{-12} \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3.856 \cdot 10^{-1} \\ 1.164 \cdot 10^0 \\ 3.605 \cdot 10^{-2} \\ -1.884 \cdot 10^{-1} \\ -1.827 \cdot 10^{-2} \\ 5.089 \cdot 10^{-4} \\ 1.098 \cdot 10^{-5} \\ -1.836 \cdot 10^{-8} \\ 1.277 \cdot 10^{-15} \end{pmatrix},$$

$$v_3 = \begin{pmatrix} 1.139 \cdot 10^{-2} \\ 1.219 \cdot 10^0 \\ 5.696 \cdot 10^{-1} \\ -2.416 \cdot 10^{-2} \\ -1.278 \cdot 10^{-2} \\ -9.433 \cdot 10^{-5} \\ 4.910 \cdot 10^{-6} \\ -2.178 \cdot 10^{-9} \\ -2.464 \cdot 10^{-19} \end{pmatrix}, \quad v_4 = \begin{pmatrix} 4.417 \cdot 10^{-3} \\ -1.152 \cdot 10^0 \\ -8.316 \cdot 10^{-1} \\ 4.686 \cdot 10^{-2} \\ 3.737 \cdot 10^{-2} \\ 7.184 \cdot 10^{-4} \\ -3.433 \cdot 10^{-5} \\ 3.438 \cdot 10^{-8} \\ 4.199 \cdot 10^{-13} \end{pmatrix}.$$

Observe that the last two components of all the vectors $v_1, \dots, v_4$ are very small, i.e., all these vectors almost lie on the subspace $E_7 \subset \mathbb{R}^9$ spanned by the vectors $\{e_1, \dots, e_7\}$ of the canonical basis of $\mathbb{R}^9$. Applying $\tilde{B}_0$ and $\tilde{B}_1$ repeatedly to $\mathcal{V}_0$ one observes that the resulting vectors also almost lie on the subspace E_7 . This means that the invariant polytope computed by Algorithm 1 is nearly degenerate, it is close to a 7-dimensional polytope. A consequence of this is a slow convergence behaviour of the algorithm and an ill-conditioning of basic linear algebra operations. The algorithm terminates and generates a centrally symmetric polytope in $\mathbb{R}^9$ of $220 \cdot 2$ vertices, in 14 iterations. The partial polytope norms of the family $\tilde{\mathcal{A}} = \{\tilde{B}_0, \tilde{B}_1\}$ are reported in Table 2. The

k	3	4	5	6	7	8	9	10	11	12	13	14
$\ \cdot\ _{P_i}$	24.856	4.693	2.990	2.237	1.743	1.414	1.140	1.064	1.027	1.025	1.001	1

TABLE 2. The partial polytope norms computed by the Algorithm 1 for $N = 10$.

largest singular value of the set of vertices is $\sigma_1 = 1.3253 \cdot 10^1$ and the smallest singular value is $\sigma_9 = 1.0620 \cdot 10^{-10}$, which gives a numerical evidence of the flatness of the polytope. To correct the behavior of the algorithm we add an extra vector along the “most narrow” (for the polytope P) direction. Adding the vector $v_5 = \frac{1}{2}e_9$, we obtain a better behaviour of the algorithm and a more balanced polytope. The results are ~~The largest singular value of the polytope is $5.7294 \cdot 10^0$ and the smallest is $1.040 \cdot 10^{-3}$, which demonstrate a much more balanced shape of the unit ball of the polytope extremal norm. The computed Hölder exponent of ψ_{10} is~~

$$\alpha_{\varphi_{10}} = 10 - \log_2 \rho(B_0, B_1) = 10 - \frac{1}{4} \log_2 \rho(B_0^2 B_1^2) = 3.361 \dots$$

i	1	2	3	4	5	6	7	8	9
$\ \cdot\ _{\mathcal{P}_i}$	9.401	1.865	1.386	1.2622	1.207	1.115	1.051	1.0169	1

TABLE 3. The partial polytope norms computed by the modified algorithm for $N = 10$.

The case $N = 12$.

In this case both B_0 and B_1 are s.m.p., i.e. $\rho(B_0, B_1) = \rho(B_0) = \rho(B_1)$ the two leading eigenvectors, say v_1 and v_2 follow:

$$v_1 = \begin{pmatrix} 0 \\ 1.346 \cdot 10^{-1} \\ 1.000 \cdot 10^0 \\ 4.393 \cdot 10^{-1} \\ -1.188 \cdot 10^{-1} \\ -4.154 \cdot 10^{-2} \\ -5.594 \cdot 10^{-4} \\ 1.229 \cdot 10^{-4} \\ 2.014 \cdot 10^{-7} \\ -3.643 \cdot 10^{-9} \\ 1.124 \cdot 10^{-13} \end{pmatrix}, \quad v_2 = \begin{pmatrix} -4.859 \cdot 10^{-1} \\ -3.283 \cdot 10^{-2} \\ 1.000 \cdot 10^0 \\ 3.519 \cdot 10^{-1} \\ -8.382 \cdot 10^{-3} \\ -5.620 \cdot 10^{-3} \\ -5.317 \cdot 10^{-5} \\ 3.007 \cdot 10^{-6} \\ -3.753 \cdot 10^{-9} \\ -1.223 \cdot 10^{-13} \\ 0 \end{pmatrix}.$$

We observe that they almost lie on the subspace $E_8 \subset \mathbb{R}^{11}$ spanned by the vectors $\{e_1, \dots, e_8\}$ of the canonical basis of $\mathbb{R}^{11}$. Applying $\tilde{B}_0$ and $\tilde{B}_1$ repeatedly to $\mathcal{V}_0$ one observes that the resulting vectors also almost lie on the subspace E_8 . Note that the last components seems to vanish exponentially in the number of iterations (from the smallest to the highest index). If we add an extra initial vector $v_3 = \frac{1}{2}e_{11}$, Algorithm 1 terminates after 6 iterations with an invariant polytope of $48 \cdot 2$ vertices. The partial polytope norms are reported in Table 4. The largest singular value is $\sigma_1 = 3.819 \cdot 10^0$ and the smallest one is $\sigma_9 = 1.541 \cdot 10^{-5}$, which demonstrate a relatively balanced shape. The computed Hölder exponent is $\alpha_{\psi_{12}} = 12 - \log_2 \rho(B_0) = 3.833 \dots$

i	1	2	3	4
$\ \cdot\ _{\mathcal{P}_i}$	5.057	1.549	1.159	1

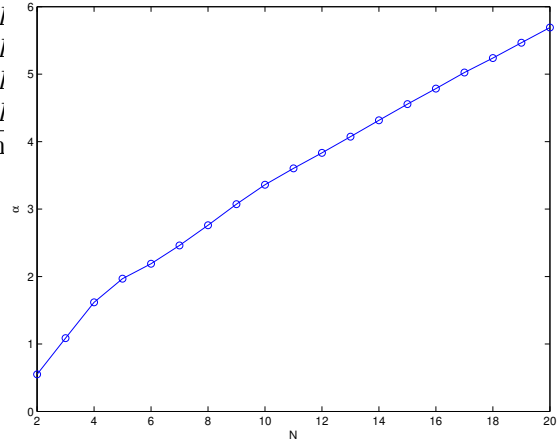
TABLE 4. The partial polytope norms computed by the modified algorithm for $N = 12$.

6. The table of results for $N \leq 20$

The values of Hölder exponents of Daubechies wavelets of order N are presented in Table 5. They are obtained by computing the joint spectral radius with Algorithm 1. We indicate the s.m.p., the extra initial vectors, the number of vertices of the final polytope ($\#V$), the number of iterations of Algorithm 1 ($\#its$) and the Hölder exponent α . In Figure 3 we α_{ψ_N} as a function of N .

N	s.m.p.	Extra vertices	#its	$\#V$	α
2	B_0	none	1	$1 \cdot 2$	$0.550 \dots$
3	B_0	none	2	$3 \cdot 2$	$1.087 \dots$
4	B_0	$0.8e_3$	1	$4 \cdot 2$	$1.617 \dots$
5	B_0 and B_1	$0.1e_4$	3	$8 \cdot 2$	$1.968 \dots$
6	B_0 and B_1	$0.1e_5$	4	$11 \cdot 2$	$2.189 \dots$
7	B_0 and B_1	$0.1e_5$	4	$12 \cdot 2$	$2.460 \dots$
8	B_0 and B_1	$0.1e_7$	4	$18 \cdot 2$	$2.760 \dots$
9	B_0 and B_1	$0.5e_8$	4	$24 \cdot 2$	$3.073 \dots$
10	$B_0^2 B_1^2$	$0.5e_9$	9	$90 \cdot 2$	$3.361 \dots$
11	B_0 and B_1	$0.5e_{10}$	9	$75 \cdot 2$	$3.603 \dots$
12	B_0 and B_1	$0.5e_{11}$	4	$48 \cdot 2$	$3.833 \dots$
13	B_0 and B_1	e_{12}	16	$73 \cdot 2$	$4.073 \dots$
14	B_0 and B_1	$0.5e_{13}, 0.25e_{12}$	13	$73 \cdot 2$	$4.316 \dots$
15	$B_0^4 B_1^2$	$10^{-3}\{e_k\}_{k=9}^{14}$	13	$376 \cdot 2$	$4.556 \dots$
16	$B_0^2 B_1^2$	$10^{-2}\{e_k\}_{k=11}^{15}$	12	$372 \cdot 2$	$4.786 \dots$
17	B_0 and i				
18	B_0 and i				
19	B_0 and i				
20	B_0 and i				

TABLE 5. Com



ets.

FIGURE 3. The function α_{ψ_N} .

7. Conclusions from the numerical results

At least two conclusions can be drawn from the numerical results in Table 5. First, we see that for $N = 10, 15$, and 16 , the spectral maximizing products have lengths bigger than one, which contradicts Conjecture A. For $N = 10$, the s.m.p. is $B_0^2 B_1^2$, hence, by Theorem 3.3, the equality $\alpha_{v(l)} = \alpha_{v(l)}(x)$ holds at all rational points x

whose dyadic expansion has the period (0011). Those are all numbers of the form $\frac{p}{5 \cdot 2^n}$, $p \in \mathbb{N}$, $n \in \mathbb{Z}$, $n \geq 0$. The same holds for $N = 16$. For $N = 15$, the s.m.p. is $B_0^4 B_1^2$, hence the points x of the worst local regularity have the dyadic period (000011). These are the points of the form $x = \frac{p}{21 \cdot 2^n}$. Thus, we have proved

Theorem 7.1. *For all $N \leq 20$, except for $N = 10, 15, 16$, the worst local regularity of the Daubechies wavelets are attained at dyadic points. For $N = 10$ and $N = 16$, it is attained at rational points of the form $x = \frac{p}{5 \cdot 2^n}$, $p \in \mathbb{N}$. For $N = 15$, it is attained at points $x = \frac{p}{21 \cdot 2^n}$.*

Another observation is the behaviour of the sequence $\{\alpha_{\psi_N}\}_{N \in \mathbb{N}}$. As it was shown in [10], this sequence grows linearly and $\alpha_{\psi_N} > 0.2N$. From Table 5 we see that this sequence approaches to a linear function indeed. The sequence of differences $\alpha_{\psi_N} - \alpha_{\psi_{N-1}}$ is not monotone, but seems to converge to a limit close to 0.226. This is just an empirical observation, it would be interesting to have theoretical results in this direction.

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