

A NEW SPLITTING TRICK FOR WAVELET PACKETS ON LOCAL FIELDS OF POSITIVE CHARACTERISTIC

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Abstract. This paper deals with a construction of wavelet packets on local fields of positive characteristic. The splitting trick here is different that the method described by Behera and Jahan in [Wavelet packets and wavelet frame packets on local fields of positive characteristic, *J. Math. Anal. Appl.* 395 (2012), 1-14], in that we split the wavelet subspaces directly instead of using the low-pass and high-pass filters associated with the multiresolution analysis. Furthermore, the method overcomes the difficulty of constructing non-orthogonal wavelet packets of the dilation factor not equal to 2.

1. Introduction

A field K equipped with a topology is called a *local field* if both the additive K^+ and multiplicative groups K^* of K are locally compact Abelian groups. The local fields are essentially of two types: zero and positive characteristic (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). Examples of local fields of characteristic zero include the p -adic field $\mathbb{Q}_p$ where as local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p -groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and multiresolution analysis theory are quite different. In recent years, local fields have attracted the attention of several mathematicians, and have found innumerable applications not only to number theory but also to representation theory, division algebras, quadratic forms and algebraic geometry. As a result, local fields are now consolidated as part of the standard repertoire of contemporary mathematics. For more about local fields and their applications, we refer to the monographs [12, 23].

In recent years there has been a considerable interest in the problem of constructing wavelet bases on various groups, namely, Cantor dyadic groups [10], locally compact Abelian groups [7], p -adic fields [9] and Vilenkin groups [11]. Recently, R. L. Benedetto

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and J. J. Benedetto [2] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. The concept of multiresolution analysis on a local field K of positive characteristic was introduced by Jiang et al.[8]. They pointed out a method for constructing orthogonal wavelets on local field K with a constant generating sequence. Subsequently, tight wavelet frames on local fields of positive characteristic were constructed by Shah and Debnath [21] using extension principles. More results in this direction can also be found in [15–18] and the references therein.

It is well known that the classical orthonormal wavelet bases have poor frequency localization. For example, if the wavelet ψ is band limited, then the measure of the supp of $(\psi_{j,k})^\wedge$ is 2^j -times that of $\text{supp } \hat{\psi}$. To overcome this disadvantage, Coifman et al.[5] introduced the notion of orthogonal univariate wavelet packets. Well known Daubechies orthogonal wavelets are a special of wavelet packets. Chui and Li [4] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be employed to the spline wavelets and so on. Shen [22] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. Other notable generalizations are the vector-valued wavelet packets [3], wavelet packets and framelet packets related to the Walsh polynomials [6,13,14,19] and M -band framelet packets [20].

The main tool used in the construction of wavelet packets is the splitting trick, which breaks the half shifts of a function into the shifts of two functions. The catalyst used in this procedure are the low-pass and high-pass filters associated with the scaling and wavelet functions. But for the non-orthogonal wavelet packets having dilation factor not equal to 2, the classical splitting trick does not work (see [22]). Recently, Behera and Jahan [1] have used the classical splitting trick to construct orthogonal wavelet packets on local fields of positive characteristic. Different from the splitting trick as used in [1], we split the wavelet subspaces directly instead of using the low-pass and high-pass filters associated with the multiresolution analysis and thus gives the Riesz basis of the wavelet subspaces. Moreover, our method overcomes the difficulty of constructing non-orthogonal wavelet packets of the dilation factor not equal to 2.

This paper is organized as follows. In Section 2, we discuss some preliminary facts about local fields of positive characteristic including the definitions of Riesz basis and multiresolution analysis on local fields. In Section 3, we prove a crucial lemma called the *splitting lemma* which decomposes the wavelet subspaces directly instead of using the low-pass and high-pass filters. By virtue of this lemma, we construct wavelet packets associated with an MRA and prove that they generate a Riesz basis for $L^2(K)$.

2. Preliminaries on Local Fields

Let K be a field and a topological space. Then K is called a *local field* if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K , respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence by a local field, we mean a field K which is locally compact, non-discrete and totally disconnected. The

p -adic fields are examples of local fields. More details are referred to [12, 23]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let K be a local field. Let dx be the Haar measure on the locally compact Abelian group K^+ . If $\alpha \in K$ and $\alpha \neq 0$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha|dx$. We call $|\alpha|$ the *absolute value* of α . Moreover, the map $x \rightarrow |x|$ has the following properties: (a) $|x| = 0$ if and only if $x = 0$; (b) $|xy| = |x||y|$ for all $x, y \in K$; and (c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$. Property (c) is called the *ultrametric inequality*. The set $\mathfrak{O} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K . Define $\mathfrak{B} = \{x \in K : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in K . The prime ideal in K is the unique maximal ideal in $\mathfrak{O}$ and hence as result $\mathfrak{B}$ is both principal and prime. Since the local field K is totally disconnected, so there exist an element of $\mathfrak{B}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{B}$. Such an element is called a *prime element* of K . Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{O}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{O}$. As it was proved in [23], the set $\mathfrak{O}$ is compact and open. Hence, $\mathfrak{B}$ is compact and open. Therefore, the residue space $\mathfrak{O}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^k$ for some prime p and $k \in \mathbb{N}$.

Let $\mathfrak{O}^* = \mathfrak{O} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Then, it can be proved that $\mathfrak{O}^*$ is a group of units in K^* and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x', x' \in \mathfrak{O}^*$. For a proof of this fact we refer to [19]. Moreover, each $\mathfrak{B}^k = \mathfrak{p}^k \mathfrak{O} = \{x \in K : |x| < q^{-k}\}$ is a compact subgroup of K^+ and usually known as the *fractional ideals* of K^+ . Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{O}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_{\ell} \mathfrak{p}^{\ell}$ with $c_{\ell} \in \mathcal{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{O}$ but is non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{O}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(yx), x \in K$. Suppose that χ_u is any character on K^+ , then clearly the restriction $\chi_u|_{\mathfrak{O}}$ is also a character on $\mathfrak{O}$. Therefore, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{O}$ in K^+ , then, as it was proved in [23], the set $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ of distinct characters on $\mathfrak{O}$ is a complete orthonormal system on $\mathfrak{O}$.

The Fourier transform $\hat{f}$ of a function $f \in L^1(K) \cap L^2(K)$ is defined by

$$\hat{f}(\xi) = \int_K f(x) \overline{\chi_{\xi}(x)} dx. \quad (2.1)$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x) \overline{\chi_{\xi}(x)} dx = \int_K f(x) \chi(-\xi x) dx.$$

Furthermore, the properties of Fourier transform on local field K are much similar to those of on the real line. In particular Fourier transform is unitary on $L^2(K)$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{O}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c -dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}\} \subset \mathfrak{O}^*$ such that $\text{span}\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \leq n < q, \quad n = a_0 + a_1 p + \dots + a_{c-1} p^{c-1}, \quad 0 \leq a_k < p, \quad \text{and } k = 0, 1, \dots, c-1,$$

we define

$$u(n) = (a_0 + a_1 \zeta_1 + \dots + a_{c-1} \zeta_{c-1}) \mathfrak{p}^{-1}. \quad (2.2)$$

Also, for $n = b_0 + b_1q + b_2q^2 + \cdots + b_sq^s$, $n \in \mathbb{N}_0$, $0 \leq b_k < q$, $k = 0, 1, 2, \dots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \quad (2.3)$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \leq s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \geq 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \dots, c-1 \text{ or } j \neq 1. \end{cases} \quad (2.4)$$

Definition 2.1. Let $\{x_n : n \in \mathbb{N}_0\}$ be a subset of a Hilbert space H . Then, $\{x_n : n \in \mathbb{N}_0\}$ is said to form a *Riesz basis* for H if

- (a) $\{x_n : n \in \mathbb{N}_0\}$ is linearly independent, and
- (b) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_2^2 \leq \sum_{n \in \mathbb{N}_0} |\langle x, x_n \rangle|^2 \leq B\|x\|_2^2 \quad \text{for every } x \in H. \quad (2.5)$$

The above definition is equivalent to the following definition. A subset $\{x_n : n \in \mathbb{N}_0\}$ of a Hilbert space H forms a Riesz basis for H if

- (a) $\overline{\text{span}} \{x_n : n \in \mathbb{N}_0\} = H$, and
- (b) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A \sum_{n \in \mathbb{N}_0} |c_n|^2 \leq \left\| \sum_{n \in \mathbb{N}_0} c_n x_n \right\|^2 \leq B \sum_{n \in \mathbb{N}_0} |c_n|^2 \quad \text{for every } \{c_n\} \in \ell^2(\mathbb{N}_0).$$

A generalization of classical theory of multiresolution analysis on local fields of positive characteristic was considered by Jiang et al.[8]. Analogous to the Euclidean case, following is a definition of *multiresolution analysis* on the local field K of positive characteristic.

Definition 2.2. Let K be a local field of positive characteristic $p > 0$ and $\mathfrak{p}$ be a prime element of K . A multiresolution analysis(MRA) of $L^2(K)$ is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$ satisfying the following properties:

- (a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
- (b) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(K)$;
- (c) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;
- (d) $f(\cdot) \in V_j$ if and only if $f(\mathfrak{p}^{-1}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
- (e) There is a function $\varphi \in V_0$, called the *scaling function*, such that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms an orthonormal basis for V_0 .

Since $\varphi \in V_0 \subset V_1$ and $\{\varphi(\mathfrak{p}^{-1}x - u(k)) : k \in \mathbb{N}_0\}$ is a Riesz basis of V_1 , there exists $\{a_k\} \in \ell^2(\mathbb{N}_0)$ such that

$$\varphi(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} a_k \varphi(\mathfrak{p}^{-1}x - u(k)). \quad (2.6)$$

On taking the Fourier transform, we have

$$\hat{\varphi}(x) = m_0(\mathfrak{p}\xi) \hat{\varphi}(\mathfrak{p}\xi), \quad (2.7)$$

where

$$m_0(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} a_k \overline{\chi_k(\xi)}.$$

Given an MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$. We denote by W the orthogonal complement of V_0 in V_1 , i.e. $W = V_1 \ominus V_0$. Assume that there exists $q-1$ functions $\{\psi_1, \psi_2, \dots, \psi_{q-1}\}$ in $L^2(K)$ such that their translates and dilations form a Riesz basis of W , i.e.,

$$W = \overline{\text{span}}\{\psi_\ell(x - u(k)) : 1 \leq \ell \leq q-1, k \in \mathbb{N}_0\}. \quad (2.8)$$

Since $\psi_\ell \in W \subset V_1$, $0 \leq \ell \leq q-1$, there exist a sequence $\{a_k^\ell : k \in \mathbb{N}_0\}$ with $\sum_{k \in \mathbb{N}_0} |a_k^\ell|^2 < \infty$ such that

$$\psi_\ell(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} a_k^\ell \varphi(\mathfrak{p}^{-1}x - u(k)). \quad (2.9)$$

Equation (2.9) can be written in the frequency domain as

$$\hat{\psi}_\ell(\xi) = m_\ell(\mathfrak{p}\xi) \hat{\varphi}(\mathfrak{p}\xi), \quad (2.10)$$

where

$$m_\ell(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} a_k^\ell \overline{\chi_k(\xi)}, \quad \ell = 1, \dots, L \quad (2.11)$$

are the integral periodic functions in $L^2(\mathfrak{D})$ and are called the *wavelet masks*.

Therefore, in view of Definition 2.2, we have

$$L^2(K) = \bigoplus_{j \in \mathbb{Z}} \Delta^j W = V_0 \oplus \left(\bigoplus_{j \geq 0} \Delta^j W \right), \quad (2.12)$$

where Δ is the dilation operator such that $\Delta f(x) = f(\mathfrak{p}^{-1}x)$, for any $f \in L^2(K)$.

For $n = 0, 1, \dots$, the *basic wavelet packets* associated with a scaling function φ on a local fields of positive characteristic are defined recursively by

$$\Gamma_n(x) = \Gamma_{qr+s}(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} a_k^s \Gamma_r(\mathfrak{p}^{-1}x - u(k)), \quad 0 \leq s \leq q-1 \quad (2.13)$$

where $r \in \mathbb{N}_0$ is the unique element such that $n = qr + s$, $0 \leq s \leq q-1$ holds.

Definition 2.3. Let $\{\Gamma_n : n \in \mathbb{N}_0\}$ be the basic wavelet packets associated with the multiresolution analysis $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$. The family of functions

$$\mathcal{F} = \{q^{j/2} \Gamma_n(\mathfrak{p}^{-j}x - u(k)) : j \in \mathbb{Z}, n \geq 0, k \in \mathbb{N}_0\}, \quad (2.14)$$

will be called the *general wavelet packets* corresponding to the MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$.

3. Splitting Trick and Wavelet Packets on Local Fields

For any $n \in \mathbb{N}_0$, we define

$$E_n = \left\{ f(x) : f(x) = \sum_{k \in \mathbb{N}_0} a_k \Gamma_n(x - u(k)), \{a_k\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0) \right\}. \quad (3.1)$$

Lemma 3.1. *Let $\{\Gamma_n : n \in \mathbb{N}_0\}$ be the basic wavelet packets associated with the MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$. Then for all $n \in \mathbb{N}_0$, $\{\Gamma_n(x - u(k)) : k \in \mathbb{N}_0\}$ forms a Riesz basis of E_n .*

Proof. We prove this result by induction on n . Since $\{\psi_1, \psi_2, \dots, \psi_{q-1}\}$ is the basic set of wavelets in W and $\{\psi_\ell(x - u(k)) : 1 \leq \ell \leq q-1, k \in \mathbb{N}_0\}$ forms a Riesz basis of W , therefore, $\{q^{1/2} \psi_\ell(\mathfrak{p}^{-1}x - u(k)) : 1 \leq \ell \leq q-1, k \in \mathbb{N}_0\}$ constitutes a Riesz basis of ΔW . By virtue of Riesz basis and Definition 2.3, it is clear that $\{\Gamma_\ell(x - u(k)) : 1 \leq \ell \leq q-1, k \in \mathbb{N}_0\}$ forms a Riesz basis of W , and hence a Riesz basis of ΔE_n . Therefore, the claim is true for all $1 \leq n \leq q-1$. Assume that the result is also true for $n < \ell, (\ell \geq q)$. Then, for $n = \ell$, there exists two unique numbers r and s such that $n = qr + s$, $0 \leq s \leq q-1, r \in \mathbb{N}_0$ and $r < n = \ell$. For this choice of ℓ , the family of functions $\{\Gamma_r(x - u(k)) : k \in \mathbb{N}_0\}$ constitutes a Riesz basis of E_n . Therefore, there exists constants C_1 and C_2 , $0 < C_1 \leq C_2 < \infty$ such that

$$C_1 \leq \sum_{k \in \mathbb{N}_0} \left| \hat{\Gamma}_r(\xi + u(k)) \right| \leq C_2. \quad (3.2)$$

Further, we have

$$\begin{aligned} \sum_{k \in \mathbb{N}_0} \left| \hat{\Gamma}_n(\xi + u(k)) \right|^2 &= q \sum_{k \in \mathbb{N}_0} \left| \hat{\Gamma}_r(\mathfrak{p}\xi + \mathfrak{p}u(k)) \right|^2 |\chi_k(\mathfrak{p}\xi)|^2 \\ &= q \sum_{k \in \mathbb{N}_0} \left| \hat{\Gamma}_r(\mathfrak{p}\xi + \mathfrak{p}u(k)) \right|^2 \\ &= q \sum_{k \in \mathbb{N}_0} \left| \hat{\Gamma}_r(\mathfrak{p}\xi + \mathfrak{p}u(k)) \right|^2 + q \sum_{k \in \mathbb{N}_0} \left| \hat{\Gamma}_r(\mathfrak{p}\xi + \mathfrak{p}u(k) + \mathfrak{p}u(s_1)) \right|^2 \\ &\quad + \dots + q \sum_{k \in \mathbb{N}_0} \left| \hat{\Gamma}_r(\mathfrak{p}\xi + \mathfrak{p}u(k) + \mathfrak{p}u(s_{q-1})) \right|^2, \end{aligned}$$

and hence

$$C_1 = q \frac{1}{q} \leq \sum_{k \in \mathbb{N}_0} \left| \hat{\Gamma}_n(\xi + u(k)) \right| \leq q \frac{1}{q} = C_2.$$

Therefore, it follows that $\{\Gamma_n(x - u(k)) : k \in \mathbb{N}_0\}$ forms a Riesz basis of E_n . $\square$

Next, we develop the splitting trick for the basic wavelet packets Γ_n on local fields of positive characteristic.

Lemma 3.2. (*Splitting lemma*). *For every $n \in \mathbb{N}_0$, the family of functions*

$$\left\{ \Gamma_{qn+s}(x - u(k)) : 0 \leq s \leq q-1, k \in \mathbb{N}_0 \right\},$$

forms Riesz basis of ΔE_n .

Proof. We first claim that

$$\Delta E_n = \left\{ f(x) : f(x) = \sum_{s=0}^{q-1} \sum_{k \in \mathbb{N}_0} a_k^s \Gamma_{qn+s}(x - u(k)), \{a_k^s\}_{k \in \mathbb{N}_0} \in \ell^2(\mathbb{N}_0) \right\}. \quad (3.3)$$

For $0 \leq s \leq q-1$, by (2.12) and (3.1), it follows that $\Gamma_{qn+s}(x - u(k)) \in \Delta E_n$. We next prove that for each $f(x) \in \Delta E_n$,

$$f(x) = q^{1/2} \sum_{k \in \mathbb{N}_0} b_k \Gamma_n(\mathfrak{p}^{-1}x - u(k)),$$

there exists a sequences $\{b_k\}_{k \in \mathbb{N}_0}$, $0 \leq s \leq q-1$, such that

$$f(x) = \sum_{s=0}^{q-1} \sum_{k \in \mathbb{N}_0} a_k^s \Gamma_{qn+s}(x - u(k)). \quad (3.4)$$

For every $k \in \mathbb{N}_0$, there exist unique numbers r and s such that $k = qr + s$, $0 \leq s \leq q-1, r \in \mathbb{N}_0$. Therefore, for this choice of $k \in \mathbb{N}_0$, we have

$$\begin{aligned} f(x) &= q^{1/2} \sum_{k \in \mathbb{N}_0} b_k \Gamma_n(\mathfrak{p}^{-1}x - u(k)) \\ &= q^{1/2} \sum_{r \in \mathbb{N}_0} b_{qr} \Gamma_n(\mathfrak{p}^{-1}x - u(qr)) + q^{1/2} \sum_{r \in \mathbb{N}_0} b_{qr+s_1} \Gamma_n(\mathfrak{p}^{-1}x - u(qr + s_1)) \\ &\quad + \cdots + q^{1/2} \sum_{r \in \mathbb{N}_0} b_{qr+s_{q-1}} \Gamma_n(\mathfrak{p}^{-1}x - u(qr + s_{q-1})) \\ &= q^{1/2} \sum_{r \in \mathbb{N}_0} b_{qr} \Gamma_n(\mathfrak{p}^{-1}x - \mathfrak{p}^{-1}u(r)) + q^{1/2} \sum_{r \in \mathbb{N}_0} b_{qr+s_1} \Gamma_n(\mathfrak{p}^{-1}x - \mathfrak{p}^{-1}u(k) - u(s_1)) \\ &\quad + \cdots + q^{1/2} \sum_{r \in \mathbb{N}_0} b_{qr+s_{q-1}} \Gamma_n(\mathfrak{p}^{-1}x - \mathfrak{p}^{-1}u(k) - u(s_{q-1})) \\ &= \sum_{r \in \mathbb{N}_0} b_{qr} \Gamma_{qn}(x - u(r)) + \sum_{r \in \mathbb{N}_0} b_{qr+s_1} \Gamma_{qn+1}(x - u(r)) \\ &\quad + \cdots + \sum_{r \in \mathbb{N}_0} b_{qr+s_{q-1}} \Gamma_{qn+q-1}(x - u(r)) \\ &= \sum_{s=0}^{q-1} \sum_{k \in \mathbb{N}_0} a_k^s \Gamma_{qn+s}(x - u(k)), \end{aligned}$$

where $a_k^s = b_{qr+s_\ell}$, $0 \leq \ell \leq q-1$ and hence the equality (3.2) follows. Finally, we show that the family of functions

$$\left\{ \Gamma_{qn+s}(x - u(k)) : 0 \leq s \leq q-1, k \in \mathbb{N}_0 \right\},$$

forms Riesz basis of ΔE_n . By Lemma 3.1, we know that $\{\Gamma_n(x - u(k)) : k \in \mathbb{N}_0\}$ form a Riesz basis of E_n . Hence $\{q^{1/2}\Gamma_n(\mathfrak{p}^{-1}x - u(k)) : k \in \mathbb{N}_0\}$ constitutes a Riesz basis of ΔE_n . However, $\{q^{1/2}\Gamma_n(\mathfrak{p}^{-1}x - u(k)) : k \in \mathbb{N}_0\}$ can be splitted into q -disjoint subsets as following:

$$\begin{aligned} &\{q^{1/2}\Gamma_n(\mathfrak{p}^{-1}x - \mathfrak{p}^{-1}u(k))\}, \{q^{1/2}\Gamma_n(\mathfrak{p}^{-1}x - \mathfrak{p}^{-1}u(k) - u(s_1))\}, \\ &\dots, \{q^{1/2}\Gamma_n(\mathfrak{p}^{-1}x - \mathfrak{p}^{-1}u(k) - u(s_{q-1}))\}, k \in \mathbb{N}_0, \quad 0 \leq s \leq q-1, \end{aligned}$$

which can further be written as

$$\{\Gamma_{qn}(x - u(k))\}, \{\Gamma_{qn+1}(x - u(k))\}, \dots, \{\Gamma_{qn+q-1}(x - u(k))\}, k \in \mathbb{N}_0, 0 \leq s \leq q-1,$$

Thus, $\{\Gamma_{qn+s}(x - u(k)) : 0 \leq s \leq q-1, k \in \mathbb{N}_0\}$ form a Riesz basis of ΔE_n . $\square$

This is the splitting trick for our method. This decomposes the wavelet subspaces directly instead of using the low-pass and high-pass filters, and thus gives the Riesz basis of the wavelet subspaces. Applying the splitting trick to the wavelet space W , we can divide W into q -subspaces. Furthermore, we have

Theorem 3.3. *Let $\{\Gamma_n : n \in \mathbb{N}_0\}$ be the basic wavelet packets associated with the scaling function $\varphi(x)$. Then, the family of functions*

$$\{\Gamma_n(x - u(k)) : q^{j-1} \leq n \leq q^j - 1, k \in \mathbb{N}_0\},$$

form a Riesz basis of $\Delta^j W$.

Proof. We prove the result inductively and use induction on j . Since $\{\Gamma_n(x) : 1 \leq n \leq q-1\}$ are the basic wavelets associated with the scaling function $\varphi(x)$ and the family $\{\Gamma_n(x - u(k)) : 1 \leq n \leq q-1, k \in \mathbb{N}_0\}$ constitutes a Riesz basis of W . Hence, the claim is true for $j = 1$.

Assume, the result is true for $j(\geq 1)$, then $\{\Gamma_n(x - u(k)) : q^{j-1} \leq n \leq q^j - 1, k \in \mathbb{N}_0\}$ forms a Riesz basis of $\Delta^j W$. Using the Splitting Lemma 3.2 for the case $j+1$, it follows that

$$\{\Gamma_{qn}(\cdot - u(k)), \Gamma_{qn+1}(\cdot - u(k)), \dots, \Gamma_{qn+q-1}(\cdot - u(k)) : q^{j-1} \leq n \leq q^j - 1, k \in \mathbb{N}_0\},$$

constitutes a Riesz basis of $\Delta^{j+1} W$. Thus $\{\Gamma_n(\cdot - u(k)) : q^j \leq n \leq q^{j+1} - 1, k \in \mathbb{N}_0\}$, constitutes a Riesz basis of $\Delta^{j+1} W$. $\square$

In the next two theorems, we provide various ways to construct Riesz basis of $L^2(K)$ which is extracted out from wavelet packets $\mathcal{F}$ (see Definition 2.3).

Theorem 3.4. *For each fixed $j > 0, k \in \mathbb{N}_0$, the family of functions*

$$\{q^{\ell/2} \Gamma_n(\mathfrak{p}^{-\ell} \cdot -u(k)) : q^{j-1} \leq n \leq q^j - 1, \ell \in \mathbb{Z}, k \in \mathbb{N}_0\}, \quad (3.5)$$

constitutes a Riesz basis of $L^2(K)$.

Proof. Since $\{\Gamma_n(x - u(k)) : q^{j-1} \leq n \leq q^j - 1, k \in \mathbb{N}_0\}$ form a Riesz basis of $\Delta^j W$. By Theorem 3.3, for each $\ell \in \mathbb{Z}$ the family

$$\{q^{\ell/2} \Gamma_n(\mathfrak{p}^{-\ell} x - u(k)) : q^{j-1} \leq n \leq q^j - 1, \ell \in \mathbb{Z}, k \in \mathbb{N}_0\}, \quad (3.6)$$

form a Riesz basis of $\Delta^{j+\ell} W$. Further, for each fixed $j > 0$, we have

$$L^2(K) = \bigoplus_{\ell \in \mathbb{Z}} \Delta^{j+\ell} W$$

Hence, the collection $\{q^{\ell/2} \Gamma_n(\mathfrak{p}^{-\ell} x - u(k)) : q^{j-1} \leq n \leq q^j - 1, \ell \in \mathbb{Z}, k \in \mathbb{N}_0\}$ constitutes a Riesz basis of $L^2(K)$. $\square$

From the above construction, it is clear that the Riesz basis of $L^2(K)$ varies with respect to the integer $j > 0$. For the case $j = 1$, the sub-collection of

$$\mathcal{F} = \left\{ q^{\ell/2} \Gamma_n(\mathfrak{p}^{-\ell} \cdot -u(k)) : n \in \mathbb{N}, \ell \in \mathbb{Z}, k \in \mathbb{N}_0 \right\},$$

gives us the known basis $\{q^{\ell/2} \psi_n(\mathfrak{p}^\ell \cdot -u(k)) : \ell \in \mathbb{Z}, k \in \mathbb{N}_0\}$. Furthermore, the integer j in the above construction is kept fixed where as the dilation parameter ℓ varies over all integers. Now, in order to construct Riesz basis from the collection $\mathcal{F}$, we allow ℓ and j in $\mathcal{F}$ to vary simultaneously.

Let $\Xi = \{(j, \ell) : j \in \mathbb{N}, \ell \in \mathbb{Z}\}$ be a disjoint covering of $\mathbb{Z}$, then for each $m \in \mathbb{Z}$, there exists a unique pair $(j, \ell) \in \Xi$ such that $m = j + \ell$. Moreover, the set Ξ is said to be an n -finite covering of $\mathbb{Z}$ if there exists a positive integer $J < \infty$ such that for all $(j, \ell) \in \Xi$, $j \leq J$.

Theorem 3.5. *Let $\{\Gamma_n : n \in \mathbb{N}_0\}$ be the wavelet packets associated with the scaling function $\varphi(x)$. Then, the collection of functions*

$$\left\{ q^{\ell/2} \Gamma_n(\mathfrak{p}^\ell \cdot -u(k)) : q^{j-1} \leq n \leq q^j - 1, k \in \mathbb{N}_0, (j, \ell) \in \Gamma \right\}, \quad (3.7)$$

form a Riesz basis of $L^2(K)$ if Ξ is an n -finite cover of $\mathbb{Z}$.

Proof. For each fixed j , the family of functions

$$\left\{ \Gamma_n(x - u(k)) : q^{j-1} \leq n \leq q^j - 1, k \in \mathbb{N}_0 \right\},$$

constitutes a Riesz basis of $\Delta^j W$ and hence

$$\left\{ q^{\ell/2} \Gamma_n(\mathfrak{p}^{-\ell} x - u(k)) : q^{j-1} \leq n \leq q^j - 1, \ell \in \mathbb{Z}, k \in \mathbb{N}_0 \right\}$$

form a Riesz basis of $\Delta^{j+\ell} W$.

For each fixed j , let $\Xi_j = \{(j, \ell) : (j, \ell) \in \Xi\}$. Since Ξ is n -finite, we can write Ξ as a finite disjoint union of Ξ_j . Therefore, by this property of Ξ_j , we can write

$$L^2(K) = \bigoplus_{j \in \mathbb{N}} \bigoplus_{(j, \ell) \in \Xi_j} \Delta^{j+\ell} W. \quad (3.8)$$

Hence, for each fixed $j > 0$, the family of functions

$$\left\{ q^{\ell/2} \Gamma_n(\mathfrak{p}^{-\ell} x - u(k)) : q^{j-1} \leq n \leq q^j - 1, k \in \mathbb{N}_0, (j, \ell) \in \Xi \right\},$$

form a Riesz basis of $\bigoplus_{(j, \ell) \in \Xi_j} \Delta^{j+\ell} W$. Using (3.8), it follows that

$$\left\{ q^{\ell/2} \Gamma_n(\mathfrak{p}^{-\ell} x - u(k)) : q^{j-1} \leq n \leq q^j - 1, k \in \mathbb{N}_0, (j, \ell) \in \Xi \right\},$$

constitutes a Riesz basis of $L^2(K)$. □

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