

## WEIGHTED ABSOLUTE CONVERGENCE OF THE SERIES OF FOURIER-HAAR COEFFICIENTS

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**Abstract.** We formulate some sharp statements on absolute convergence of the series of Fourier-Haar coefficients. The two-dimensional analogs of some one-dimensional results are also formulated.

### 1. Introduction

The Haar orthonormal system  $\{\chi_n\}_{n=1}^\infty$  had been constructed in 1909 (see [1]). By this system A. Haar gave positive answer on the question of D. Hilbert: is there an orthogonal system such that Fourier series with respect to this system of any continuous function converges uniformly to that function.

Let us remind the definition of Haar system. We set  $\chi_1(x) \equiv 1$  on  $[0, 1]$ . After that we introduce the open dyadic intervals  $I_i^k = ((i-1)/2^k, i/2^k)$ ,  $i = 1, 2, \dots, 2^k$ ,  $k = 0, 1, \dots$ , and represent the natural number  $n \geq 2$  in the form  $n = 2^k + i$ ,  $i = 1, 2, \dots, 2^k$ ,  $k = 0, 1, \dots$ . Then we set  $\chi_n(x) = 2^{k/2}$  for  $x \in I_{2i-1}^{k+1}$ ,  $\chi_n(x) = -2^{k/2}$  for  $x \in I_{2i}^{k+1}$  and  $\chi_n(x) = 0$   $x \in [0, 1] \setminus \overline{I_i^k}$ , where  $\overline{I_i^k}$  is closure of the interval  $I_i^k$ . If the Haar function  $\chi_n(x)$  has a jump in some point  $x \in (0, 1)$ , then  $\chi_n(x) = (\chi_n(x-0) + \chi_n(x+0))/2$ . In the endpoints of the interval  $[0, 1]$  we set  $\chi_n(0) = \chi_n(+0)$  and  $\chi_n(1) = \chi_n(1-0)$ . The Haar functions  $\chi_n(x)$  are step functions.

The principal information on Fourier-Haar series may be found in the book [2].

For the function  $f \in L_p[0, 1]$ ,  $1 \leq p < \infty$ , we introduce integral modulus of continuity

$$\omega(\delta, f)_p = \sup_{0 \leq h \leq \delta} \left( \int_0^{1-h} |f(t+h) - f(t)|^p dt \right)^{1/p} \quad (0 \leq \delta \leq 1) \quad (1.1)$$

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and Fourier-Haar coefficients  $\hat{f}(n) = \int_0^1 f(t)\chi_n(t) dt$ ,  $n \in \mathbb{N}$ . Z.Ciesielski and J.Musielak [3] proved the following

**Theorem 1.1.** *If  $\beta > 0$ ,  $\gamma \geq 0$ ,  $p = \max(\beta, 1)$ ,  $f \in L_p[0, 1]$  and  $\sum_{n=1}^{\infty} n^{\gamma-\beta/2} \omega^\beta(1/n, f)_p < \infty$ , then the series  $\sum_{n=1}^{\infty} n^\gamma |\hat{f}(n)|^\beta$  converges.*

Let us observe that in the paper [3] the authors introduced another definition of integral modulus of continuity in the space  $L_p[0, 1]$ ,  $1 \leq p < \infty$ . They extended the function to all real axis by the equality  $f(x) = 0$  for  $x \notin [0, 1]$  and the integral in the right-hand side of (1) was evaluated over the interval  $[0, 1]$ . But if we analyze the proof of the Theorem 2 from the paper [3], we see that the statement of the Theorem 1.1 is valid.

Let us define the Wiener's class  $V_p[0, 1]$ ,  $1 \leq p < \infty$ , of functions of bounded  $p$ -variation on the interval  $[0, 1]$  (see [4]). We set  $f \in V_p[0, 1]$ , if

$$V_p(f) = \sup_{\tau} \left( \sum_{i=1}^n |f(x_i) - f(x_{i-1})|^p \right)^{1/p} < \infty,$$

where  $\tau = \{0 = x_0 < x_1 < \dots < x_n = 1\}$  is an arbitrary partition of the interval  $[0, 1]$ . Let us note that the inclusion  $Lip(1/p) \subset V_p[0, 1]$  holds for  $p \in [1, \infty)$ .

P.L. Ulyanov [5] proved the following theorem.

**Theorem 1.2.** *For functions  $f \in V_1[0, 1]$  the series*

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^\beta \quad \text{or} \quad \sum_{n=1}^{\infty} n^{\gamma-1/2} |\hat{f}(n)| \quad (1.2)$$

*converge, if  $\beta > 2/3$  or  $\gamma < 1$  respectively. But this statement is not true for  $\beta = 2/3$  or  $\gamma = 1$  respectively.*

The first author [6] generalized Theorem 1.2 for functions  $f \in V_p[0, 1]$ ,  $1 \leq p < \infty$ .

**Theorem 1.3.** *For the function  $f \in V_p[0, 1]$ ,  $1 \leq p < \infty$ , the series (2) converge if  $\beta > 2p/(p+2)$  or  $\gamma < 1/p - 1/2$  respectively. But this statement is not true if  $\beta = 2p/(p+2)$  and  $\gamma = 1/p - 1/2$  respectively.*

In the paper [7] the first author proved a two-dimensional analog of Theorem 1.3, while the second author [8] gave a generalization of the Theorem 1.3 by using the modulus of continuity  $\omega_{1-1/p}(f, \delta)$  of fractional order.

P.L. Ulyanov [5] proved the following result

**Theorem 1.4.** *If the function  $f \in L_p[0, 1]$ ,  $1 \leq p < \infty$ , satisfies the condition*

$$\sum_{n=1}^{\infty} n^{1/p-1} \omega(1/n, f)_p < \infty,$$

*then the series*

$$\sum_{n=1}^{\infty} |\hat{f}(n)\chi_n(x)|$$

*converges uniformly on the interval  $[0, 1]$ .*

In this paper we formulate some sharp generalizations of Theorems 1.1 and 1.3 and give a two-dimensional analog of Theorem 1.4. We use the weight sequences belonging to the class  $A(\gamma)$ ,  $\gamma \geq 1$ , which was introduced by L.Gogoladze and R.Meskhia [9]. Moreover, we shall formulate a sharp two-dimensional analog of Theorem 1.1.

## 2. Main results

We shall say that the positive sequence  $\gamma = \{\gamma_k\}_{k=1}^\infty$  belongs to the class  $A(\alpha)$ ,  $1 \leq \alpha < \infty$ , if there is a constant  $C > 0$  such that

$$\left( \sum_{k=2^{n+1}}^{2^{n+1}} \gamma_k^\alpha \right)^{1/\alpha} \leq C 2^{n(1-\alpha)/\alpha} \Gamma_n, \quad \text{where} \quad \Gamma_n = \sum_{k=2^{n-1}+1}^{2^n} \gamma_k, \quad n \in \mathbb{N}.$$

For  $n = 0$  we assume that the above inequality for  $\Gamma_0 = \gamma_1$  is valid. This definition was introduced by L. Gogoladze and R. Meskhia [9]. Let us introduce also the class  $A(\infty)$ . We say that a positive sequence  $\gamma = \{\gamma_k\}_{k=1}^\infty$  belongs to the class  $A(\infty)$ , if there is a constant  $C > 0$  such that

$$\max_{2^n < k \leq 2^{n+1}} \gamma_k \leq C 2^{-n} \sum_{k=2^{n-1}+1}^{2^n} \gamma_k, \quad n \in \mathbb{N}.$$

We observe that the inclusions  $A(\infty) \subset A(\alpha_1) \subset A(\alpha_2)$  for  $\alpha_1 > \alpha_2 \geq 1$  hold.

Let us recall that for a function  $f \in L_p[0, 1]$ ,  $1 \leq p < \infty$ , the norm is defined by the equality  $\|f\|_p = \left( \int_0^1 |f(x)|^p dx \right)^{1/p}$ . Below we shall use the best approximation  $E_n(f)_p = \inf_{\{a_k\}} \|f - t_n\|_p$  of the function  $f \in L_p[0, 1]$  by Haar polynomials  $t_n(x) = \sum_{k=1}^n a_k \chi_k(x)$  of order  $n$ .

**Theorem 2.1.** *Let  $f \in L_p[0, 1]$ ,  $1 \leq p < \infty$ ,  $0 < \beta \leq p$  and  $\gamma \in A(p/(p - \beta))$  for  $0 < \beta < p$  and  $\gamma \in A(\infty)$  for  $\beta = p$ . If*

$$\sum_{k=1}^{\infty} \gamma_k \left[ k^{-1/2} E_k(f)_p \right]^\beta < \infty, \quad (2.1)$$

*then the series*

$$\sum_{k=1}^{\infty} \gamma_k |\hat{f}(k)|^\beta \quad (2.2)$$

*converges.*

From Theorem 1 and the inequality  $E_n(f)_p \leq 2^{1+1/p} \omega(n^{-1}, f)$ ,  $1 \leq p < \infty$ ,  $n \in \mathbb{N}$ , of P.L. Ulyanov (see [5] or [10], Theorem 10.2.3) it follows

**Theorem 2.2.** *The statement of the Theorem 1 is also valid, if instead of the condition (3) we assume*

$$\sum_{k=1}^{\infty} \gamma_k \left[ k^{-1/2} \omega(k^{-1}, f)_p \right]^\beta < \infty.$$

For the function  $f \in V_p[0, 1]$ ,  $1 \leq p < \infty$ , we set

$$\omega_{1-1/p}(\delta, f) = \sup_{\lambda(\tau) \leq \delta} \left( \sum_{i=1}^n |f(x_i) - f(x_{i-1})|^p \right)^{1/p},$$

where  $\tau = \{x_i\}_{i=0}^n$  is a partition of the interval  $[0, 1]$   $\lambda(\tau) = \max_{1 \leq i \leq n} (x_i - x_{i-1})$ . This quantity was introduced by A.P.Terekhin [11]. It is known the inequality  $\omega(\delta, f)_p \leq \delta^{1/p} \omega_{1-1/p}(\delta, f)$  (see [11], Theorem 2.5). Therefore from the Theorem 1 it follows

**Corollary 2.3.** *If  $1 < p < \infty$ ,  $f \in V_p[0, 1]$ , and  $0 < \beta \leq p$ ,  $\gamma \in A(p/(p - \beta))$ ,*

$$\sum_{k=1}^{\infty} \gamma_k \left[ k^{-(1/2+1/p)} \omega_{1-1/p}(k^{-1}, f) \right]^\beta < \infty, \quad (2.3)$$

*then the series  $\sum_{k=1}^{\infty} \gamma_k |\hat{f}(k)|^\beta$  converges.*

For the function  $f \in V_p[0, 1]$ ,  $1 \leq p < \infty$ , we define the norm  $\|f\|_{V_p} = \max(V_p(f), \|f\|_\infty)$ , where  $\|f\|_\infty = \sup\{|f(x)| : x \in [0, 1]\}$ , and the best approximation  $E_n(f)_{V_p} = \inf_{\{a_k\}} \|f - t_n\|_{V_p}$  by Haar polynomials  $t_n(x) = \sum_{k=1}^n a_k \chi_k(x)$  of order  $n$ . It is easy to prove the inequality

$$E_n(f)_p \leq C_p n^{-1/p} E_n(f)_{V_p}, \quad n \in \mathbb{N}.$$

Therefore from Theorem 2.2 it follows

**Corollary 2.4.** *The statement of the Corollary 2.3 is also valid, if instead of the condition (2.3) we assume*

$$\sum_{k=1}^{\infty} \gamma_k \left[ k^{-(1/2+1/p)} E_k(f)_{V_p} \right]^\beta < \infty.$$

The following two theorems show that under some conditions the statement of Theorem 2.1 is sharp.

**Theorem 2.5.** *Let  $1 \leq p < \infty$ ,  $0 < \beta \leq p$ ,  $\gamma \in A(p/(p - \beta))$  (i.e.  $\gamma \in A(\infty)$  for  $\beta = p$ ) and let be given some decreasing and tending to zero sequence  $\varepsilon = \{\varepsilon_i\}_{i=1}^\infty$  satisfying Bary condition*

$$\sum_{i=k}^{\infty} i^{-1} \varepsilon_i = O(\varepsilon_k), \quad k \in \mathbb{N}, \quad (2.4)$$

*and  $\sum_{i=1}^{\infty} \gamma_i i^{-\beta/2} \varepsilon_i^\beta = \infty$ . Then there exists a function  $f_0 \in L_p[0, 1]$  such that*

*$E_n(f_0)_p \leq \varepsilon_n$ ,  $n \in \mathbb{N}$ , and the series  $\sum_{k=1}^{\infty} \gamma_k |\hat{f}(k)|^\beta$  diverges.*

**Theorem 2.6.** *Let  $0 < \beta \leq 1 \leq p < \infty$ . Moreover, let be given a sequence  $\gamma \in A(p/(p - \beta))$  such that*

$$(1 - \alpha) \Gamma_{n+1} 2^{-\beta/2} \geq \Gamma_n := \sum_{k=2^{n-1}+1}^{2^n} \gamma_k, \quad n \in \mathbb{N},$$

for some  $\alpha \in (0, 1)$  and some decreasing and tending to zero sequence  $\varepsilon = \{\varepsilon_k\}_{k=1}^{\infty}$  such that

$$\sum_{k=1}^{\infty} \gamma_k (k^{-1/2} \varepsilon_k)^{\beta} = \infty.$$

Then there exists a function  $f_0 \in L^p[0, 1]$ , such that  $E_n(f_0)_p \leq \varepsilon_n$ ,  $n \in \mathbb{N}$ , and the series  $\sum_{k=1}^{\infty} \gamma_k |\hat{f}(k)|^{\beta}$  diverges.

The following theorem shows that under some conditions the statement of the Corollary 2.4 is sharp.

**Theorem 2.7.** Let  $1 < p < \infty$ ,  $0 < \beta < p$ ,  $\gamma \in A(p/(p - \beta))$ , and let be given a decreasing and tending to zero sequence  $\{\varepsilon_k\}_{k=1}^{\infty}$  satisfying Bary condition (2.4) and such that  $\sum_{k=1}^{\infty} \gamma_k [k^{-(1/2+1/p)} \varepsilon_k]^{\beta} = \infty$ . Then there exists a function  $f_0 \in V_p[0, 1]$ , such that  $E_n(f)_{V_p} \leq \varepsilon_n$ ,  $n \in \mathbb{N}$ , and the series  $\sum_{k=1}^{\infty} \gamma_k |\hat{f}(k)|^{\beta}$  diverges.

### 3. Two-dimensional results

Let be given a function of two variables  $f \in L_p(\Delta)$ ,  $1 \leq p < \infty$ , where  $\Delta = [0, 1] \times [0, 1]$  is the unit square. We introduce two partial and one mixed modulus of continuity in  $L_p$ -metric

$$\omega_1(\delta, f)_p = \sup_{0 \leq h \leq \delta} \left( \int_0^{1-h} \left( \int_0^1 |f(x+h, y) - f(x, y)|^p dy \right) dx \right)^{1/p},$$

$$\omega_2(\delta, f)_p = \sup_{0 \leq h \leq \delta} \left( \int_0^1 \left( \int_0^{1-h} |f(x, y+h) - f(x, y)|^p dx \right) dy \right)^{1/p},$$

$$\omega_{1,2}(\delta_1, \delta_2, f)_p = \sup_{0 \leq u \leq \delta_1} \sup_{0 \leq v \leq \delta_2} \left( \int_0^{1-u} \left( \int_0^{1-v} |\Delta_{u,v} f(x, y)|^p dy \right) dx \right)^{1/p},$$

where  $0 \leq \delta \leq 1$ ,  $0 \leq \delta_i \leq 1$ ,  $i = 1, 2$ , and

$$\Delta_{u,v} f(x, y) = f(x+u, y+v) - f(x+u, y) - f(x, y+v) + f(x, y)$$

is mixed difference of the function  $f$  with steps  $u, v$ . Let us set

$$\hat{f}(m, n) = \int \int_{\Delta} f(x, y) \chi_m(x) \chi_n(y) dx dy,$$

i.e.  $\hat{f}(m, n)$  are Fourier coefficients of the function  $f$  with respect to two-dimensional Haar system  $\{\chi_m(x) \chi_n(y)\}_{m,n=1}^{\infty}$ .

F.Moricz and A.Veres [12] introduced the foolowing two-dimensional analog  $A^*(\alpha, 2)$  of one-dimensional class  $A(\alpha)$ ,  $1 \leq \alpha < \infty$ . A double positive sequence  $\gamma = \{\gamma_{ij}\}_{i,j=1}^{\infty}$  belongs to the class  $A^*(\alpha, 2)$  if for every  $k, l \in \mathbb{Z}_+$  the inequality

$$\left( \sum_{i=2^k+1}^{2^{k+1}} \sum_{j=2^l+1}^{2^{l+1}} \gamma_{ij}^{\alpha} \right)^{1/\alpha} \leq C 2^{(k+l)(1-\alpha)/\alpha} \sum_{i=2^{k-1}+1}^{2^k} \sum_{j=2^{l-1}+1}^{2^l} \gamma_{ij} =: C 2^{(k+l)(1-\alpha)/\alpha} \Gamma_{kl}^* (3.1)$$

holds. In the case  $k = 0$ ,  $l \in \mathbb{N}$ , or  $l = 0$ ,  $k \in \mathbb{N}$ , we set  $\Gamma_{0,l}^* = \sum_{j=2^{l-1}+1}^{2^l} \gamma_{1,j}$  or  $\Gamma_{k,0}^* = \sum_{j=2^{k-1}+1}^{2^k} \gamma_{j,1}$  respectively. For  $k = l = 0$  we set  $\Gamma_{00}^* = \gamma_{11}$ . The class  $A^*(\infty, 2)$  is defined in a similar manner.

The following theorem is a two-dimensional analog of Theorem 2.2.

**Theorem 3.1.** *Let us set  $1 \leq p < \infty$ ,  $0 < \beta \leq p$ ,  $f \in L_p[0, 1]^2$ ,  $\gamma = \{\gamma_{mn}\}_{m,n=1}^\infty \in A^*(p/(p-\beta), 2)$ ,*

$$\gamma^{(1)} := \{\gamma_{m1}\}_{m=1}^\infty \in A(p/(p-\beta)), \quad \gamma^{(2)} := \{\gamma_{1n}\}_{n=1}^\infty \in A(p/(p-\beta)),$$

*and suppose that the conditions*

$$\sum_{m=1}^\infty \gamma_{m1} m^{-\beta/2} \omega_1^\beta(m^{-1}, f)_p < \infty, \quad \sum_{n=1}^\infty \gamma_{1n} n^{-\beta/2} \omega_2^\beta(n^{-1}, f)_p < \infty,$$

*and*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \gamma_{mn} (mn)^{-\beta/2} \omega_{1,2}^\beta(m^{-1}, n^{-1}, f)_p < \infty$$

*hold. Then*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \gamma_{mn} |\hat{f}(m, n)|^\beta < \infty.$$

From this theorem two-dimensional analogs of Corollaries 1 and 2 follow. Let us introduce the definition of the functions of bounded  $p$ -variation in the Hardy sense on the unit square  $\Delta = [0, 1]^2$ . Let be given two systems of points

$$\tau_1 = \{x_0 < x_1 < \dots < x_r = 1\}, \quad \text{and} \quad \tau_2 = \{y_0 < y_1 < \dots < y_l = 1\}.$$

If for a function  $f$  defined on the square  $[0, 1]^2$  the following three quantities

$$V_p(f)_{1,0} = \sup_{0 \leq y \leq 1} \sup_{\tau_1} \left( \sum_{j=1}^r |f(x_j, y) - f(x_{j-1}, y)|^p \right)^{1/p},$$

$$V_p(f)_{0,1} = \sup_{0 \leq x \leq 1} \sup_{\tau_2} \left( \sum_{i=1}^l |f(x, y_i) - f(x, y_{i-1})|^p \right)^{1/p},$$

$$V_p(f)_{1,1} = \sup_{\tau_1, \tau_2} \left( \sum_{j=1}^r \sum_{i=1}^l |f(x_j, y_i) - f(x_{j-1}, y_i) - f(x_j, y_{i-1}) + f(x_{j-1}, y_{i-1})|^p \right)^{1/p},$$

are finite for some  $p \in [1, \infty)$ , then we shall write  $f \in V_p(\Delta)$ . Such function  $f$  is called the function of bounded  $p$ -variation in Hardy sense on the square  $[0, 1]^2$ . Let us note that the class  $V_1(\Delta)$  was introduced by G.H.Hardy [13].

**Corollary 3.2.** *Let  $f \in V_p(\Delta)$ ,  $1 \leq p < \infty$  and  $\beta > 0$ ,  $\gamma + 1 < \beta(1/2p + 1/2)$ . Then*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty (mn)^\gamma |\hat{f}(m, n)|^\beta < \infty.$$

This statement was proved by the first author in the paper [7], where also it was shown that for  $\beta > 0$ ,  $\gamma + 1 = \beta(1/2p + 1/2)$ , it is not true.

As usually we write  $f \in Lip(\alpha, \Delta)$ ,  $0 < \alpha \leq 1$ , for the function  $f$  defined on the unit square  $\Delta = [0, 1]^2$ , if

$$\sup_{\rho(A', A'') \leq \delta} |f(A') - f(A'')| \leq C\delta^\alpha,$$

where

$$\rho(A', A'') = ((x' - x'')^2 + (y' - y'')^2)^{1/2} \quad \text{for } A'(x', y'), A''(x'', y'') \in \Delta.$$

It is easy to see that  $Lip(\alpha, \Delta) \subset V_{2/\alpha}(\Delta)$ ,  $0 < \alpha \leq 1$ . Therefore from the Corollary 3 it follows

**Corollary 3.3.** *If  $f \in Lip(\alpha, \Delta)$ ,  $0 < \alpha \leq 1$  and  $\gamma + 1 < \beta(\alpha + 1)/2$ ,  $\beta > 0$ , then*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^\gamma |\hat{f}(m, n)|^\beta < \infty.$$

The statement of this Corollary is known (see [14], where it is proved that in the case  $\gamma + 1 = \beta(\alpha + 1)/2$ ,  $\beta > 0$ , it is not true.) For  $\gamma = 0$  Corollary 4 was proved earlier in the paper [15], where also it is proved that in the case  $\beta(\alpha + 1)/2 = 1$ ,  $\beta > 0$ , it does not true.

Let us introduce the classes  $PBV_p[0, 1]^2$ ,  $1 \leq p < \infty$ , of functions of partial bounded  $p$ -variation on the square  $\Delta$  (see [16]). By definition,  $f \in PBV_p[0, 1]^2$ , if  $V_p(f)_{1,0} < \infty$  and  $V_p(f)_{0,1} < \infty$ . It is evident that  $V_p[0, 1]^2 \subset PBV_p[0, 1]^2$ , if  $1 \leq p < \infty$ .

For the function  $f \in B(\Delta)$ ,  $1 \leq p < \infty$ , we define three moduli of continuity of fractional order as follows

$$\omega_{1-1/p}(f, \delta)_{1,0} = \sup_{0 \leq y \leq 1} \sup_{\lambda(\tau_1) \leq \delta} \left( \sum_{j=1}^r |f(x_j, y) - f(x_{j-1}, y)|^p \right)^{1/p},$$

$$\omega_{1-1/p}(f, \delta)_{0,1} = \sup_{0 \leq x \leq 1} \sup_{\lambda(\tau_2) \leq \delta} \left( \sum_{i=1}^l |f(x, y_i) - f(x, y_{i-1})|^p \right)^{1/p},$$

According to the inequality by A.P.Terekhin for functions  $f \in PBV_p[0, 1]^2$  and  $0 < h \leq \delta$  the inequalities

$$\int_0^{1-h} |f(x+h, y) - f(x, y)|^p dx \leq \delta \omega_{1-1/p}^p(\delta, f(\cdot, y)) \leq \delta \omega_{1-1/p}^p(\delta, f)_{1,0}, \quad y \in [0, 1],$$

$$\int_0^{1-h} |f(x, y+h) - f(x, y)|^p dx \leq \delta \omega_{1-1/p}^p(\delta, f(x, \cdot)) \leq \delta \omega_{1-1/p}^p(\delta, f)_{0,1}, \quad x \in [0, 1],$$

hold. From these inequalities we obtain

$$\omega_1(\delta, f)_p \leq \delta^{1/p} \omega_{1-1/p}(\delta, f)_{1,0}, \quad \omega_2(\delta, f)_p \leq \delta^{1/p} \omega_{1-1/p}(\delta, f)_{0,1}, \quad \delta \in [0, 1],$$

$$\omega_1(\delta, f)_p \leq \delta^{1/p} V_p(f)_{1,0}, \quad \omega_2(\delta, f)_p \leq \delta^{1/p} V_p(f)_{0,1}, \quad \delta \in [0, 1].$$

These inequalities and Theorem 3.1 imply

**Corollary 3.4.** *If  $f \in PBV_p[0, 1]^2$ ,  $1 \leq p < \infty$ ,  $0 < \beta \leq p$ ,*

*$\gamma = \{\gamma_{mn}\}_{m,n=1}^\infty \in A^*(p/(p-\beta), 2)$ ,  $\gamma^{(1)} = \{\gamma_{m1}\}_{m=1}^\infty \in A(p/(p-\beta))$ ,  $\gamma^{(2)} = \{\gamma_{1n}\}_{n=1}^\infty \in A(p/(p-\beta))$ , and the series*

$$\sum_{m=1}^\infty \gamma_{m1} m^{-\beta/2-\beta/p} \omega_{1-1/p}^\beta(1/m, f)_{1,0}, \quad \sum_{n=1}^\infty \gamma_{1n} n^{-\beta/2-\beta/p} \omega_{1-1/p}^\beta(1/n, f)_{0,1},$$

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \gamma_{mn} (mn)^{-\beta/2-\beta/p} (\omega_{1-1/p}(1/m, f)_{1,0} \omega_{1-1/p}(1/n, f)_{0,1})^{\beta/2},$$

*converge. Then*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \gamma_{mn} |\hat{f}(m, n)|^\beta < \infty.$$

**Corollary 3.5.** *If  $f \in PBV_p[0, 1]^2$ ,  $1 \leq p < \infty$ ,  $0 < \beta \leq p$ ,*

*$\gamma = \{\gamma_{mn}\}_{m,n=1}^\infty \in A^*(p/(p-\beta), 2)$ ,  $\gamma^{(1)} = \{\gamma_{m1}\}_{m=1}^\infty \in A(p/(p-\beta))$ ,  $\gamma^{(2)} = \{\gamma_{1n}\}_{n=1}^\infty \in A(p/(p-\beta))$ , and the series*

$$\sum_{m=1}^\infty \gamma_{m1} m^{-\beta/2-\beta/p}, \quad \sum_{n=1}^\infty \gamma_{1n} n^{-\beta/2-\beta/p}, \quad \sum_{m=1}^\infty \sum_{n=1}^\infty \gamma_{mn} (mn)^{-\beta/2-\beta/p}$$

*converge. Then*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \gamma_{mn} |\hat{f}(m, n)|^\beta < \infty.$$

**Corollary 3.6.** *Let  $f \in PBV_p[0, 1]^2$ ,  $1 \leq p < \infty$  and  $0 < \beta \leq p$ ,  $\alpha + 1 < \beta(1/2p + 1/2)$ .*

*Then  $\sum_{m=1}^\infty \sum_{n=1}^\infty (mn)^\alpha |\hat{f}(m, n)|^\beta < \infty$ .*

Corollary 7 was proved earlier by A. Aplakov [17].

If  $\mathcal{P}_{m,n} = \{f \in L^1[0, 1]^2 : \hat{f}(i, j) = 0, \text{ for } i > m \text{ or } j > n\}$ , then we set  $E_{m,n}(f)_p = \inf\{\|f - g\|_p : g \in \mathcal{P}_{m,n}\}$ .

In the paper [18] also the class  $A(\alpha, 2)$  was introduced as follows. Let us set  $R_l = \{(i, j) : 1 \leq i, j \leq 2^l\} \setminus \{(i, j) : 1 \leq i, j \leq 2^{l-1}\}$  for  $l \in \mathbb{N}$ ,  $R_0 = \{(1, 1)\}$  and  $\Gamma_l^{(2)} = \sum_{(i,j) \in R_l} \gamma_{i,j}$   $l \in \mathbb{Z}_+ = \{0, 1, \dots\}$ . A positive double sequence  $\gamma = \{\gamma_{i,j}\}_{i,j=1}^\infty$

belongs to the class  $A(\alpha, 2)$ ,  $1 \leq \alpha < \infty$ , if there is a constant  $C > 0$  such that

$\left( \sum_{(i,j) \in R_l} \gamma_{i,j}^\alpha \right)^{1/\alpha} \leq C 2^{l(1/\alpha-1)} \Gamma_{l-1}^{(2)}$ ,  $l \in \mathbb{N}$ , and  $\gamma_{2,2} \leq C \gamma_{1,1}$ . In the paper [18] the inclusion  $\{i^\gamma j^\delta\}_{i,j=1}^\infty \in A(\alpha, 2)$  was proved for  $\gamma, \delta \geq 0$ .

**Theorem 3.7.**  $1 \leq p < \infty$ ,  $r = \min(p, 2)$ ,  $1/r + 1/r' = 1$ ,  $0 < \beta < 2$ ,  $f \in L^p[0, 1]^2$ ,  $\gamma \in A(2/(2-\beta), 2)$  and the series  $\sum_{l=0}^\infty 2^{-2l\beta/r'} E_{2^l, 2^l}^\beta(f)_p \Gamma_l^{(2)}$  converges. Then the assertion

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \gamma_{mn} |\hat{f}(m, n)|^\beta < \infty. \quad (3.2)$$

*is valid.*

The following theorem shows that the statement of the Theorem 3.7 under some conditions is sharp.

**Theorem 3.8.** *Let  $0 < \beta < 2$ ,  $\gamma \in A(2/(2 - \beta), 2)$  and  $\{\varepsilon_n\}_{n=1}^\infty$  be a decreasing and tending to zero sequence satisfying Bary condition (2.4) and such that  $\sum_{l=0}^\infty 2^{-l\beta} \varepsilon_{2^l}^\beta \Gamma_l^{(2)} = \infty$ . Then for each  $p \in [1, \infty)$  there exists a function  $f_0 \in L_p[0, 1]^2$ , such that  $E_{k,k}(f_0)_p \leq \varepsilon_k$ ,  $k \in \mathbb{N}$ , for which the condition (3.2) is not valid.*

**Theorem 3.9.** *Let  $1 \leq p < \infty$ ,  $0 < \beta, r < p$ ,  $f \in C[0, 1]^2 \cap V_{p-r}[0, 1]^2$ , and the following inclusions*

$$\gamma \in A^*(p/(p - \beta), 2), \quad \gamma^{(1)}, \gamma^{(2)} \in A(p/(p - \beta)),$$

*hold. If the series*

$$\sum_{m=1}^\infty \gamma_m m^{-\beta(1/2+1/p)} \omega_1^{r\beta/p}(m^{-1}, f)_\infty < \infty, \quad \sum_{n=1}^\infty \gamma_n n^{-\beta(1/2+1/p)} \omega_2^{r\beta/p}(n^{-1}, f)_\infty < \infty$$

*and*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \gamma_{mn} (mn)^{-\beta(1/p+1/2)} \omega_{1,2}^{r\beta/p}(m^{-1}, n^{-1}, f)_\infty < \infty,$$

*converge, then the inequality (3.2) is valid.*

Let us recall that the double series  $\sum_{i=1}^\infty \sum_{j=1}^\infty a_{i,j}$  converges by squares if there exists the finite limit  $\lim_{n \rightarrow +\infty} \sum_{i=1}^n \sum_{j=1}^n a_{i,j}$ . The following theorem is a two-dimensional analog of the Theorem D of P.L.Ulyanov, which was formulated in the Introduction.

**Theorem 3.10.** *Let  $1 \leq p < \infty$ ,  $f \in L_p[0, 1]^2$ , and the series*

$$\sum_{i=1}^\infty \sum_{j=1}^\infty (ij)^{1/p-1} E_{i,j}(f)_p$$

*converges. Then the series  $\sum_{i=1}^\infty \sum_{j=1}^\infty |\hat{f}(i, j)| |\chi_i(x) \chi_j(y)|$  uniformly converges by squares on  $[0, 1]^2$ .*

In the following theorem we give an inequality for double Haar polynomials, which is an analog of the Nikolskii inequality for trigonometric polynomials (see [19], Theorem 3.3.5).

**Theorem 3.11.**  *$g \in \mathcal{P}_{m,n}$ ,  $m, n \in \mathbb{N}$ ,  $1 \leq p < q \leq \infty$ .  $\|g\|_q \leq C(mn)^{1/p-1/q} \|g\|_p$ .*

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