

VANISHING MOMENTS OF HILBERT TRANSFORM OF WAVELETS

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Abstract. A wavelet is a localised function having a prescribed number of vanishing moments. Hilbert transform of a wavelet is again a wavelet. In this paper, we discuss the higher vanishing moments of Hilbert transform of wavelets under certain conditions on wavelet and its Hilbert transform.

1. Introduction

Hilbert transform and its basic properties were developed mainly by G.H. Hardy [2, 3] and simultaneously by E.C. Titchmarsh [9, 10]. Hilbert transforms were initially defined for periodic functions. Hilbert transform is a basic tool in Fourier Analysis which provides a definite means for obtaining the harmonic conjugate of a given function or fourier series. It plays an important role in Fluid Mechanics, Aerodynamics, Signal Processing and Electronics. The theory of Hilbert transform also finds a number of applications in pure mathematics. It has been observed in [8] that if $\psi(t)$ is a real wavelet, then Hilbert transform of $\psi(t)$ is also a real wavelet with same energy and admissibility coefficient of its generating wavelet. The higher vanishing moments of wavelets under various conditions on $\psi(x)$ have been investigated in [1, 4, 11].

The aim of this paper is to study the higher vanishing moments of Hilbert transform of wavelets under certain conditions. We begin with the following definition of Hilbert transform of order n given in [6].

For a non-negative integer n , the Hilbert transform of n^{th} order is given by

$$\mathcal{H}^n f(x) = \begin{cases} (-1)^{n/2} f(x), & \text{if } n \text{ is even} \\ (-1)^{(n-1)/2} \mathcal{H} f(x), & \text{if } n \text{ is odd,} \end{cases} \quad (1.1)$$

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where $\mathcal{H}f(x) \in \mathcal{L}^p(\mathbb{R})$ for $f(x) \in \mathcal{L}^p(\mathbb{R})$, $p > 1$.

If $n = 1$, we obtain Hilbert transform of order 1.

One may observe that the Hilbert transform of n^{th} order, $n \in \mathbb{N}$ satisfies the following properties:

- (1) It commutes with complex conjugation i.e., $\mathcal{H}^n f^* = (\mathcal{H}f^*)^n$, where $*$ denotes the complex conjugate operation.
- (2) It commutes with both the translation operator T_a and dialation operator D_a , $a > 0$ i.e., $\mathcal{H}^n T_a f(x) = T_a \mathcal{H}^n f(x)$ and $\mathcal{H}^n D_a f(x) = D_a \mathcal{H}^n f(x)$, where the translation operator T_a and the dialation operator D_a are respectively given by $T_a f(x) = f(x - a)$ and $D_a f(x) = f(ax)$.
- (3) It commutes with the m^{th} order differential operator $\frac{d^m}{dx^m} \mathcal{H}^n f(x) = \mathcal{H}^n \frac{d^m}{dx^m} f(x)$.
- (4) It preserves the orthogonality of wavelets in $\mathcal{L}^2(\mathbb{R})$.

Remark 1.1. Relationship between Fourier transform and Hilbert transform is as follows:

(for details one may refer to [6, 11])

- (i) For $f \in \mathcal{L}^p(\mathbb{R})$, $1 < p \leq 2$, $\widehat{\mathcal{H}f}(x) = -i \operatorname{sgn}(x) \hat{f}(x)$, where $\operatorname{sgn}(x)$ denotes the signum function.
- (ii) $x^k \mathcal{H}\psi(x)(\xi) = \left(\frac{i}{2\pi}\right)^k \frac{d^k}{d\xi^k} \widehat{\mathcal{H}\psi}(\xi)$, where $k \in \mathbb{N}$.

2. Vanishing Moments of n^{th} order Hilbert Transform

We begin this section with the following definition of vanishing moments given in [11].

A function $f(x)$ is said to have k *vanishing moments* if $\int_{\mathbb{R}} x^j f(x) dx = 0$, $0 \leq j \leq k-1$, where the given integral is called the j^{th} moment of $f(x)$.

Vanishing moments are important in compression. The wavelet series of a smooth function f given by

$$f = \sum_{j,k \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k},$$

converges very rapidly to the function f if the wavelet is with a lot of vanishing moments. Vanishing moments are important because a wavelet having k vanishing moments forces all wavelet coefficients $\langle f, \psi_{j,k} \rangle$ of any polynomial of degree k or less to vanish. In fact, if a smooth function is interrupted by occasional singularity (or discontinuity), then the wavelet coefficients on the smooth parts will be very small or zero if at that point the function behaves like a polynomial of certain order. This property is sometimes useful for data compression. Also in case of statistical estimation problem, vanishing moments are advantageous as only a few coefficients are needed to be estimated.

In this section, we shall first investigate the following problem:

Problem: Let $\mathcal{H}^n$ be an n^{th} order Hilbert transform such that $\psi, \hat{\psi} \in \mathcal{L}^1(\mathbb{R})$ and

$\{\mathcal{H}^n \psi_{j,k}\}_{j,k \in \mathbb{Z}}$ be an orthogonal system on $\mathbb{R}$. Is

$$\int_{\mathbb{R}} \mathcal{H}^n \psi(x) dx = 0, \text{ for all } n \in \mathbb{N}.$$

We shall investigate this problem via some examples given below.

Example 2.1. The Haar wavelet

$$\psi(x) = \begin{cases} 1, & \text{if } 0 \leq x < \frac{1}{2} \\ -1, & \text{if } \frac{1}{2} \leq x < 1 \\ 0, & \text{if otherwise.} \end{cases}$$

Example 2.2. The Daubechies wavelet of order 3.

The scaling function, φ is defined by $\varphi(x) = \sum_{k=0}^3 h_k \varphi(2x - k)$, where $\varphi(x)$ has its support in $[0, 3]$.

Note that the wavelet, ψ is defined by $\psi(x) = \sum_{k=-2}^1 (-1)^k h_{1-k} \varphi(2x - k)$, where $\psi(x)$ has its support in $[-1, 2]$ and $\sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} h_{2k} = 1 = \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} h_{2k+1}$.

In Examples 2.1 and 2.2, one may note that both Haar wavelet and Daubechies wavelet of order 3 have finite support.

In the following example, we shall investigate the problem in case of a wavelet having infinite support and found that the answer is still affirmative.

Example 2.3. The Meyer Wavelet is infinitely differentiable with infinite support and defined in frequency domain in terms of function ν by

$$\hat{\psi}(\omega) = \begin{cases} \frac{1}{\sqrt{(2\pi)}} e^{\frac{i\omega}{2}} \sin\left[\frac{\pi\nu}{2} \left(\frac{3|\omega|}{2\pi} - 1\right)\right], & \text{if } \frac{2\pi}{3} < |\omega| < \frac{4\pi}{3} \\ \frac{1}{\sqrt{(2\pi)}} e^{\frac{i\omega}{2}} \cos\left[\frac{\pi\nu}{2} \left(\frac{3|\omega|}{4\pi} - 1\right)\right], & \text{if } \frac{4\pi}{3} < |\omega| < \frac{8\pi}{3} \\ 0, & \text{if otherwise,} \end{cases}$$

where

$$\nu(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 1 \end{cases}$$

and $\nu(x) + \nu(1-x) = 1$.

It is now natural to ask whether the Examples 2.1-2.3 are true in general.

The following result addresses this query and gives the sufficient conditions for the zeroth order vanishing moment of $\mathcal{H}^n \psi$.

Theorem 2.4. Let $\{\mathcal{H}^n \psi_{j,k}\}_{j,k \in \mathbb{Z}}$ be an orthogonal system on $\mathbb{R}$, where $\mathcal{H}^n$ denotes the n^{th} order Hilbert transform, $n \in \mathbb{N}$ and $\psi, \hat{\psi} \in \mathcal{L}^1(\mathbb{R})$. Then

$$\int_{\mathbb{R}} \mathcal{H}^n \psi(x) dx = 0, \text{ for all } n \in \mathbb{N}$$

Proof. Suppose first that n is even.

By hypothesis $\int_{\mathbb{R}} \mathcal{H}^n \psi(x) \overline{\mathcal{H}^n \psi_{j,k}(x)} dx = 0$, if $(j, k) \neq (0, 0)$.

This gives

$$(-1)^n \int_{\mathbb{R}} \psi(x) 2^{\frac{j}{2}} \overline{\psi(2^j x - k)} dx = 0. \quad (2.1)$$

Taking $k = 2^{j-j_1} k_1$, for all $j \in \mathbb{N}$ and $j > j_1$, where $x_0 = 2^{-j_1} k_1$ is a dyadic point such that $(j_1, k_1) \neq (0, 0)$.

This gives

$$\int_{\mathbb{R}} \psi(x) \overline{\psi(2^j(x - 2^{-j_1} k_1))} dx = 0.$$

Write $2^j(x - 2^{-j_1} k_1) = t$. Then

$$\int_{\mathbb{R}} \psi(x_0 + 2^{-j} t) \overline{\psi(t)} dt = 0.$$

Taking $j \rightarrow \infty$, we have

$$\psi(x_0) \int_{\mathbb{R}} \overline{\psi(t)} dt = 0.$$

Now since ψ is continuous so $\psi(x_0) \neq 0$. Thus

$$\int_{\mathbb{R}} \mathcal{H}^n \psi(x) dx = 0.$$

Now suppose that n is odd. Then

$$\int_{\mathbb{R}} \mathcal{H}^n \psi(x) dx = 0.$$

Alternate proof of Theorem 2.4. Let if possible $\int_{\mathbb{R}} \mathcal{H}^n \psi(x) dx \neq 0$ if n is even.

Then by normalisation, we can assume that

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{H}^n \psi(x) dx &= 1 \\ \implies (-1)^{\frac{n}{2}} \int_{\mathbb{R}} \psi(x) dx &= 1. \end{aligned}$$

Then the collection of functions $\{(-1)^{\frac{n}{2}} 2^{\frac{j}{2}} \overline{\psi_{j,0}(x)} : j \in \mathbb{N}\}$ forms an approximate identity on $\mathbb{R}$.

This yields

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}} \psi(x) (-1)^{\frac{n}{2}} 2^{\frac{j}{2}} \overline{\psi_{j,0}(x)} dx = \psi(0). \quad (2.2)$$

Using the orthogonality of $\{\mathcal{H}^n \psi_{j,k}\}_{j,k \in \mathbb{Z}}$ on $\mathbb{R}$, we get

$$\int_{\mathbb{R}} \mathcal{H}^n \psi(x) \overline{\mathcal{H}^n [(-1)^{\frac{n}{2}} 2^{\frac{j}{2}} \psi_{j,0}(x)]} dx = 0.$$

This gives

$$\int_{\mathbb{R}} \psi(x) (-1)^{\frac{n}{2}} 2^{\frac{j}{2}} \overline{\psi_{j,0}(x)} dx = 0. \quad (2.3)$$

Then, by (2.2) and (2.3), $\psi(0) = 0$. Since dyadic rationals are dense in $\mathbb{R}$, we shift the argument to the dyadic rational $x_0 = 2^{j_0} k_0$, where $j_0, k_0 \in \mathbb{Z}$. Again using the orthogonality of $\{\mathcal{H}^n \psi_{j,k}\}_{j,k \in \mathbb{Z}}$, $\int_{\mathbb{R}} \mathcal{H}^n \psi(x) \mathcal{H}^n [(-1)^{\frac{n}{2}} 2^{\frac{j}{2}} \overline{\psi_{j,2^{j_0} k_0}(x)}] dx = 0$.

This gives

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}} \psi(x) 2^{j/2} \overline{\psi_{j,0}(2^j x - 2^{j+j_0} k_0)} dx = 0.$$

Therefore, $\psi(x_0) = 0$ at every dyadic point. This is a contradiction since ψ is continuous. The case when n is odd can be worked out similarly.

Hence $\int_{\mathbb{R}} \mathcal{H}^n \psi(x) dx = 0$, for all $n \in \mathbb{N}$. $\square$

The following result extends Theorem 2.4 to general moments of the first order Hilbert transform of wavelets, $\mathcal{H}\psi$.

Theorem 2.5. *Let $\psi, \hat{\psi} \in \mathcal{L}^1(\mathbb{R})$ be such that for some $p \in \mathbb{N}$, both $x^p \mathcal{H}\psi(x)$ and $\gamma^{p+1}(\widehat{\mathcal{H}\psi})(\gamma) \in \mathcal{L}^1(\mathbb{R})$. If $\{\mathcal{H}\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ is an orthogonal system on $\mathbb{R}$, then*

$$\int_{\mathbb{R}} x^q \mathcal{H}\psi(x) dx = 0 \text{ for } 0 \leq q \leq p. \quad (2.4)$$

Proof. Since $(\widehat{\mathcal{H}\psi})^{(p+1)}(\gamma) = (2\pi\gamma)^{p+1} \widehat{(\mathcal{H}\psi)}(\gamma)$, $\mathcal{H}\psi$ has $p+1$ continuous derivatives and $(\widehat{\mathcal{H}\psi})^{(p+1)}(\gamma)$ vanishes at infinity.

We prove the theorem by principle of mathematical induction on q . For $q = 0$, the result is obvious. Assume that

$$\int_{\mathbb{R}} x^q \mathcal{H}\psi(x) dx = 0, \quad 0 \leq q \leq k-1 \text{ for some } k \leq p. \quad (2.5)$$

Since $\mathcal{H}\psi$ has $p+1$ continuous derivatives, $\widehat{\mathcal{H}\psi}$ can be expanded using Taylor's theorem. Again since $(\hat{f})^{(n)}(0) = (-2\pi i)^n \int_{\mathbb{R}} x^n f(x) dx$, $n \in \mathbb{N}$, we obtain

$$(\widehat{\mathcal{H}\psi})(\gamma) = \frac{1}{k!} \gamma^k (\widehat{\mathcal{H}\psi})^{(k)}(0) + R_k(\gamma), \quad (2.6)$$

where $R_k(\gamma) = \frac{1}{(k+1)!} \gamma^{k+1} (\widehat{\mathcal{H}\psi})^{(k+1)}(\xi)$ for some ξ between 0 and γ .

Note that

$$\begin{aligned} |R_k(\gamma)| &\leq \max_{\gamma \in \mathbb{R}} \frac{|(\widehat{\mathcal{H}\psi})^{(k+1)}(\xi)| |\gamma|^{k+1}}{(k+1)!} \\ &= K_{k,\psi} |\gamma|^{k+1}, \text{ where } K_{k,\psi} = \max_{\gamma \in \mathbb{R}} \frac{|(\widehat{\mathcal{H}\psi})^{(k+1)}(\xi)| |\gamma|^{k+1}}{(k+1)!}. \end{aligned} \quad (2.7)$$

Choose a dyadic point $x_0 = 2^{j_0}k_0$ such that $(j_0, k_0) \neq (0, 0)$. Since $\mathcal{H}\psi$ has continuous derivatives,

$$(\mathcal{H}\psi)^{(k)}(x_0) \neq 0. \quad (2.8)$$

Using the orthogonality of $\{\mathcal{H}\psi_{j,k}\}_{j,k \in \mathbb{Z}}$, we have

$$\int_{\mathbb{R}} \mathcal{H}\psi(x) \overline{\mathcal{H}\psi_{j,k}(x)} dx = 0 \text{ for } j \neq 0 \text{ or } k \neq 0$$

Also, using Parseval's formula, we obtain

$$\begin{aligned} & \int_{\mathbb{R}} (\widehat{\mathcal{H}\psi})(\gamma) (-i \operatorname{sgn}(\gamma)) \overline{\widehat{\psi_{j,k}}(\gamma)} d\gamma = 0 \\ \implies & \int_{\mathbb{R}} (\widehat{\mathcal{H}\psi})(\gamma) 2^{\frac{-j}{2}} e^{2\pi i(2^{-j}k)\gamma} \overline{(\widehat{\mathcal{H}\psi})(2^{-j}\gamma)} d\gamma = 0. \end{aligned}$$

Choose $k_j \in \mathbb{Z}$ such that $2^{-j}k_j = 2^{j_0}k_0 = x_0$, for all $j \in \mathbb{N}$, where x_0 is a dyadic point. This is true for $j > -j_0$. Then using (2.6), we get

$$\int_{\mathbb{R}} (\widehat{\mathcal{H}\psi})(\gamma) 2^{\frac{-j}{2}} e^{2\pi i(2^{-j}k_j)\gamma} \overline{\left[\frac{1}{k!} 2^{-jk} \gamma^k (\widehat{\mathcal{H}\psi})^{(k)}(0) + R_k(2^{-j}\gamma) \right]} d\gamma = 0 \quad (2.9)$$

This gives

$$2^{\frac{-j}{2}} \left[\frac{1}{k!} 2^{-jk} \overline{(\widehat{\mathcal{H}\psi})^{(k)}(0)} \int_{\mathbb{R}} (\widehat{\mathcal{H}\psi})(\gamma) \gamma^k e^{2\pi i x_0 \gamma} d\gamma + \int_{\mathbb{R}} (\widehat{\mathcal{H}\psi})(\gamma) \overline{R_k(2^{-j}\gamma)} e^{2\pi i x_0 \gamma} d\gamma \right] = 0. \quad (2.10)$$

Note that

$$\begin{aligned} |J_2| &= \left| \int_{\mathbb{R}} (\widehat{\mathcal{H}\psi})(\gamma) \overline{R_k(2^{-j}\gamma)} e^{2\pi i x_0 \gamma} d\gamma \right| \\ &\leq K_{k,\psi} 2^{-j(k+1)} \int_{\mathbb{R}} |\gamma|^{k+1} |(\widehat{\mathcal{H}\psi})(\gamma)| d\gamma \\ &= K_{J_2} 2^{-j(k+1)}, \end{aligned}$$

Hence

$$|J_2| \leq K_{J_2} 2^{-j(k+1)}. \quad (2.11)$$

Also

$$\begin{aligned} J_1 &= \frac{1}{k!} 2^{-jk} \overline{(\widehat{\mathcal{H}\psi})^{(k)}(0)} \int_{\mathbb{R}} (\widehat{\mathcal{H}\psi})(\gamma) \gamma^k e^{2\pi i x_0 \gamma} d\gamma \\ &= \frac{1}{(2\pi i)^k} \frac{1}{k!} 2^{-jk} \overline{(\widehat{\mathcal{H}\psi})^{(k)}(0)} \int_{\mathbb{R}} (\widehat{\mathcal{H}\psi})(\gamma) (2\pi i \gamma)^k e^{2\pi i x_0 \gamma} d\gamma. \end{aligned}$$

Since $\widehat{f^{(p)}}(\gamma) = (2\pi i\gamma)^p \widehat{f}(\gamma)$,

$$\begin{aligned} J_1 &= \frac{1}{(2\pi i)^k} \frac{1}{k!} 2^{-jk} \overline{\widehat{(\mathcal{H}\psi)}^{(k)}(0)} \int_{\mathbb{R}} (\widehat{\mathcal{H}\psi}^{(k)}(\gamma)) e^{2\pi i x_0 \gamma} d\gamma \\ &= \frac{1}{(2\pi i)^k} \frac{1}{k!} 2^{-jk} \overline{\widehat{(\mathcal{H}\psi)}^{(k)}(0)} (\mathcal{H}\psi)^{(k)}(x_0) d\gamma. \end{aligned}$$

This gives

$$J_1 = K_{J_1} 2^{-jk} \overline{\widehat{(\mathcal{H}\psi)}^{(k)}(0)} (\mathcal{H}\psi)^{(k)}(x_0), \quad (2.12)$$

where $K_{J_1} = \frac{1}{(2\pi i)^k} \frac{1}{k!}$. From (2.10) and (2.12), we obtain

$$K_{J_1} 2^{-jk} \overline{\widehat{(\mathcal{H}\psi)}^{(k)}(0)} (\mathcal{H}\psi)^{(k)}(x_0) + J_2 = 0. \quad (2.13)$$

Also, using (2.11), we get

$$\begin{aligned} |2^{jk} J_2| &\leq 2^{jk} K_{J_2} 2^{-j(k+1)} \\ &= K_{J_2} 2^{-j} \\ &\rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned}$$

Taking $j \rightarrow \infty$ in (2.13), we obtain

$$|K_{J_1} \overline{\widehat{(\mathcal{H}\psi)}^{(k)}(0)} (\mathcal{H}\psi)^{(k)}(x_0)| = 0.$$

This gives $\int_{\mathbb{R}} x^k \mathcal{H}\psi(x) dx = 0$. So the result is true for $q = k$. $\square$

The next result discusses the relationship between the regularity of orthonormal wavelets $\mathcal{H}\psi$ and their vanishing moments.

Theorem 2.6. *Let $\mathcal{H}\psi_{j,k}(x) = \mathcal{H}[2^{-\frac{j}{2}} \psi(2^{-j}x - k)]$, $j, k \in \mathbb{Z}$ constitutes an orthonormal set in $\mathcal{L}^2(\mathbb{R})$ satisfying*

$$|\mathcal{H}\psi(x)| \leq K(1 + |x|)^{-p-1-\epsilon}, \text{ where } \epsilon > 0 \quad (2.14)$$

and $\mathcal{H}\psi \in \mathcal{C}^p(\mathbb{R})$ such that $(\mathcal{H}\psi)^{(r)}$ is bounded for $r \leq p$. Then

$$\int_{\mathbb{R}} x^r \mathcal{H}\psi(x) dx = 0 \text{ for } r = 0, 1, 2, \dots, p. \quad (2.15)$$

Proof. We prove the result by the principle of mathematical induction on n . Assume that

$$\int_{\mathbb{R}} x^n \mathcal{H}\psi(x) dx = 0 \text{ for } 0 \leq n \leq r-1. \quad (2.16)$$

Since $\{\mathcal{H}\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ constitute an orthonormal set in $\mathcal{L}^2(\mathbb{R})$,

$$\langle \mathcal{H}\psi_{j,k} \mathcal{H}\psi_{j',k'} \rangle = \begin{cases} 0, & \text{if } j \neq j' \text{ or } k \neq k' \\ 1, & \text{if } j = j' \text{ or } k = k'. \end{cases}$$

For $j' = 0$ ($j \neq 0$ or $k \neq k'$), we have

$$\int_{\mathbb{R}} \mathcal{H}\psi_{j,k}(t) \overline{\mathcal{H}\psi_{0,k'}(t)} dt = 0,$$

$$\text{or} \quad \int_{\mathbb{R}} \mathcal{H}[2^{\frac{-j}{2}} \psi(2^{-j}t - k)] \overline{\mathcal{H}\psi(t - k')} dt = 0.$$

Write $k' = 2^j k + 2^{j-J} K \in \mathbb{Z}$, where $J, K \in \mathbb{Z}$ and let $j > J$. Then

$$\int_{\mathbb{R}} 2^{\frac{-j}{2}} \mathcal{H}\psi(2^{-j}t - k) \overline{\mathcal{H}\psi(t - (2^j k + 2^{j-J} K))} dt = 0,$$

$$\text{or} \quad \int_{\mathbb{R}} 2^{\frac{-j}{2}} \mathcal{H}\psi(2^{-j}t - k) \overline{\mathcal{H}\psi(2^j(2^{-j}t - k) - 2^{j-J} K)} dt = 0.$$

This gives $2^j \int_{\mathbb{R}} 2^{\frac{-j}{2}} \mathcal{H}\psi(x) \overline{\mathcal{H}\psi(2^j x - 2^{j-J} K)} dx = 0$, where $x = 2^{-j}t - k$.

Thus $\int_{\mathbb{R}} \mathcal{H}\psi(x) \overline{\mathcal{H}\psi[2^j x - 2^{j-J} K]} dx = 0$. This yields

$$\begin{aligned} \int_{\mathbb{R}} \left[\mathcal{H}\psi(x) - \sum_{n=0}^r \frac{(x - 2^{-J} K)^n}{n!} (\mathcal{H}\psi)^{(n)}(2^{-J} K) \right] \overline{\mathcal{H}\psi[2^j x - 2^{j-J} K]} dx \\ + \int_{\mathbb{R}} \left[\sum_{n=0}^r \frac{(x - 2^{-J} K)^n}{n!} (\mathcal{H}\psi)^{(n)}(2^{-J} K) \right] \overline{\mathcal{H}\psi[2^j x - 2^{j-J} K]} dx = (0.17) \end{aligned}$$

Let

$$\begin{aligned} I &= \int_{\mathbb{R}} \left[\mathcal{H}\psi(x) - \sum_{n=0}^r \frac{(x - 2^{-J} K)^n}{n!} (\mathcal{H}\psi)^{(n)}(2^{-J} K) \right] \overline{\mathcal{H}\psi[2^j x - 2^{j-J} K]} dx \quad (2.18) \\ II &= \int_{\mathbb{R}} \left[\sum_{n=0}^r \frac{(x - 2^{-J} K)^n}{n!} (\mathcal{H}\psi)^{(n)}(2^{-J} K) \right] \overline{\mathcal{H}\psi[2^j x - 2^{j-J} K]} dx. \quad (2.19) \end{aligned}$$

Since $(\mathcal{H}\psi)^{(r)}$ is continuous for $r \leq p$ and since the dyadic rationals $2^{-j}k, j, k \in \mathbb{Z}$ are dense in $\mathbb{R}$, there exists $J, K \in \mathbb{Z}$ such that $(\mathcal{H}\psi)^{(r)}(2^{-J} K) \neq 0$. Also $(\mathcal{H}\psi)^{(r)}(x) \in C^r(\mathbb{R})$ for $r \leq p$. So $\mathcal{H}\psi(x)$ can be expanded using Taylor's expansion about the point $2^{-J} K$ i.e.,

$$\begin{aligned} \mathcal{H}\psi(x) &= (\mathcal{H}\psi)(2^{-J} K) + (x - 2^{-J} K)(\mathcal{H}\psi)^{(1)}(2^{-J} K) \\ &\quad + \dots + \frac{(x - 2^{-J} K)^r}{r!} (\mathcal{H}\psi)^{(r)}(2^{-J} K) + R_r(x), \end{aligned}$$

where $R_r(x) = \frac{(x - 2^{-J} K)^{r+1}}{(r+1)!} (\mathcal{H}\psi)^{(r+1)}(\xi)$ for some number ξ between $2^{-J} K$ and x .

By continuity of $\mathcal{H}\psi$, for any $\epsilon > 0$, there exist $\delta > 0$ such that

$$\begin{aligned} \left| \mathcal{H}\psi(x) - \sum_{n=0}^r \frac{(x - 2^{-J} K)^n}{n!} (\mathcal{H}\psi)^{(n)}(2^{-J} K) \right| &\leq \epsilon |x - 2^{-J} K|^r \\ \text{if } |x - 2^{-J} K| &\leq \delta. \end{aligned} \quad (2.20)$$

From (2.16) and (2.19), we get

$$\begin{aligned} II &= \frac{1}{r!} (\mathcal{H}\psi)^{(r)}(2^{-J}K) \int_{\mathbb{R}} (x - 2^{-J}K)^r \overline{\mathcal{H}\psi[2^j x - 2^{j-J}K]} dx. \\ &= \frac{1}{r!} 2^{-j(r+1)} (\mathcal{H}\psi)^{(r)}(2^{-J}K) \int_{\mathbb{R}} t^r \overline{\mathcal{H}\psi(t)} dt, \text{ where } x - 2^{-J}K = 2^{-j}t. \end{aligned} \quad (2.21)$$

From (2.18), we obtain

$$\begin{aligned} |I| &= \left| \int_{\mathbb{R}} [\mathcal{H}\psi(x) - \sum_{n=0}^r \frac{(x - 2^{-J}K)^n}{n!} (\mathcal{H}\psi)^{(n)}(2^{-J}K)] \overline{\mathcal{H}\psi[2^j x - 2^{j-J}K]} dx \right| \\ &\leq \epsilon \int_{|x - 2^{-J}K| \leq \delta} |x - 2^{-J}K|^r |\mathcal{H}\psi(2^j x - 2^{j-J}K)| dx \\ &\quad + \int_{|x - 2^{-J}K| > \delta} \left(|\mathcal{H}\psi(x)| + \sum_{n=0}^r |(\mathcal{H}\psi)^{(n)}(2^{-J}K)| |x - 2^{-J}K|^n \right) |\mathcal{H}\psi(2^j x - 2^{j-J}K)| dx. \end{aligned}$$

Write $x - 2^{-J}K = z$. Then

$$\begin{aligned} |I| &\leq \epsilon \int_{|z| \leq \delta} |z|^r |\mathcal{H}\psi(2^j z)| dz + \int_{|z| > \delta} |\mathcal{H}\psi(z + 2^{-J}K)| |\mathcal{H}\psi(2^j z)| dz \\ &\quad + \int_{|z| > \delta} \sum_{n=0}^r |(\mathcal{H}\psi)^{(n)}(2^{-J}K)| |z|^n |\mathcal{H}\psi(2^j z)| dz. \end{aligned}$$

Let K_r be the bounds of $(\mathcal{H}\psi)^{(r)}$, $r \leq p$. Then

$$\begin{aligned} |I| &\leq \epsilon \int_{|z| \leq \delta} |z|^r |\mathcal{H}\psi(2^j z)| dz + K \int_{|z| > \delta} (1 + |z + 2^{-J}K|^{-p-1-\epsilon}) |\mathcal{H}\psi(2^j z)| dz \\ &\quad + K_r \int_{|z| > \delta} |z|^r |\mathcal{H}\psi(2^j z)| dz \\ &\leq \epsilon \int_{|z| \leq \delta} |z|^r |\mathcal{H}\psi(2^j z)| dz + K \int_{|z| > \delta} |\mathcal{H}\psi(2^j z)| dz \\ &\quad + K_r \int_{|z| > \delta} |z|^r |\mathcal{H}\psi(2^j z)| dz \\ &\leq \epsilon \int_{|z| \leq \delta} |z|^r |\mathcal{H}\psi(2^j z)| dz + K' \int_{|z| > \delta} (1 + |z|^r) |\mathcal{H}\psi(2^j z)| dz, \\ &\quad \text{where } K' = \max_r \{K, K_r\}, \quad r \leq p \\ &\leq K\epsilon \int_{|z| < \delta} |z|^r (1 + |2^j z|)^{-p-1-\epsilon} dz + K' \int_{|z| > \delta} (1 + |z|^r) (1 + |2^j z|)^{-p-1-\epsilon} dz. \end{aligned}$$

For $|2^j z| = t$ in the first integral and for $|z| = t$ in the second integral respectively, we get

$$\begin{aligned} |I| &\leq 2K\epsilon \int_0^{2^j\delta} \left(\frac{t}{2^j}\right)^r (1+t)^{-p-1-\epsilon} \frac{dt}{2^j} + K' \int_\delta^\infty (1+t^r)(1+2^j t)^{-p-1-\epsilon} dt \\ &\leq 2K\epsilon 2^{-j(r+1)} \int_0^\infty t^r (1+t)^{-p-1-\epsilon} dt + K' \int_\delta^\infty (1+t^r)(1+2^j t)^{-p-1-\epsilon} dt \\ &\leq K''\epsilon 2^{-j(r+1)} + K' 2^{-j(p+1+\epsilon)} \frac{(1+\delta)^{p+1+\epsilon}}{\delta^{p+1+\epsilon}} \int_\delta^\infty (1+t)^r (1+t)^{-p-1-\epsilon} dt, \end{aligned}$$

$$\text{where } (1+2^j t)^{-1} \leq \frac{(1+\delta)(1+t)^{-1}}{(1+2^j\delta)} \leq \frac{(2^{-j})(1+\delta)(1+t)^{-1}}{\delta}.$$

This gives

$$\begin{aligned} |I| &\leq K''\epsilon 2^{-j(r+1)} - K' 2^{-j(p+1+\epsilon)} \frac{(1+\delta)^{p+1+\epsilon}}{\delta^{p+1+\epsilon}} \frac{(1+\delta)^{r-p-\epsilon}}{(r-p-\epsilon)} \\ &= K''\epsilon 2^{-j(r+1)} + K''' 2^{-j(p+1)} \delta^{-p-1} (1+\delta)^{r+1}. \end{aligned} \quad (2.22)$$

In view of (2.17), $|II| = |I|$. So, by using (2.21) and (2.22), we get

$$\left| \int_{\mathbb{R}} t^r \overline{\mathcal{H}\psi(t)} dt \right| \leq r! \left| (\mathcal{H}\psi)^{(r)}(2^{-J}K) \right|^{-1} \left[K''\epsilon + K''' 2^{-j(p-r)} \delta^{-p-1} (1+\delta)^{r+1} \right].$$

Here ϵ can be made arbitrarily small and for the corresponding δ , one can choose j sufficiently large to make the second term on the right hand side of the inequality arbitrarily small so that

$$\int_{\mathbb{R}} t^r \overline{\mathcal{H}\psi(t)} dt = 0.$$

Hence the result holds for $n = r$. □

Remark 2.7. If $\{\mathcal{H}\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ is an orthonormal system on $\mathbb{R}$ such that $\mathcal{H}\psi \in C^\infty$. Then in view of 4.6.6 in [4], $\mathcal{H}\psi$ cannot have exponential decay.

Recall from [7] that a function f is said to have fast decay with decay rate $m \in \mathbb{N}$, if there exists a constant C_m such that $|f(x)| \leq \frac{C_m}{1+|x|^m}$, for all $x \in \mathbb{R}$.

The next result discusses the relationship between the vanishing moments and the fast decay of wavelet $\mathcal{H}\psi$.

Theorem 2.8. Let $\psi, \hat{\psi} \in \mathcal{L}^1(\mathbb{R})$ and suppose that $\mathcal{H}\psi \in C^n(\mathbb{R})$ have fast decay with decay rate $m \in \mathbb{N}$ and $(\mathcal{H}\psi)^{(r)} \in \mathcal{L}^\infty(\mathbb{R})$, $r = 1, 2, \dots, n$. If $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ be an orthonormal system on $\mathbb{R}$, then

$$\int_{\mathbb{R}} x^r \mathcal{H}\psi(x) dx = 0, \text{ for all } r = 0, 1, 2, \dots, n, \quad (2.23)$$

where $r+1 < m$.

Proof. For $r = 0$, the result is obvious. For $1 \leq r \leq n$, we shall prove the result by induction on r . Let us prove the result when $r = 1$.

Since $\psi \in \mathcal{L}^1(\mathbb{R})$, $p_1(x) = \int_{-\infty}^x \mathcal{H}\psi(y) dy$ approaches to 0. Also since $\mathcal{H}\psi(y)$ has fast decay,

$$|p_1(x)| \leq \frac{C_1}{1 + |x|^{m-1}}.$$

Further, we have

$$\int_{\mathbb{R}} p_1(x) dx = - \int_{\mathbb{R}} x \mathcal{H}\psi(x) dx. \quad (2.24)$$

Let $x' = 2^{-j'} k'$, for some $j', k' \in \mathbb{Z}$. Then using the orthogonality of $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$, we have

$$\int_{\mathbb{R}} \mathcal{H}\psi(x) \overline{\mathcal{H}\psi(2^j x - k)} dx = 0.$$

Write $k = 2^{j-j'} k', j > j'$. Then

$$\begin{aligned} & \int_{\mathbb{R}} (\mathcal{H}\psi)^{(1)}(x) \overline{p_1(2^j(x - x'))} dx = 0, \\ \text{or} \quad & \int_{\mathbb{R}} (\mathcal{H}\psi)^{(1)}(x' + 2^{-j} z) \overline{p_1(z)} dz = 0, \\ \text{or} \quad & (\mathcal{H}\psi)^{(1)}(x') \int_{\mathbb{R}} \overline{p_1(z)} dz \rightarrow 0 \text{ as } j \rightarrow \infty, \text{ where } z = 2^j(x - x'). \end{aligned} \quad (2.25)$$

Since $(\mathcal{H}\psi)^{(1)}$ is continuous, $(\mathcal{H}\psi)^{(1)}(x') \neq 0$. Therefore, we obtain

$$\int_{\mathbb{R}} x \mathcal{H}\psi(x) dx = 0. \quad (2.26)$$

Now assume that

$$\int_{\mathbb{R}} x^r \mathcal{H}\psi(x) dx = 0, \text{ for } r = 1, 2, \dots, n-1.$$

Using integration, we obtain functions p_r , $r = 1, 2, \dots, n$ satisfying $p_r^{(1)} = p_{r-1}$ and $p_1, p_2, \dots, p_n$ all tend to 0 at $\pm\infty$. Also, there exists constants C_r ($1 \leq r \leq n$) such that

$$|p_r(x)| \leq \frac{C_r}{1 + |x|^{m-r}}, \quad r = 1, 2, \dots, n.$$

Thus

$$\int_{\mathbb{R}} x^n \mathcal{H}\psi(x) dx = 0 \text{ iff } \int_{\mathbb{R}} p_n(x) dx = 0.$$

Since $(\mathcal{H}\psi)^{(n)}$ is continuous, there exists $x' = 2^{-j'} k'$ such that $(\mathcal{H}\psi)^{(n)}(x') \neq 0$.

Using (2.25), we obtain

$$\int_{\mathbb{R}} x^n \mathcal{H}\psi(x) dx = 0. \quad \square$$

Conclusion

The smoothness (regularity) of wavelets is connected with the vanishing moments of scaling and wavelet functions. It may be noted that in view of the admissibility condition, the Haar wavelet has only zeroth vanishing moment. In this article, we study Hilbert transform of wavelets $\psi \in \mathcal{L}^2(\mathbb{R})$ and observe that orthogonality, smoothness, regularity and fast decay are important for the vanishing moments of $\mathcal{H}\psi$. We investigate this behaviour on wavelets with finite as well as infinite support and find that support plays no significant role in zeroth order vanishing moments of $\mathcal{H}^n\psi$.

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