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Properties of almost $\mathcal{I}$-continuous functions

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Abstract. The aim of this paper is to introduce and characterize a new class of functions called almost $\mathcal{I}$-continuous functions in ideal topological spaces by using $\mathcal{I}$-open sets.

1. Introduction

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [11] and Vaidyanathaswamy, [19]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [19] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. A Kuratowski closure operator $\mathrm{Cl}^\star(\cdot)$ for a topology $\tau^\star(\tau, \mathcal{I})$ called the $\star$-topology, finer than τ is defined by $\mathrm{Cl}^\star(A) = A \cup A^\star(\tau, \mathcal{I})$ when there is no chance of confusion, $A^\star(\mathcal{I})$ is denoted by $A^\star$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space. The aim of this paper is to introduce and characterize a new class of functions called almost $\mathcal{I}$-continuous functions in ideal topological spaces by using $\mathcal{I}$-open sets.

2. Preliminaries

Let A be a subset of a topological space (X, τ). We denote the closure of A and the interior of A by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be regular open [18] if $A = \mathrm{Int}(\mathrm{Cl}(A))$. A set $A \subset X$ is said to be δ-open [20] if it is the union of regular open sets of X. The complement of a regular open (resp. δ-open) set is called regular closed (resp. δ-closed). The intersection of all δ-closed sets

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of (X,τ) containing A is called the δ-closure [20] of A and is denoted by $\mathrm{Cl}_\delta(A)$. A subset A of a topological space (X,τ) is said to be semiopen [12] (resp. preopen [13], β-open [2]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$). The set of all regular open (resp. regular closed, δ-open, δ-closed, preopen) sets of (X,τ) is denoted by $RO(X)$ (resp. $RC(X)$, $\delta O(X)$, $\delta C(X)$, $PO(X)$). A subset S of an ideal topological space $(X,\tau,\mathcal{I})$ is called $\mathcal{I}$-open [9] if $S \subset \mathrm{Int}(S^\star)$. The complement of an $\mathcal{I}$-open set is called an $\mathcal{I}$-closed set. The intersection of all $\mathcal{I}$-closed sets containing S is called the $\mathcal{I}$-closure of S and is denoted by $\mathcal{I}\,\mathrm{Cl}(S)$. The $\mathcal{I}$-interior of S is defined by the union of all $\mathcal{I}$-open sets contained in S and is denoted by $\mathcal{I}\,\mathrm{Int}(S)$. The set of all $\mathcal{I}$-open sets of $(X,\tau,\mathcal{I})$ is denoted by $\mathcal{I}O(X)$. The set of all $\mathcal{I}$-open sets of $(X,\tau,\mathcal{I})$ containing a point $x \in X$ is denoted by $\mathcal{I}O(X,x)$.

Definition 2.1. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be almost precontinuous [14] if $f^{-1}(V)$ is preopen in X for every regular open set V of Y.

Definition 2.2. A function $f:(X,\tau,\mathcal{I})\to(Y,\sigma)$ is said to be:

(1) $\mathcal{I}$-continuous [1] if $f^{-1}(V)$ is $\mathcal{I}$-open in X for every open set V of Y,
(2) weakly $\mathcal{I}$-continuous [7] if for each $x \in X$ if for each open subset V in Y containing $f(x)$, there exists $U \in \mathcal{I}O(X,x)$ such that $f(U) \subset \mathrm{Cl}(V)$.

Definition 2.3. An ideal topological space $(X,\tau,\mathcal{I})$ is said to be:

(1) $\mathcal{I}$-T_1 (resp. r-T_1 [6]) if for each pair of distinct points x and y of X, there exist $\mathcal{I}$-open (resp. regular open) sets and U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.
(2) $\mathcal{I}$-T_2 [5] (resp. r-T_2 [6]) if for each pair of distinct points x and y of X, there exist $\mathcal{I}$-open (resp. regular open) sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Lemma 2.4. *The following statements are true:*

(1) Let A be a subset of a space (X,τ). Then $A \in PO(X)$ if and only if $s\,\mathrm{Cl}(A) = \mathrm{Int}(\mathrm{Cl}(A))$ [8].
(2) A subset A of a space (X,τ) is β-open if and only if $\mathrm{Cl}(A)$ is regular closed [3].

3. Almost $\mathcal{I}$-continuous functions

Definition 3.1. A function $f:(X,\tau,\mathcal{I})\to(Y,\sigma)$ is said to be:

(1) almost $\mathcal{I}$-continuous at a point $x \in X$ if for each open subset V of Y containing $f(x)$, there exists $U \in \mathcal{I}O(X,x)$ such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$;
(2) almost $\mathcal{I}$-continuous if it has this property at each point of X.

Remark 3.2. Almost $\mathcal{I}$-continuity implies weak $\mathcal{I}$-continuity and it is obvious that almost $\mathcal{I}$-continuity implied by $\mathcal{I}$-continuity. However, the converses of these implications is not true in general as the following examples show.

Example 3.3. Let $X = \{a,b,c\}$, $\tau = \{\varnothing, X\}$, $\sigma = \{\varnothing, \{a\}, X\}$ and $\mathcal{I} = \{\varnothing, \{a\}\}$. Then the identity function $f:(X,\tau,\mathcal{I})\to(Y,\sigma)$ is almost $\mathcal{I}$-continuous but not $\mathcal{I}$-continuous.

Example 3.4. Let $X = \{a,b,c\}$, $\tau = \{\varnothing, \{a,c\}$, $\sigma = \{\varnothing, \{a\}, \{b\}, \{a,b\}, X\}$ and $\mathcal{I} = \{\varnothing, \{b\}\}$. Then the identity function $f:(X,\tau,\mathcal{I})\to(X,\sigma)$ is weakly $\mathcal{I}$-continuous but not almost $\mathcal{I}$-continuous.

Theorem 3.5. *For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following statements are equivalent:*

(1) f is almost $\mathcal{I}$-continuous at $x \in X$;
(2) $x \in \mathrm{Int}(f^{-1}(s\,\mathrm{Cl}(V)))^\star$ for every open set V of Y containing $f(x)$;
(3) $f^{-1}(V) \subset \mathcal{I}\,\mathrm{Int}(f^{-1}(s\,\mathrm{Cl}(V)))$ for every open set V of Y.

Proof. (1)⇒(2): Let V be an open set of Y containing $f(x)$. Then there exists $U \in \mathcal{I}O(X, x)$ such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V)) = s\,\mathrm{Cl}(V)$. It follows that $U \subset f^{-1}(s\,\mathrm{Cl}(V))$. Since $U \in \mathcal{I}O(X, x)$, $x \in U \subset \mathrm{Int}(f^{-1}(s\,\mathrm{Cl}(V)))^\star$.
(2)⇒(3): Let V be an open set of Y containing $f(x)$. Since $x \in \mathrm{Int}(f^{-1}(s\,\mathrm{Cl}(V)))^\star)$, we have $x \in f^{-1}(s\,\mathrm{Cl}(V)) \cap \mathrm{Int}(f^{-1}(s\,\mathrm{Cl}(V)))^\star = \mathcal{I}\,\mathrm{Int}(f^{-1}(s\,\mathrm{Cl}(V)))$ by [[10], Theorem 4.1(3)]. Hence $f^{-1}(V) \subset \mathcal{I}\,\mathrm{Int}(f^{-1}(s\,\mathrm{Cl}(V)))$.
(3)⇒(1): Let V be an open set of Y containing $f(x)$, then $x \in f^{-1}(V) \subset \mathcal{I}\,\mathrm{Int}(f^{-1}(s\,\mathrm{Cl}(V)))$. Set $U = \mathcal{I}\,\mathrm{Int}(f^{-1}(s\,\mathrm{Cl}(V)))$, then $U \in \mathcal{I}O(X, x)$ such that $f(U) \subset s\,\mathrm{Cl}(V)$. This shows that f is almost $\mathcal{I}$-continuous at x.

□

Theorem 3.6. *For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following statements are equivalent:*

(1) f is almost $\mathcal{I}$-continuous;
(2) $f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)) \in \mathcal{I}O(X)$ for every open set V of Y;
(3) $f^{-1}(\mathrm{Cl}(\mathrm{Int}(V)) \in \mathcal{I}C(X)$ for every closed set V of Y;
(4) $f^{-1}(V) \in \mathcal{I}O(X)$ for every $V \in RO(Y)$;
(5) $f^{-1}(F) \in \mathcal{I}C(X)$ for every $F \in RC(Y)$;
(6) for each $x \in X$ and each open set V of Y containing $f(x)$ there exists $U \in \mathcal{I}O(X, x)$ such that $f(U) \subset s\,\mathrm{Cl}(V)$;
(7) $\mathcal{I}\,\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(F))) \subset f^{-1}(F)$ for every closed set F of Y;
(8) $f(\mathcal{I}\,\mathrm{Cl}(A)) \subset \mathrm{Cl}_\delta(f(A))$ for every subset A of X;
(9) $\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_\delta(B))$ for every subset B of Y;
(10) $f^{-1}(F) \in \mathcal{I}C(X)$ for every $F \in \delta C(Y)$;
(11) $f^{-1}(V) \in \mathcal{I}O(X)$ for every $V \in \delta O(Y)$.

Proof. (4)⇒(5): Let $F \in RC(Y)$. Then $Y\backslash F \in RO(Y)$. Take $x \in f^{-1}(Y\backslash F)$, then $f(x) \in Y\backslash F$ and since f is almost $\mathcal{I}$-continuous, there exists $W_x \in \mathcal{I}O(X, x)$ and $f(W_x) \subset Y\backslash F$. Then $x \in W_x \subset f^{-1}(Y\backslash F)$ so that $f^{-1}(Y\backslash F) = \bigcup_{x \in f^{-1}(Y\backslash F)} W_x$. Since any union of $\mathcal{I}$-open sets is $\mathcal{I}$-open, $f^{-1}(Y\backslash F)$ is $\mathcal{I}$-open in X and hence $f^{-1}(F) \in \mathcal{I}C(X)$.
(5)⇒(8): Let A be a subset of X. Since $\mathrm{Cl}_\delta(f(A))$ is δ-closed in Y, it is equal to $\cap\ \{F_\alpha$: F_α is regular closed in Y, $\alpha \in \Lambda\}$, where Λ is an index set. From (5), we have $A \subset f^{-1}(\mathrm{Cl}_\delta(f(A))) = \cap\{f^{-1}(F_\alpha) : \alpha \in \Lambda\} \in \mathcal{I}C(X)$ and hence $\mathcal{I}\,\mathrm{Cl}(A) \subset f^{-1}(\mathrm{Cl}_\delta(f(A)))$. Therefore, we obtain $f(\mathcal{I}\,\mathrm{Cl}(A)) \subset \mathrm{Cl}_\delta(f(A))$.
(8)⇒(9): Set $A = f^{-1}(B)$ in (8), then $f(\mathcal{I}\,\mathrm{Cl}(f^{-1}(B))) \subset \mathrm{Cl}_\delta(f(f^{-1}(B))) \subset \mathrm{Cl}_\delta(B)$ and hence $\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_\delta(B))$.
(9)⇒(10): Let F be a δ-closed set of Y, then $\mathcal{I}\,\mathrm{Cl}(f^{-1}(F)) \subset f^{-1}(F)$ so $f^{-1}(F) \in \mathcal{I}C(X)$.
(10)⇒(11): Let V be a δ-open set of Y, then $Y\backslash V$ is δ-closed set in Y. This gives $f^{-1}(Y\backslash V) \in \mathcal{I}C(X)$ and hence $f^{-1}(V) \in \mathcal{I}O(X)$.

(11)⇒(1): Let V be any regular open set of Y. Since V is δ-open in Y, $f^{-1}(V) \in \mathcal{I}O(X)$ and hence from $f(f^{-1}(V)) \subset V = \mathrm{Int}(\mathrm{Cl}(V))$. Then f is almost $\mathcal{I}$-continuous.
The other implications are obvious. □

Theorem 3.7. *For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following statements are equivalent:*

(1) f is almost $\mathcal{I}$-continuous;
(2) $\mathcal{I}\,\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B)))))) \subset f^{-1}(\mathrm{Cl}(B)))$ for every open subset B of Y;
(3) $\mathcal{I}\,\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(F))))) \subset f^{-1}(F))$ for every closed subset F of Y;
(4) $\mathcal{I}\,\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V)))) \subset f^{-1}(\mathrm{Cl}(V))$ for every open subset V of Y;

Proof. (1)⇒(2): Let B be any subset of Y. Assume that $x \in X\backslash f^{-1}(\mathrm{Cl}(B))$. Then $f(x) \in Y \setminus \mathrm{Cl}(B)$ and there exists an open set V containing $f(x)$ such that $V \cap B = \emptyset$; hence $\mathrm{Int}(\mathrm{Cl}(V)) \cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))) = \emptyset$. Since f is almost $\mathcal{I}$-continuous, there exists $U \in \mathcal{I}O(X, x)$ such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Therefore, we have $U \cap f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B)))) = \emptyset$ and hence $x \in X\backslash\mathcal{I}\,\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))$. Thus, we obtain $\mathcal{I}\,\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B)))))) \subset f^{-1}(\mathrm{Cl}(B)))$.
(2)⇒(3): Let F be any closed set of Y. Then we have $\mathcal{I}\,\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F))))))$ $= \mathcal{I}\,\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(F)))) \subset f^{-1}(\mathrm{Cl}(\mathrm{Int}(F))) \subset f^{-1}(F)$.
(3)⇒(4): For any open set V of Y, $\mathrm{Cl}(V)$ is regular closed in Y and we have $\mathcal{I}\,\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))) = \mathcal{I}\,\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V))))) \subset f^{-1}(\mathrm{Cl}(V))$.
(4)⇒(1): Clear. □

Theorem 3.8. *A function $f : (X, \tau, \{\varnothing\}) \to (Y, \sigma)$ is almost $\mathcal{I}$-continuous if and only if it is almost precontinuous.*

Proof. It follows from Theorem 2.1 of [1]. □

Definition 3.9. [5] Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$ such that $A \subset B \subset X$. Then $(B, \tau_{|_B}, \mathcal{I}_{|_B})$ is an ideal topological space with an ideal $\mathcal{I}_{|_B} = \{\mathbf{I} \in \mathcal{I} | \mathbf{I} \subset B\} = \{\mathbf{I} \cap B | \mathbf{I} \in \mathcal{I}\}$.

Lemma 3.10. [5] *Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$.*

(1) If $A \in \mathcal{I}O(X)$ and $B \in \tau$, then $A \cap B \in \mathcal{I}O(B)$.
(2) If $A \in \mathcal{I}O(X)$ and $B \in \tau$, then $A \cap B \in \mathcal{I}O(X)$.

Theorem 3.11. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an almost $\mathcal{I}$-continuous function and $A \subset X$. If $A \in \tau$, then $f_{|A}$: $(A, \tau_{|A}, \mathcal{I}_{|A}) \to (Y, \sigma)$ is almost $\mathcal{I}_{|\mathcal{A}}$-continuous.*

Proof. It follows from Lemma 3.10. □

Theorem 3.12. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function and $\Lambda = \{U_i : i \in \Delta\}$ be a family such that $U_i \in \tau$ for each $i \in \Delta$. If $f|_{U_i}$ is almost $\mathcal{I}$-continuous for each $i \in \Delta$, then f is almost $\mathcal{I}$-continuous.*

Proof. Suppose that V is any regular open subset of (Y, σ). Since $f|_{U_i}$ is almost $\mathcal{I}$-continuous for each $i \in \Delta$, it follows that $(f|_{U_i})^{-1}(V)$ is $\mathcal{I}$-open in U_i. We have $f^{-1}(V)$ $= \cup_{i\in\Delta}\ (f^{-1}(V) \cap U_i) = \cup_{i\in\Delta}(f|_{U_i})^{-1}(V)$. Since any union of $\mathcal{I}$-open sets is $\mathcal{I}$-open, $f^{-1}(V) \in \mathcal{I}O(X)$. Hence f is $\mathcal{I}$-continuous. □

Definition 3.13. A filterbase Λ is said to be

(1) $\mathcal{I}$-convergent to a point x in X if for any $U \in \mathcal{I}O(X, x)$, there exists $B \in \Lambda$ such that $B \subset U$.
(2) r-convergent to a point x in X if for any regular open set U of X containing x, there exists $B \in \Lambda$ such that $B \subset U$.

Theorem 3.14. *If a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is almost $\mathcal{I}$-continuous, then for each point $x \in X$ and each filter base Λ in X $\mathcal{I}$-converging to x, the filter base $f(\Lambda)$ is r-convergent to $f(x)$.*

Proof. Let $x \in X$ and Λ be any filter base in X $\mathcal{I}$-converging to x. Since f is $\mathcal{I}$-continuous, then for any open set V of (Y, σ) containing $f(x)$, there exists $U \in \mathcal{I}O(X, x)$ such that $f(U) \subset V$. Since Λ is $\mathcal{I}$-converging to x, there exists $B \in \Lambda$ such that $B \subset U$. This means that $f(B) \subset V$ and hence the filter base $f(\Lambda)$ is convergent to $f(x)$. $\square$

Definition 3.15. A sequence (x_n) is said to be $\mathcal{I}$-convergent to a point x if for every $\mathcal{I}$-open set V containing x, there exists an index n_0 such that for $n \geq n_0$, $x_n \in V$.

Theorem 3.16. *If a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is almost $\mathcal{I}$-continuous, then for each point $x \in X$ and each net (x_n) which is $\mathcal{I}$-convergt to x, the net $(f(x_n))$ is r-convergent to $f(x)$.*

Proof. The proof is similar to that of Theorem 3.14. $\square$

Theorem 3.17. *If an injective function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is almost $\mathcal{I}$-continuous and (Y, σ) is r-T_1, then $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-T_1.*

Proof. Suppose that Y is r-T_1. For any distict points x and y in X, there exist regular open sets V and W such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$ and $f(y) \in W$. Since f is almost $\mathcal{I}$-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are $\mathcal{I}$-open subsets of $(X, \tau, \mathcal{I})$ such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-T_1. $\square$

Theorem 3.18. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a almost $\mathcal{I}$-continuous injective function and (Y, σ) is r-T_2, then (X, τ) is $\mathcal{I}$-T_2.*

Proof. For any pair of distinct points x and y in X, there exist disjoint regular open sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is almost $\mathcal{I}$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $\mathcal{I}$-open sets in X containing x and y, respectively. Therefore, $f^{-1}(U) \cap f^{-1}(V) = \varnothing$ because $U \cap V = \varnothing$. This shows that $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-T_2. $\square$

Theorem 3.19. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a almost continuous function and $g : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a almost $\mathcal{I}$-continuous function and Y is a r-T_2-space, then the set $E = \{x \in X : f(x) = g(x)\}$ is $\mathcal{I}$-closed set in $(X, \tau, \mathcal{I})$.*

Proof. If $x \in X \backslash E$, then it follows that $f(x) \neq g(x)$. Since Y is r-T_2, there exist disjoint regular open sets V and W of Y such that $f(x) \in V$ and $g(x) \in W$. Since f is almost continuous and g is almost $\mathcal{I}$-continuous, then $f^{-1}(V)$ is open and $g^{-1}(W)$ is $\mathcal{I}$-open in X with $x \in f^{-1}(V)$ and $x \in g^{-1}(W)$. Put $A = f^{-1}(V) \cap g^{-1}(W)$. By Lemma 3.10, A is $\mathcal{I}$-open in X. Therefore, $f(A) \cap g(A) = \varnothing$ and it follows that $x \notin \mathcal{I}\,\mathrm{Cl}(E)$. This shows that E is $\mathcal{I}$-closed in X. $\square$

Theorem 3.20. *If a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is $\mathcal{I}$-preopen and weakly $\mathcal{I}$-continuous, then f is almost $\mathcal{I}$-continuous.*

Proof. Let $x \in X$ and V be an open set of Y containing $f(x)$. Since f is weakly $\mathcal{I}$-continuous, there exists $U \in \mathcal{I}O(X,x)$ such that $f(U) \subset \mathrm{Cl}(V)$. Since f is $\mathcal{I}$-preopen, $f(U) \subset \mathrm{Int}(\mathrm{Cl}(f(U))) \subset \mathrm{Int}(\mathrm{Cl}(V))$; hence f is almost $\mathcal{I}$-continuous. □

Theorem 3.21. *Let $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ be a function and $g : X \to X \times Y$ the graph function defined by $g(x) = (x, f(x))$ for every $x \in X$. Then g is almost $\mathcal{I}$-continuous if and only if f is almost $\mathcal{I}$-continuous.*

Proof. Let x be any point of X and V any regular open set of Y containing $f(x)$. Then we have $g(x) = (x, f(x)) \in X \times V$ is regular open in $X \times Y$. Since g is almost $\mathcal{I}$-continuous, there exists $U \in \mathcal{I}O(X,x)$ such that $g(U) \subset X \times V$. Therefore, we obtain $f(U) \subset V$; hence f is almost $\mathcal{I}$-continuous. Conversely, let $x \in X$ and W be a regular open set of $X \times Y$ containing $g(x)$. There exist a regular open set U_1 in X and a regular open set V in Y such that $U_1 \times V \subset W$. Since f is almost $\mathcal{I}$-continuous, there exists $U_2 \in \mathcal{I}O(X,x)$ such that $f(U_2) \subset V$. Put $U = U_1 \cap U_2$, then we obtain $x \in U \in \mathcal{I}O(X)$ and $g(U) \subset U \times V \subset W$. This shows that g is almost $\mathcal{I}$-continuous. □

Definition 3.22. A topological space (X,τ) is said to be:

(1) almost regular [16] if for any regular closed set F of X and any point $x \in X \backslash F$ there exist disjoint open sets U and V such that $x \in U$ and $F \subset V$.

(2) semi-regular if for any open set U of X and each point $x \in U$ there exists a regular open set V of X such that $x \in V \subset U$.

Theorem 3.23. *If $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is a weakly $\mathcal{I}$-continuous function and Y is almost regular, then f is almost $\mathcal{I}$-continuous.*

Proof. Let $x \in X$ and let V be any open set of Y containing $f(x)$. By the almost regularity of Y, there exists a regular open set G of Y such that $f(x) \in G \subset \mathrm{Cl}(G) \subset \mathrm{Int}(\mathrm{Cl}(V))$ [[16], Theorem 2.2]. Since f is weakly $\mathcal{I}$-continuous, there exists $U \in \mathcal{I}O(X,x)$ such that $f(U) \subset \mathrm{Cl}(G) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Therefore, f is almost $\mathcal{I}$-continuous. □

Theorem 3.24. *If $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is an almost $\mathcal{I}$-continuous function and Y is semi-regular, then f is $\mathcal{I}$-continuous.*

Proof. Let $x \in X$ and let V be any open set of Y containing $f(x)$. By the semi-regularity of Y, there exists a regular open set G of Y such that $f(x) \in G \subset V$. Since f is almost $\mathcal{I}$-continuous, there exists $U \in \mathcal{I}O(X,x)$ such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(G)) = G \subset V$ and hence f is $\mathcal{I}$-continuous. □

Theorem 3.25. *If $g : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is almost $\mathcal{I}$-continuous and S is δ-closed set of $X \times Y$, then $p_X(S \cap G(g))$ is $\mathcal{I}$-closed in X, where p_X represents the projection of $X \times Y$ onto X and $G(g)$ denotes the graph of g.*

Proof. Let S be any δ-closed set of $X \times Y$ and $x \in \mathcal{I}\,\mathrm{Cl}(p_X(S \cap G(g))$. Let U be any open set of X containing x and V any open set of Y containing $g(x)$. Since g is almost $\mathcal{I}$-continuous, we have $x \in g^{-1}(V) \subset \mathcal{I}\,\mathrm{Int}(g^{-1}(\mathrm{Int}(\mathrm{Cl}(V))))$ and $U \cap \mathcal{I}\,\mathrm{Int}(g^{-1}(\mathrm{Int}(\mathrm{Cl}(V))))) \in \mathcal{I}O(X,x)$. Since $x \in \mathcal{I}\,\mathrm{Cl}(p_X(S \cap G(g))$, $(U \cap \mathcal{I}\,\mathrm{Int}(g^{-1}(\mathrm{Int}(\mathrm{Cl}(V))))) \cap p_X(S \cap G(g))$ contains some point u of X. This implies that $(u, g(u)) \in S$ and $g(u) \in \mathrm{Int}(\mathrm{Cl}(V))$. Thus, we have $\emptyset \neq (U \times \mathrm{Int}(\mathrm{Cl}(V)) \cap S \subset$

$\mathrm{Int}(\mathrm{Cl}(U\times V))\cap S$ and hence $(x, g(x)) \in \mathrm{Cl}_\delta(S)$. Since S is δ-closed, $(x, g(x)) \in (S\cap G(g))$ and $x \in p_X(S \cap G(g))$. Then $p_X(S \cap G(g))$ is $\mathcal{I}$-closed. □

Corollary 3.26. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ has a δ-closed graph and $g : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is almost $\mathcal{I}$-continuous, then the set $\{x \in X : f(x) = g(x)\}$ is $\mathcal{I}$-closed in X.

Proof. Since $G(f)$ is δ-closed and $p_X(G(f) \cap G(g)) = \{x \in X : f(x) = g(x)\}$, it follows from Theorem 3.25 that $\{x \in X : f(x) = g(x)\}$ is $\mathcal{I}$-closed in X. □

Theorem 3.27. *If for each pair of distinct x_1 and x_2 in an ideal topological space $(X, \tau, \mathcal{I})$ there exists a function f of X into a Hausdorff space Y such that $f(x_1) \neq f(x_2)$, f is weakly $\mathcal{I}$-continuous and f is almost $\mathcal{I}$-continuous at x_2, then $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-T_2.*

Proof. Since Y is Hausdorff, if for each pair of distinct point x_1 and x_2 there exist disjoint open sets V_1 and V_2 of Y containing $f(x_1)$ and $f(x_2)$, respectively; hence $\mathrm{Cl}(V_1) \cap \mathrm{Int}(\mathrm{Cl}(V_2)) = \emptyset$. Since f is weakly $\mathcal{I}$-continuous at x_1, there exists $U_1 \in \mathcal{I}O(X, x_1)$ such that $f(U_1) \subset \mathrm{Cl}(V_1)$. Since f is almost $\mathcal{I}$-continuous at x_2, there exists $U_2 \in \mathcal{I}O(X, x_2)$ such that $f(U_2) \subset \mathrm{Int}(\mathrm{Cl}(V_2))$. Therefore, we obtain $U_1 \cap U_2 = \emptyset$. This shows that $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-T_2. □

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