Poincare Journal of Analysis & Applications
Vol. 2018 (1), 9-23

General decay of solution for a porous-elastic system with weak nonlinear dissipation

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Date of Receiving	:	14.01.2018
Date of Revision	:	28.04.2018
Date of Acceptance	:	14.06.2018

Abstract. In this work the asymptotic behaviour of solution for one-dimensional equations of an homogeneous and isotropic porous elastic solid is analyzed. We use a classic result of P. Martinez [1] to obtain the general decay result. We give some example to illustrate the energy decay rates and consider also the case of the polynomial growth.

1. Introduction

Elastic solids with voids is one of the simple extensions of the theory of the classical elasticity. It allows the treatment of porous solids in which the matrix material is elastic and the interstices are void of material (see Goodman and Cowin [2] and Nunziato and Cowin [3] for details).

It is worth noting that in the last decades, research on porous elastic systems has been made considering various types of dissipative mechanisms. We describe here some of the main results. To do this, let us consider the following evolution equations in one-dimensional case

$$\begin{cases} \rho u_{tt} = T_x, \\ J\phi_{tt} = H_x + G. \end{cases} \tag{1.1}$$

2010 *Mathematics Subject Classification.* 35Q70, 35B40, 74F05.
Key words and phrases. Porous-elasticity, exponential decay, localized damping.
The authors thanks the anonymous referees for their valuable comments as well as suggestions that helped to improve the presentation of this paper. The first author has been partially supported by the CNPq Grant 163428/2014-0. The second author has been partially supported by the CNPq Grant 311553/2013-3 and the third author was partially supported by PNPD/UFBA/CAPES(Brazil).
Communicated by: Mauricio S. Cortes and Shiv K. Kaushik
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Here T is the stress, H is the equilibrated stress and G is the equilibrated body force. The variables u and ϕ represent the displacement of a solid elastic material and the volume fraction, respectively. The constitutive equations are

$$\begin{cases} T = \mu u_x + b\phi, \\ H = \delta\phi_x, \\ G = -bu_x - \xi\phi - \tau\phi_t. \end{cases} \tag{1.2}$$

Then substituting (1.2) into (1.1), we have

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 \quad \text{in} \quad (0, L) \times (0, \infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \tau\phi_t = 0 \quad \text{in} \quad (0, L) \times (0, \infty). \end{cases} \tag{1.3}$$

We added to system (1.3) the initial conditions given by

$$\begin{cases} (u(x, 0), \phi(x, 0)) = (u_0(x), \phi_0(x)), \quad \text{in} \quad (0, L), \\ (u_t(x, 0), \phi_t(x, 0)) = (u_1(x), \phi_1(x)), \quad \text{in} \quad (0, L) \end{cases} \tag{1.4}$$

and Dirichlet-Dirichlet boundary conditions

$$u(0, t) = u(L, t) = \phi(0, t) = \phi(L, t) = 0, \quad t > 0 \tag{1.5}$$

or Dirichlet-Neumann boundary conditions

$$u(0, t) = u(L, t) = \phi_x(0, t) = \phi_x(L, t) = 0, \quad t > 0. \tag{1.6}$$

Here ρ, μ, $J = \rho\kappa$, δ, b, κ, τ and ξ are the constitutive coefficients whose physical meaning is well known. The constitutive coefficients, in one-dimensional case, satisfies

$$\xi > 0,\ \delta > 0,\ \mu > 0,\ \rho > 0,\ J > 0,\ \kappa > 0,\ \tau > 0, \text{ and } \mu\xi \geq b^2. \tag{1.7}$$

It is worth mentioning that system (1.3)-(1.4) with boundary conditions (1.6) has been studied by R. Quintanilla in [4]. There, the author used the Hurtwitz theorem to prove the lack of exponential decay when $\frac{\rho}{\mu} \neq \frac{J}{\delta}$.

In [5] A. Magaña and R. Quintanilla considered the system:

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x - \gamma u_{xxt} = 0 \quad \text{in} \quad (0, L) \times (0, \infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \tau\phi_t = 0 \quad \text{in} \quad (0, L) \times (0, \infty), \\ u(0, t) = u(L, t) = \phi_x(0, t) = \phi_x(L, t) = 0, \quad t > 0, \\ (u(x, 0), \phi(x, 0)) = (u_0(x), \phi_0(x)), \quad \text{in} \quad (0, L) \\ (u_t(x, 0), \phi_t(x, 0)) = (u_1(x), \phi_1(x)), \quad \text{in} \quad (0, L) \end{cases} \tag{1.8}$$

with positive constant γ. They proved that system (1.8) is exponentially stable using semigroup arguments due to Liu and Zheng [6]. Also, they proved that when $\tau = 0$ the system is not exponentially stable. In [7] J. Muñoz Rivera and R. Quintanilla proved that when $\tau = 0$ the energy is controlled by a rate decay of type $\frac{1}{t}$. Moreover, using a result due to Prüss [8] they improved the polynomial rate of decay by taking more regular initial data. In [9] Pamplona et. al., studied the analyticity of the solutions for one-dimensional porous elasticity problem with temperatures and microtemperatures when viscoelasticity and porous viscosity effects are also present. They showed the lack of analyticity when the porous dissipation is weak, and the analyticity when it is strong. Other problems associated with porous elastic systems can be found in references [10, 11, 12, 13].

On the other hand, localized frictional damping has been studied by several authors in one or more space dimension, (see e.g. [14, 15, 16, 17, 18, 19]). The main result of the above articles is that localized frictional damping produces exponential decay in time of the solution. A more general result occurs in one-dimensional space where the solution always decays exponentially to zero for any localized frictional damping active over an open subset of the domain. This result is no longer valid for materials configured over bounded domain $\Omega \subset \mathbb{R}^n$ for $n \geq 2$ where the position of the frictional effect is important. See for example [20], where necessary and sufficient conditions are given to get stabilization of the wave equation with localized frictional damping. That is, to get the exponential stability, the damping mechanism must be present in a sufficient large neighborhood of the boundary, see also [15].

In [21] Santos et al considered the full damped system, with weakly dissipation in both components, displacement of a solid elastic material and the volume fraction. The authors that the corresponding semigroup is exponentially stable if and only if the wave speeds of the system are equal. In the case of lack of exponential stability they also showed that the solution is polynomially stable with optimal rate of decay.

In [22] was studied the action of viscoelastic damping and was proved that the viscoelasticity is not strong enough to make the solutions decay in an exponential way, independently of any relationship between the coefficients of wave propagation speed. In this case the solution decays polynomially with optimal rate. Another important analysis was given, that is, when the porous damping is coupled with microtemperatures, there exists a explicit characterization on the decay rate that can be exponential or polynomial type, depending on the relation between the coefficients of wave propagation speed.

In [23] Freitas et al considered the porous-elastic system with nonlinear dissipation and source terms. Again was considered the damping in both components. By employing nonlinear semigroups and the theory of monotone operators, was obtained several results on the existence of local and global weak solutions, and uniqueness of weak solutions. Applying the potential well theory, the authors showed that the total energy of the system decays exponentially or algebraically, depending on the behavior of the dissipation in the system near the origin. They also proved the existence of a global attractor.

In this work we consider the damping just in the volume fraction. Taking into account the technical condition $\frac{\rho}{\mu} = \frac{J}{\delta}$ the purpose of this article is to prove the general decay for one-dimensional equations of the homogeneous and isotropic porous elastic solid given by

$$\left\{ \begin{array}{c} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 \quad \text{in} \quad (0,L) \times (0,\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \gamma(t)g(\phi_t) = 0 \quad \text{in} \quad (0,L) \times (0,\infty), \\ (u(x,0), \phi(x,0)) = (u_0(x), \phi_0(x)), \quad \text{in} \quad (0,L), \\ (u_t(x,0), \phi_t(x,0)) = (u_1(x), \phi_1(x)), \quad \text{in} \quad (0,L). \end{array} \right. \tag{1.9}$$

This paper is organized as follows. In the section 2 we give the appropriate hypothesis about the functions γ and g and introduce the classic result of P. Martinez on energy decay rate for the dissipative system. In the section 3, we give our main result and present some examples taking in account the comparative case in which g has exponential growth. Finally in the section 4 we consider the case where g has polynomial growth.

2. Preliminary results

We consider the following hypotheses on γ and g:

$(H1)$ $\gamma : [0, +\infty) \to \mathbb{R}_+$ is a nonincreasing differentiable function satisfying

$$\int_0^\infty \gamma(t)dt = +\infty.$$

$(H2)$ $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^0 function such that there exist a strictly increasing function $g_0 \in C^1([0, \infty)$, with $g_0(0) = 0$, and positive constants c_1, c_2 and m such that

$$g_0(|s|) \leq |g(s)| \leq g_0^{-1}(|s|) \quad \text{for all} \;\; |s| \leq m,$$

$$c_1|s| \leq |g(s)| \leq c_2|s| \quad \text{for all} \;\; |s| \geq m.$$

Remark 2.1. Hypothesis $(H1)$ implies that γ is bounded and $(H2)$ implies that $sg(s) > 0$ for all $s \neq 0$.

We have the following result concerning existence and uniqueness of solutions of the problem (1.9)

Proposition 2.2. Let $(u_0, u_1), (\phi_0, \phi_1) \in H_0^1(0, L) \times L^2(0, L)$ be given. Assume that $(H1)$ and $(H2)$ are satisfied; then problem (1.9) has a unique global (weak) solution

$$u, \phi \;\in C((0, \infty); H_0^1(0, L)) \cap C^1((0, \infty); L^2(0, L)).$$

Moreover, if

$$(u_0, u_1), (\phi_0, \phi_1) \in (H^2(0, L) \cap H_0^1(0, L)) \times H_0^1(0, l),$$

then the solution satisfies

$$u, \phi \;\in L^\infty((0, \infty); H^2(0, L) \cap H_0^1(0, L)) \cap W^{1,\infty}((0, \infty); H_0^1(0, L)) \cap W^{2,\infty}((0, \infty); L^2(0, L)).$$

Remark 2.3. This result can be proved using standard arguments such as the nonlinear semi-group method or the Galerkin method.

The following lemma will be of essential use in establishing our main result.

Lemma 2.4. *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ a strictly increasing C^1 function, with $\alpha(t) \to +\infty$ as $t \to +\infty$.*

Assume that there exist $p, q \geq 0$ and $c > 0$ such that

$$\int_S^\infty \alpha'(t)E(t)^{1+p}dt \leq cE(t)^{1+p} + \frac{cE(t)}{\alpha^q}, \quad 1 \leq S < \infty$$

Then, there exist positive constants K and ω such that

$$E(t) \leq ke^{-\omega\alpha(t)} \quad \forall\, t \geq 1 \;\; if \;\; p = q = 0$$

and

$$E(t) \leq \frac{k}{\alpha(t)^{\frac{(1+q)}{p}}} \quad \forall\, t \geq 1 \;\; if \;\; p > 0.$$

Proof. See, [1]. ☐

3. Asymptotic behaviour

In this section we study the asymptotic behavior of the solution of system (1.9). For this purpose we establish several lemmas.

Lemma 3.1. *Let (u, ϕ) be the solution of* (1.9). *Then the energy functional $E(t)$ defined by*

$$\frac{1}{2}\frac{d}{dt}\int_0^L \Big(\rho|u_t|^2 + \mu|u_x|^2 + J|\phi_t|^2 + \delta|\phi_x|^2 + \xi|\phi|^2 + 2bu_x\phi\Big)dx \tag{3.1}$$

satisfies

$$E'(T) = -\gamma(t)\int_0^L \phi_t g(\phi_t)dx. \tag{3.2}$$

Proof. By multiplying equations in (1.9) by u_t and ϕ_t, respectively, and integrating over $(0, L)$, using integration by parts, hypotheses $(H1)$ - $(H2)$, we obtain (3.2). □

Now we are going to construct a Lyapunov functional $\mathcal{L}$ equivalent to E, with which we can show the desired result. For this, we define several functionals that allow us to obtain the needed estimates.

Lemma 3.2. *Under the assumptions* $(H1)$ *and* $(H2)$, *the functional F_1 defined by*

$$F_1(t) := -\int_0^L \Big(\rho u u_t + J\phi\phi_t\Big)dx$$

satisfies, along with the solution, the estimate

$$\begin{aligned} F_1'(t) \quad &\leq \quad -\int_0^L \Big(\rho u_t^2 + J\phi_t^2 + \mu u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi\Big)dx + \epsilon C_p\int_0^L \phi_x^2 dx \\ &+ \quad c\int_0^L g^2(\phi_t)dx. \end{aligned} \tag{3.3}$$

Proof. Making the derivatives of $F_1(t)$, we have

$$F_1'(t) = -\int_0^L \Big(\rho u_t^2 + J\phi_t^2 + \mu u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi\Big)dx - \int_0^L \gamma(t)\phi g(\phi_t)dx,$$

using Young's and Poincaré's inequalities, gives the desired result. □

Lemma 3.3. *Assume that* $(H1)$ *and* $(H2)$ *hold. Then, the functional F_2 defined by*

$$F_2(t) := \int_0^L \Big(J\phi_t u_x + J\phi_x u_t\Big)dx$$

satisfies, along with the solution, the estimate

$$\begin{aligned} F_2'(t) \quad &\leq \quad \left(\frac{J\mu - \rho\delta}{\rho}\right)\int_0^L \phi_x u_{xx}dx - \int_0^L \left(bu_x^2 + \frac{Jb}{\rho}\phi_x^2\right)dx + c\epsilon\int_0^L u_x^2 dx \\ &+ \quad c\int_0^L g^2(\phi_t)dx. \end{aligned} \tag{3.4}$$

Proof. Using equations (1.9) and integrating by parts yield

$$F_2'(t) = \left(\frac{J\mu-\rho\delta}{\rho}\right)\int_0^L \phi_x u_{xx}dx - b\int_0^L u_x^2dx - \xi\int_0^L \phi u_x dx - \int_0^L \gamma(t)g(\phi_t)u_x dx - \frac{Jb}{\rho}\int_0^L \phi_x^2 dx$$

then, by Young's inequality, we obtain

$$F_2'(t) \le \left(\frac{J\mu-\rho\delta}{\rho}\right)\int_0^L \phi_x u_{xx}dx - b\int_0^L u_x^2dx + \frac{\xi}{\epsilon}\int_0^L \phi^2 dx + c\epsilon\int_0^L u_x^2 dx$$
$$-\frac{Jb}{\rho}\int_0^L \phi_x^2dx + \frac{\xi}{\epsilon}\int_0^L \phi^2dx + c\int_0^L g^2(\phi_t)^2dx.$$

This gives (3.4). □

Lemma 3.4. *Assume that* $\chi = \left(\frac{J\mu-\rho\delta}{\rho}\right) = 0$, *(H1) and (H2) hold. Then, the functional* F *defined by*

$$F(t) := F_1 + F_2,$$

satisfies, along with the solution, the estimate

$$\begin{aligned} F'(t) \quad &\le \quad -\int_0^L \left(\rho u_t^2 + J\phi_t^2 + c\mu u_x^2 + \delta\phi_x^2 + c\xi\phi^2 + 2bu_x\phi\right)dx \\ &+ \quad \epsilon C_p\int_0^L \phi_x^2dx + c\epsilon\int_0^L u_x^2dx + \int_0^L g^2(\phi_t)dx. \end{aligned} \tag{3.5}$$

Proof. By using **Lemmas 3.2-3.3**, we obtain (3.5). □

As in [14], we use the multiplier w, which solves

$$-w_{xx} = \phi_x, \qquad w(0) = w(L). \tag{3.6}$$

Lemma 3.5. *The solution os* (3.6) *satisfies*

$$\int_0^L w_x^2dx \le \int_0^L \phi^2dx, \qquad \int_0^L w_t^2dx \le \int_0^L \phi_t^2dx.$$

Proof. We multiply (3.6) by w, integrate by parts, and use the Cauchy-Schwarz inequality to obtain

$$\int_0^L w_x^2dx \le \int_0^L \phi^2dx.$$

Next, we differentiate (3.6) with respect to t then multiply by w_t, to obtain, by similar calculations.

$$\int_0^L w_{xt}^2dx \le \int_0^L \phi_t^2dx.$$

Poincaré's inequality then yields

$$\int_0^L w_t^2dx \le \int_0^L \phi_t^2dx.$$

This completes the proof of **Lemma 3.4**. □

Lemma 3.6. *Assume that $(H1)$ and $(H2)$ hold. Then,for any $0 < \sigma < 1$, the functional K defined by*

$$K(t) := \int_0^L \Big(J\phi\phi_t + \rho w u_t \Big) dx$$

satisfies, along with the solution, the estimate

$$\begin{aligned} K'(t) &\leq \int_0^L \Big\{ \Big(J + \frac{c}{\sigma} \Big) \phi_t^2 + c\sigma u_t^2 + c_p \phi_x^2 + \frac{c}{\sigma} u_x^2 + c\sigma \phi^2 \Big\} dx - \int_0^L \Big(\delta\phi_x^2 + \xi\phi^2 \Big) dx \\ &+ c\int_0^L g^2(\phi_t) dx. \end{aligned} \tag{3.7}$$

Proof. By deriving $K(t)$, substituting the equation (1.9), integrating by parts, using Young's and Poincaré's inequalities, we obtain

$$\begin{aligned} K'(t) &= J\int_0^L \phi_t^2 dx - \delta \int_0^L \phi_x^2 dx - b\int_0^L u_x \phi dx - \xi \int_0^L \phi^2 dx - \int_0^L \gamma(t)\phi g(\phi_t) dx \\ &+ \rho \int_0^L w_t u_t dx - \mu \int_0^L w_x u_x dx - b\int_0^L w_x \phi dx \\ &\leq J\int_0^L \phi_t^2 dx - \delta \int_0^L \phi_x^2 dx + \frac{c}{\sigma}\int_0^L u_x^2 dx + c\sigma \int_0^L \phi^2 dx - \xi \int_0^L \phi^2 dx \\ &+ c\sigma \int_0^L \phi^2 dx + \frac{c}{\sigma}\int_0^L g^2(t) dx + \rho\sigma \int_0^L u_t^2 dx + \frac{c}{\sigma}\int_0^L w_t^2 dx + \frac{c}{2}\int_0^L w_x^2 dx \\ &+ \frac{c}{\sigma}\int_0^L u_x^2 dx + \frac{c}{2}\int_0^L w_x^2 dx + c\sigma \int_0^L \phi^2 dx. \end{aligned}$$

Then **Lemma 3.4** gives the desired result. □

Now, with $N_1, N_2 > 0$, we introduce the Lyapunov functional $\mathcal{L}(t)$ given by

$$\mathcal{L}(t) := N_1 E(t) + N_2 K(t) + F(t).$$

By combining (3.2), (3.5) and (3.7), we arrive at

$$\begin{aligned} \mathcal{L}'(t) &\leq -N_1\gamma(t) \int_0^L \phi_t g(\phi_t) dx - \big(C_p - N_2\big) \int_0^L \phi_x^2 dx - \big(c - c\sigma N_2\big) \int_0^L u_t^2 dx \\ &- \big(J - c\frac{N_2}{\sigma}\big) \int_0^L \phi_t^2 dx - \big(c\mu - c\frac{N_2}{\sigma}\big) \int_0^L u_x^2 dx - \big(c\xi - c\sigma N_2\big) \\ &- \int_0^L \phi^2 dx \int_0^L 2bu_x\phi dx + J\int_0^L \phi_t^2 + \big(cN_2 + c\big) \int_0^L g^2(\phi_t) dx. \end{aligned} \tag{3.8}$$

We choose N_2 large enough and σ small enough so that

$$k_1 := \big(C_p - N_2\big) > 0,$$

$$k_2 := \big(c - c\sigma N_2\big) > 0,$$

$$k_3 := \big(J - c\frac{N_2}{\sigma}\big) > 0,$$

$$k_4 := \big(c\mu - c\frac{N_2}{\sigma}\big) > 0,$$

$$k_5 := \big(c\xi - c\sigma N_2\big) > 0.$$

Thus, (3.8) becomes

$$\begin{aligned} \mathcal{L}'(t) &\leq -k_1\int_0^L \phi_x^2 dx - k_2\int_0^L u_t^2 dx - k_3\int_0^L \phi_t^2 dx - k_4\int_0^L u_x^2 dx \\ &- k_5\int_0^L \phi^2 dx - \int_0^L 2bu_x\phi dx + c\int_0^L \Big(\phi_t^2 + g^2(\phi_t)\Big)dx \\ &\leq -d_1E(t) + c\int_0^L \Big(\phi_t^2 + g^2(\phi_t)\Big)dx \end{aligned} \tag{3.9}$$

or

$$E(t) \leq -d_2\mathcal{L}'(t) + c\int_0^L \Big(\phi_t^2 + g^2(\phi_t)\Big)dx. \tag{3.10}$$

On the other hand, we choose N_1 large enough so that

$$\mathcal{L} \sim E(t). \tag{3.11}$$

We are now ready to state and prove our main result.

Theorem 3.7. *Assume that (H1) and (H2) hold, then, there exists a constant $c > 0$ such tha, for t large, the solution of (1.9) satisfies*

$$E(t) \leq c\bigg(G_0^{-1}\bigg(\frac{1}{\int_0^t \gamma(s)ds}\bigg)\bigg)^2, \tag{3.12}$$

where

$$G_0(s) = sg_0(s).$$

Let κ, defined by $\kappa(s) = \frac{g_0(s)}{s}$, a strictly increasing function on $[0, r)$, for some $r > 0$, with $\kappa(0) = 0$, then we have the improved estimate

$$E(t) \leq c\bigg(g_0^{-1}\bigg(\frac{1}{\int_0^t \gamma(s)ds}\bigg)\bigg)^2. \tag{3.13}$$

Proof. We define

$$\psi(t) := 1 + \int_1^t \left(\frac{1}{g_0\left(\frac{1}{s}\right)}\right) ds \qquad t \geq t',$$

for some $t' > max\{1, \frac{1}{m}\}$. Then

$$\psi'(t) := \frac{1}{g_0\left(\frac{1}{t}\right)} > 0 \quad t \geq t', \quad \psi'(t) \to +\infty \quad as \;\; t \to +\infty$$

and $\psi'(t)$ is strictly increasing.

Thus, ψ is a convex and strictly increasing C^2 function, with $\psi'(t) \to +\infty \quad as \;\; t \to +\infty$.

If we set $\alpha_0 := \psi^{-1}$, then α_0 is strictly increasing, $\alpha_0'(t) = g_0\left(\frac{1}{\alpha_0(t)}\right)$ is nonincreasing, and $\alpha_0(t) \to +\infty \quad as \;\; t \to +\infty$.

Let us define

$$\alpha(t) := \alpha_0\left(\int_0^t \gamma(s)\, ds\right), \qquad t \geq t_1$$

for some $t_1 \geq t'$ with $\int_0^t \gamma(s)ds \geq t'$ and $\alpha_0\left(\int_0^t \gamma(s)\, ds\right) < m$. Using the properties of α_0 and γ, it is easy to chech that α is strictly increasing and concave twice differentiable function, with $\alpha_0(t) \to +\infty \quad as \quad t \to +\infty$. Using (3.10) and (3.11), we have for $T \geq S \geq t_1$,

$$\begin{aligned}
\int_S^T \alpha'(t)E(t)^2\, dt &\leq c\int_S^T \alpha'(t)E\mathcal{L}\, dt \\
&\leq -c'\int_S^T \alpha'(t)\mathcal{L}'\mathcal{L}\, dt + c\int_S^T \alpha'(t)\mathcal{L}\left(\int_0^L (\phi_t^2 + g^2(\phi_t))\, dx\right) dt \\
&\leq cE(S)^2 + c\int_S^T \alpha'(t)E\left(\int_0^L (\phi_t^2 + g^2(\phi_t))\, dx\right) dt. \qquad (3.14)
\end{aligned}$$

Now, using (H2), (3.2), and the properties of α, α_0 and γ, we estimate the last integral as follows.

(1) Estimate for $\int_S^T \alpha'(t)E\left(\int_0^L \phi_t^2\, dx\right) dt$

We consider the following partition of $(0, L)$:

$$\begin{aligned}
\Omega_1 &= \left\{x \in (0,L); |\phi_t| > m\right\}, \\
\Omega_2 &= \left\{x \in (0,L); |\phi_t| \leq m \text{ and } |\phi_t| \leq g_0^{-1}\left(\alpha_0'\left(\int_0^t \gamma(s)\, ds\right)\right)\right\}, \qquad (3.15) \\
\Omega_3 &= \left\{x \in (0,L); |\phi_t| \leq m \text{ and } |\phi_t| > g_0^{-1}\left(\alpha_0'\left(\int_0^t \gamma(s)\, ds\right)\right)\right\}.
\end{aligned}$$

Then, we have

$$\alpha'(t)\int_{\Omega_1} \phi_t^2\, dx \leq c\alpha_0'\left(\int_0^t \gamma(s); ds\right)\gamma(t)\int_0^L \phi_t g(\phi_t)\, dx \leq -cE'(t)$$

$$\alpha'(t)\int_{\Omega_2}\phi_t^2\,dx \le \alpha_0'\Big(\int_0^t\gamma(s);ds\Big)\gamma(t)\Big(g_0^{-1}\Big(\alpha_0'\Big(\int_0^t\gamma(s)\,ds\Big)\Big)\Big)^2$$

$$\begin{aligned}\alpha'(t)\int_{\Omega_3}\phi_t^2\,dx &= c\alpha_0'\Big(\int_0^t\gamma(s);ds\Big)\gamma(t)\int_{\Omega_3}\phi_t^2\,dx\\ &\le m\gamma(t)\int_{\Omega_3}g_0(|\phi_t|)|\phi_t|\,dx\\ &\le \int_0^L\phi_t g(\phi_t)\,dx=-E'(t).\end{aligned}$$

Consequently, we obtain

$$\alpha'(t)\int_0^L\phi_t^2\,dx \le -cE'(t)+\gamma(t)\alpha_0'\Big(\int_0^t\gamma(s);ds\Big)\Big(g_0^{-1}\Big(\alpha_0'\Big(\int_0^t\gamma(s)\,ds\Big)\Big)\Big)^2$$

which gives

$$\begin{aligned}&\int_S^T\alpha'(t)E\Big(\int_0^L\phi_t^2\,dx\Big)\,dt\\ &\quad\le cE(S)^2+cE(S)\int_S^T\gamma(t)\alpha_0'\Big(\int_0^t\gamma(s);ds\Big)\Big(g_0^{-1}\Big(\alpha_0'\Big(\int_0^t\gamma(s)\,ds\Big)\Big)\Big)^2\,dt.\end{aligned}\tag{3.16}$$

(2) Estimate for $\int_S^T\alpha'(t)E\Big(\int_0^L g^2(\phi_t)\,dx\Big)\,dt$

We consider the following partition of $(0,L)$:

$$\begin{aligned}\Theta_1&=\Big\{x\in(0,L);|\phi_t|>m\Big\},\\ \Theta_2&=\Big\{x\in(0,L);|\phi_t|\le m\text{ and }|\phi_t|\le\alpha_0'\Big(\int_0^t\gamma(s)\,ds\Big)\Big\},\\ \Theta_3&=\Big\{x\in(0,L);|\phi_t|\le m\text{ and }|\phi_t|>\alpha_0'\Big(\int_0^t\gamma(s)\,ds\Big)\Big\}.\end{aligned}\tag{3.17}$$

Then, we have

$$\alpha'(t)\int_{\Theta_1}g^2(\phi_t)\,dx\le c\alpha_0'\Big(\int_0^t\gamma(s);ds\Big)\gamma(t)\int_0^L\phi_t g(\phi_t)\,dx\le -cE'(t)$$

$$\begin{aligned}\alpha'(t)\int_{\Theta_2}g^2(\phi_t)\,dx &\le \gamma(t)\alpha_0'\Big(\int_0^t\gamma(s);ds\Big)\int_{\Theta_2}(g_0^{-1}(|\phi_t|))^2\,dx\\ &\le \gamma(t)\alpha_0'\Big(\int_0^t\gamma(s);ds\Big)\Big(g_0^{-1}\Big(\alpha_0'\Big(\int_0^t\gamma(s)\,ds\Big)\Big)\Big)^2\end{aligned}$$

$$\begin{aligned}\alpha'(t)\int_{\Theta_3}g^2(\phi_t)\,dx &= \gamma(t)\alpha_0'\Big(\int_0^t\gamma(s);ds\Big)\int_{\Theta_3}g^2(\phi_t)\,dx\\ &\le g_0^{-1}(m)\gamma(t)\int_0^L\phi_t g(\phi_t)\,dx\le -cE'(t)\end{aligned}$$

which implies

$$\int_S^T \alpha'(t)E\Big(\int_0^L g^2(\phi_t)\,dx\Big)\,dt \le cE(S)^2$$
$$+cE(S)\int_S^T \gamma(t)\alpha_0'\Big(\int_0^t \gamma(s)ds\Big)\Big(g_0^{-1}\Big(\alpha_0'\Big(\int_0^t \gamma(s)\,ds\Big)\Big)\Big)^2 dt.$$

A combination of (3.14), (3.16) and (3.18) yields

$$\begin{aligned}\int_S^\infty \alpha'(t)E(t)^2\,dt &\le cE(S)^2\\ &+ cE(S)\int_S^\infty \gamma(t)\alpha_0'\Big(\int_0^t \gamma(s)ds\Big)\Big(g_0^{-1}\Big(\alpha_0'\Big(\int_0^t \gamma(s)\,ds\Big)\Big)\Big)^2 dt\\ &= cE(S)^2 + cE(S)\int_{(\int_0^S \gamma(s)ds)}^\infty \alpha_0'(\tau)(g_0^{-1}(\alpha_0'(\tau)))^2 d\tau\\ &= cE(S)^2 + cE(S)\int_{\alpha_0(\int_0^S \gamma(s)ds)}^\infty \Big(g_0^{-1}\Big(g_0'\Big(\frac{1}{s}\Big)\Big)\Big)^2 ds\\ &= cE(S)^2 + \frac{cE(S)}{\alpha_0(\int_0^S \gamma(s)ds)} = cE(S)^2 + \frac{cE(S)}{\alpha(S)}.\end{aligned}$$

Lemma 2.4 then gives

$$E(t) \le \frac{c}{\alpha(t)^2} = \frac{c}{(\alpha_0(\int_0^t \gamma(s)ds))^2} \quad \forall\, t \ge t_1. \tag{3.18}$$

To obtain (3.12), take $s_0 > t'$ such that $g_0\big(\frac{1}{s_0}\big) \le 1$. Since g_0 is increasing and $G_0(s) = sg_0(s)$, we have

$$\alpha_0{-}1(s) \le 1 + (s-1)\frac{1}{g_0\big(\frac{1}{s}\big)} \le \frac{s}{g_0\big(\frac{1}{s}\big)} = \frac{1}{G_0\big(\frac{1}{s}\big)}, \quad \forall\, s \ge s_0.$$

Therefore, with $t = \frac{1}{G_0\left(\frac{1}{s}\right)}$, we easily see that

$$\frac{1}{\alpha_0(t)} \le \frac{1}{G_0^{-1}\big(\frac{1}{t}\big)}, \quad \forall\, t \ge t'.$$

This yields (3.12) by virtue of (3.18).

To prove (3.13), we assume that $r = m$. In fact, if $r < m$ and $r \le |s| \le m$, then, using $(H2)$, we have

$$|g(s)| \le \frac{g_0^{-1}(|s|)}{|s|}|s| \le \frac{g_0^{-1}(m)}{r}|s| \quad \text{and} \quad |g(s)| \ge \frac{g_0(|s|)}{|s|}|s| \le \frac{g_0(m)}{r}|s|.$$

This implies that

$$\begin{array}{rll} g_0(|s|) \le & |g(s)| & \le g_0^{-1}(|s|) \quad \text{for all } \ |s| \le r,\\ c_1'|s| \le & |g(s)| & \le c_2'|s| \quad \text{for all} |s| \ge r,\end{array}$$

which justifies our assumption.

Now, we take

$$\alpha_0 := \psi^{-1} \quad \text{where} \;\; \psi(t) := 1 + \int_1^t \frac{1}{\kappa\big(\frac{1}{s}\big)} ds, \quad t \geq t'.$$

In this case, we replace (3.15) and (3.17) by

$$\begin{aligned}
\Omega_1 &= \Big\{ x \in (0,L); |\phi_t| > m \Big\}, \\
\Omega_2 &= \Big\{ x \in (0,L); |\phi_t| \leq m \text{ and } |\phi_t| \leq \kappa^{-1}\Big(\alpha_0'\Big(\int_0^t \gamma(s)\, ds\Big)\Big) \Big\}, \\
\Omega_3 &= \Big\{ x \in (0,L); |\phi_t| \leq m \text{ and } |\phi_t| > \kappa^{-1}\Big(\alpha_0'\Big(\int_0^t \gamma(s)\, ds\Big)\Big) \Big\}.
\end{aligned}$$

and

$$\begin{aligned}
\Theta_1 &= \Big\{ x \in (0,L); |\phi_t| > m \Big\}, \\
\Theta_2 &= \Big\{ x \in (0,L); |\phi_t| \leq m \text{ and } g_0^{-1}(|\phi_t|) \leq \kappa^{-1}\Big(\alpha_0'\Big(\int_0^t \gamma(s)\, ds\Big)\Big) \Big\}, \\
\Theta_3 &= \Big\{ x \in (0,L); |\phi_t| \leq m \text{ and } g_0^{-1}(|\phi_t|) > \kappa^{-1}\Big(\alpha_0'\Big(\int_0^t \gamma(s)\, ds\Big)\Big) \Big\}.
\end{aligned}$$

Therefore, we find that

$$\begin{aligned}
\alpha'(t) \int_{\Omega_3} \phi_t^2 \, dx &= c\alpha_0'\Big(\int_0^t \gamma(s); ds\Big)\gamma(t) \int_{\Omega_3} \phi_t^2 \, dx \\
&\leq \gamma(t) \int_{\Omega_3} \kappa^{-1}(|\phi_t|)|\phi_t| \, dx \\
&= \gamma(t) \int_{\Omega_3} g_0(|\phi_t|)|\phi_t| \, dx \\
&\leq \int_0^L \phi_t g(\phi_t) \, dx = -E'(t)
\end{aligned}$$

and

$$\begin{aligned}
\alpha'(t) \int_{\Theta_3} g^2(\phi_t) \, dx &= \gamma(t)\alpha_0'\Big(\int_0^t \gamma(s); ds\Big) \int_{\Theta_3} g^2(\phi_t) \, dx \\
&\leq \gamma(t) \int_{\Theta_3} \kappa(g_0^{-1}(|\phi_t|)) g_0^{-1}(|\phi_t|)|g(\phi_t)| \, dx \\
&= \gamma(t) \int_0^L \phi_t g(|\phi_t|) dx \leq -cE'(t).
\end{aligned}$$

The other cases can be dealt with similarly and the same reason leads to (3.13). □

Examples

We give some examples to illustrate the energy decay rates obtained by *Theorem 3.7.*

(1) Between polynomial and exponential growth: If $g_0(s) = e^{-lns}$ near zero, then we have the following energy decay rate:

$$E(t) \leq c e^{-2\sqrt{ln(\int_0^t \gamma(s) ds)}}.$$

(2) Exponential growth: If $g_0(s) = e^{-1/s}$ near zero,then we have the following energy decay rate:

$$E(t) \leq \frac{c}{(ln(\int_0^t \gamma(s)ds))^2}.$$

(3) Faster than exponential growth: If$g_0(s) = e^{-\sqrt[s]{e}}$ near zero. Then, we have the following energy decay rate: $E(t) \leq \frac{c}{(ln(ln(\int_0^t \gamma(s)ds)))^2}$.

4. Polynomial growth

As a special case of $(H2)$, we assume that there exist constants $c_1, c_2 > 0$ and $q \geq 1$ such that

$$c_1 min\{|s|, |s|^q\} \leq |g(s)| \leq c_2 max\{|s|, |s|^{1/q}\}. \tag{4.1}$$

According to *Theorem 3.7*, we have the following estimate:

$$E(t) \leq \frac{c}{(\int_0^L \gamma(s)ds)^{2/q}}.$$

However, we can obtain a better decay rate as follows.

We use (3.9), (4.1) and Holder's inequality to obtain

$$\begin{aligned}
\gamma(t)\mathcal{L}'(t) &\leq -d_1\gamma(t)E(t) + c\gamma(t)\int_0^L \left(\phi_t^2 + g^2(\phi_t)\right)dx \\
&\leq -d_1\gamma(t)E(t) + c\gamma(t)\int_0^L \left[(\phi_t g(\phi_t))^{2/q+1} + \phi_t g(\phi_t)\right]dx \\
&\leq -d_1\gamma(t)E(t) + c\gamma(t)\left(\int_0^L (\phi_t g(\phi_t))dx\right)^{2/q+1} + c\gamma(t)\int_0^L \phi_t g(\phi_t)dx
\end{aligned}$$

that is,

$$F'(t) \quad \leq \quad -d_1\gamma(t)E(t) + c\gamma(t)\left(\int_0^L (\phi_t g(\phi_t))dx\right)^{2/q+1}, \tag{4.2}$$

where $F(t) = \gamma(t)\mathcal{L}(t) + cE(t)$ is clearly equivalent to E.

Case 1: If $q = 1$. In this case, we obtain

$$F'(t) = -d_1\gamma(t)E(t) - cE'(t),$$

that is,

$$(F + cE)'(t) = -d_1\gamma(t)E(t).$$

Then, using the fact that $(F + cE) \sim E$, we easily obtain

$$E(t) \leq ce^{-c'\int_0^t \gamma(s)ds}. \tag{4.3}$$

Case 2: If $q > 1$. Multiplying (4.2) by $E^{(q-1)/2}$ yields

$$F'(t)E^{(q-1)/2} \leq -d_1\gamma(t)E^{(q-1)/2} + c\gamma(t)E^{(q-1)/2}\Big(\int_0^L \phi_t g(\phi_t)dx\Big)^{2/(q+1)}.$$

Using Young's inequality, we obtain

$$F'(t)E^{(q-1)/2} \leq -d_1\gamma(t)E^{(q-1)/2} + \epsilon\gamma(t)E^{(q-1)/2} + C_\epsilon\gamma(t)\int_0^L \phi_t g(\phi_t)dx$$

which leads to

$$\Big(F(t)E^{(q-1)/2} + C_\epsilon E(t)\Big)' \leq -d_1\gamma(t)E^{(q-1)/2} + \epsilon\gamma(t)E^{(q-1)/2}.$$

Choosing ϵ small enough, we obtain

$$\Big(F(t)E^{(q-1)/2} + C_\epsilon E(t)\Big)' \leq -d_1\gamma(t)E^{(q-1)/2},$$

Consequently, using the fact that $FE^{(q-1)/2} + cE \sim E$, we easily obtain the estimate

$$E(t) \leq \frac{c}{(\int_0^t \gamma(s)ds)^{2/(q-1)}}. \tag{4.4}$$

Here, we note that there was no need of the condition $(\int_0^\infty \gamma(t)dt = +\infty)$ to obtain estimates (4.3) and (4.4). However, the presence of such a condition serves as a guarantee for the uniform stability.

Conclusion

From the point of view of the asymptotic behavior we prove energy decay rate for the system of homogeneous and isotropic porous elastic solid with a nonlinear indirect control ϕ_t on volume fraction ϕ given by $\gamma(t)g(\phi_t)$ being $\gamma : [0,\infty) \to \mathbb{R}_+$ a nonincreasing differentiable function with time dependence and $g : \mathbb{R} \to \mathbb{R}$ a nondecreasing C^0 function which may have a polynomial or exponential growth.

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