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Some results on the existence of frames in Banach spaces

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Abstract. The notion of $\mathcal{J}$-frames in Banach spaces has been defined and studied. A necessary and sufficient condition for the existence of a $\mathcal{J}$-frame in a Banach space has been given. Also, a sufficient condition, in terms of $\mathcal{J}$-frame, the existence of bounded approximation property in a Banach space has been given. Further, we study the interplay between the existence of retro Banach frames in conjugate Banach spaces and the existence of a support cone. Finally, we have given conditions for the existence of a retro Banach frame in the conjugate space of a Banach space with Fréchet differentiable norm.

1. Introduction

Dennis Gabor [12] in 1946 introduced a fundamental approach to signal decomposition in terms of elementary signals. Later, in 1952, While addressing some deep problems in non-harmonic Fourier series, Duffin and Schaeffer [9] abstracted Gabors method to define frames for Hilbert spaces. These ideas did not generate much interest outside of non-harmonic Fourier series and signal processing for more than three decades until Daubechies, Grossmann and Meyer [8] reintroduced frames. After this land mark paper the theory of frames begin to be studied widely and found new applications to wavelet and Gabor transforms in which frames played an important role. Frames are generalizations of orthonormal bases in Hilbert spaces. The main property of frames which makes them useful is their redundancy.

Representation of signals using frames is advantageous over basis expansions in a variety of practical applications in pure and applied mathematics. The theory of frame has grown rapidly in the past one decade in view of wide range of its applications in signal processing, wireless communications(CDMA system) image processing, differential equations, filter banks, geophysics, quantum computing, wireless sensor network, multiple-antenna code design and many more. The reason for such wide applications is that frames provide both great liberties in design of vector space. For a nice and

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comprehensive introduction to frames for Hilbert spaces one may refer to [4, 7, 23]. The notion of frames has been extended to Banach spaces by Feichtinger and Gröchenig [11]. They introduced the notion of atomic decomposition for Banach spaces. A more general notion called Banach frames for Banach spaces was introduced by Gröchenig [13]. Casazza, Han and Larson [6] carried out a study of atomic decompositions and Banach frames. Jain et. al [17] introduced and studied the concept of retro Banach frame for the conjugate Banach spaces. For more details regarding frames for Banach spaces and conjugate Banach spaces one may refer to [5, 6, 7, 16, 17, 18, 19]. Recently Shah et. al [14] studied the existence of non-linear frames in Banach Spaces and proved that if a Banach space $\mathcal{X}$ has approximative Schauder frame, then $\mathcal{X}^*$ has Strong Retro Banach frame. Poumai and Jahan [20] studied K-atomic decompositions in Banach spaces and obtained various results related to the existence of K-atomic decompositions. The notion of cone in Banach spaces or normed linear spaces had been studied by many authors in various contexts [2, 1].
In this paper, we have defined $\mathcal{J}$-frames in Banach spaces and studied some characterizations of $\mathcal{J}$-frames in Banach spaces. Further, we have related the concept of retro Banach frame with the geometric notion of a cone in Banach spaces.
The paper is organised as follows. In the first section we have given some basic definitions and notations that will be used throughout this paper. In the 2nd section we have define $\mathcal{J}$-frames in Banach spaces and followed by an example of $\mathcal{J}$-frame, a necessary and sufficient condition for the existence of $\mathcal{J}$-frame is given. Also, a sufficient condition in term of $\mathcal{J}$-frame for the existence of a bounded approximation property in a Banach space $\mathcal{X}$ has been given. In 3rd section we have related the notation of retro Banach frame with the geometric notion of a support cone in Banach spaces. In fact, we study the interplay between the existence of retro Banach frames in conjugate Banach spaces and the existence of a support cone. Also, we construct a Banach space and prove a result related to the existence of a Banach frame in it and some geometric property of the space. Further, we have given conditions for the existence of a retro Banach frame in the conjugate space $\mathcal{X}^*$ of a Banach space $\mathcal{X}$ with Fréchet differentiable norm.

2. Preliminaries

Throughout this paper $\mathcal{X}$ will denotes an infinite dimensional Banach space over the scalar field $\mathbb{K}(\mathbb{R},\mathbb{C})$, $\mathcal{X}^*$ denotes the conjugate space of $\mathcal{X}$ and $L(\mathcal{X},\mathcal{X})$ denote the Banach space of all continuous linear mappings of $\mathcal{X}$ into $\mathcal{X}$. For a sequence $\{x_n\} \in \mathcal{X}$ and $\{f_n\} \in \mathcal{X}^*$, $[x_n]$ denotes the closure of linear span of $\{x_n\}$ in the norm topology of $\mathcal{X}$ and $[\overline{f_n}]$ the closure of $\{f_n\}$. A sequence space S is called a *BK-space* if it is a Banach space and the co-ordinate functionals are continuous on S. That is the relations $x_n = \{\alpha_j{}^{(n)}\}$, $x = \{\alpha_j\} \in S$, $\lim\limits_{n\longrightarrow\infty} x_n = x$ imply $\lim\limits_{n\longrightarrow\infty} \alpha_j^{(n)} = \alpha_j$ $(j = 1, 2, 3, ...)$. The notion of Banach frames was introduced and studied by Gröcheing [13]. He gave the following definition:

Definition 2.1. [13] Let $\mathcal{X}$ be a Banach space over $\mathbb{K}$ $(\mathbb{R},\mathbb{C})$and $\mathcal{X}_d$ be an associated Banach space of Scalar-valued sequences, indexed by $\mathbb{N}$. Let $\{f_n\} \subset \mathcal{X}^*$ and $S : \mathcal{X}_d \longrightarrow \mathcal{X}$ be given. Then $\Phi = (\{f_n\}, S)$ is called a $Banach\ frame$ for $\mathcal{X}$ with respect to $\mathcal{X}_d$ if the following statements hold:

(i) $\{f_n(x)\} \in \mathcal{X}_d$, for each $x \in \mathcal{X}$.

(ii) There exist positive constants A and B with $0 < A \le B < \infty$ such that
$$A\|x\|_{\mathcal{X}} \le \|\{f_n(x)\}\|_{\mathcal{X}_d} \le B\|x\|_{\mathcal{X}},\ x \in \mathcal{X}. \tag{2.1}$$
(iii) S is a bounded linear operator such that
$$S(\{f_n(x)\}) = x,\ x \in \mathcal{X}.$$
The positive constants A and B, respectively are called lower and upper frame bounds of the Banach frame $\Phi = (\{f_n\}, S)$. The operator $S : \mathcal{X}_d \longrightarrow \mathcal{X}$ is called the reconstruction operator (or the pre-frame operator). The inequality (2.1) is called the frame inequality. The Banach frame $\Phi = (\{f_n\}, S)$ is called tight if $A = B$ and is called normalized tight if $A = B = 1$.

Later on the concept of retro Banach frames in conjugate Banach spaces was introduced and studied by Jain, Kaushik and Vashisht [17].

Definition 2.2. [17] Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_d$ be a BK-space. Let $\{x_n\} \subset \mathcal{X}$ and $J : \mathcal{X}_d^* \to \mathcal{X}^*$ be given. The pair $(\{x_n\}, J)$ is called a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{X}_d^*$ if the following statements hold:
(a) $\{f(x_n)\} \in \mathcal{X}_d^*$, for all $f \in \mathcal{X}^*$.
(b) There exists positive constant A and B with $0 < A \le B < \infty$ such that
$$A\|f\|_{\mathcal{X}^*} \le \|\{f(x_n)\}\|_{\mathcal{X}_d^*} \le B\|f\|_{\mathcal{X}^*},\ \text{for all } f \in \mathcal{X}^*. \tag{2.2}$$
(c) $J(\{f(x_n)\}) = f$, for all $f \in \mathcal{X}^*$.
A and B are lower and upper bounds of retro Banach frames $(\{x_n\}, J)$. The inequality (2.2) is called retro Banach frame inequality. Retro Banach frames $(\{x_n\}, J)$ is said to be exact if on removal of one x_j renders the collection $\{x_n\}_{n \ne j}$ no longer a retro Banach frame for $\mathcal{X}^*$.

Definition 2.3. [22] A Banach space $\mathcal{X}$ is said to have approximation property if for every compact set $K \subset \mathcal{X}$ and every $\epsilon > 0$, there is an operator $T : \mathcal{X} \to \mathcal{X}$ of finite rank so that $\|Tx - x\| \le \epsilon$, for every $x \in K$.

Definition 2.4. Let $\mathcal{X}$ be a Banach space and let $1 \le \lambda < \infty$. We say that $\mathcal{X}$ has the λ-approximative property if for every compact set $K \subset \mathcal{X}$ and every $\epsilon > 0$, there is an operator $T : \mathcal{X} \to \mathcal{X}$ of finite rank so that $\|Tx - x\| \le \epsilon$, for every $x \in K$ and $\|T\| \le \lambda$. A Banach space is said to have bounded approximative property if it has the λ-approximative for some λ.

Definition 2.5. A Banach space $\mathcal{X}$ is said to have the metric approximation property if for every compact set $K \subset \mathcal{X}$ and every $\epsilon > 0$, there is an operator $T \in L(\mathcal{X}, \mathcal{X})$ with $\|T\| \le 1$ such that $\|Tx - x\| \le \epsilon$, for all $x \in \mathcal{X}$.

Lemma 2.6. [16] *If $\{f_n\} \subset \mathcal{X}^*$ is total over $\mathcal{X}$, then there exists an associated Banach space $\mathcal{X}_d = \{\{f_n(x)\} : x \in \mathcal{X}\}$ and a reconstruction operator $T : \mathcal{X}_d \longrightarrow \mathcal{X}$ such that $(\{f_n\}, T)$ is a normalized tight Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$.*

Lemma 2.7. [10] *Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and let $x \in S_{\mathcal{X}}$. Then $\|\cdot\|$ is Fréchet differentiable at x if and only if $\lim\limits_{n\to\infty} \|f_n - g_n\| = 0$ whenever $f_n,\ g_n \in S_{\mathcal{X}^*}$ satisfy $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} g_n(x) = 1$ if and only if $\{f_n\} \subset S_{\mathcal{X}^*}$ is convergent whenever $\lim\limits_{n\to\infty} (f_n(x)) = 1$.*

Theorem 2.8. [10] *Let $\mathcal{X}$ be a Banach space. The set of all functionals in $\mathcal{X}^*$ that attains their norm is dense in $\mathcal{X}^*$.*

3. $\mathcal{J}$-frames in Banach Spaces

We begin this section with the following definition of $\mathcal{J}$-frame

Definition 3.1. Let $\mathcal{X}$ be a Banach space and let $\{u_n\} \subset L(\mathcal{X}, \mathcal{X})$ be a sequence of finite rank endomorphism. Then $\{u_n\}$ is called a $\mathcal{J}$-frame for $\mathcal{X}$ if $x = \lim\limits_{n\to\infty} u_n(x)$, $x \in \mathcal{X}$ and $\sup\limits_{1\leq n<\infty} \|u_n\| \leq \lambda < \infty$.

Next, we give an example of $\mathcal{J}$-frame in a Banach space $\mathcal{X}$.

Example 3.2. Let $\mathcal{X} = l^1$. Let $\{e_n\}$ be the sequence of unit vectors in $\mathcal{X}$. Define $\{x_n\} \subset \mathcal{X}$ and $\{f_n\} \subset \mathcal{X}^*$ by

$$x_1 = \frac{e_1}{2}, \quad x_2 = \frac{e_1}{2}, \quad x_n = e_{n-1},\ n = 3, 4, \ldots$$

$$f_1(x) = \xi_1, \quad f_2(x) = \xi_1, \quad f_n(x) = \xi_{n-1}, \quad x = \{\xi_n\} \in \mathcal{X}.$$

Now, for each n, define $\{h_{n,i}\}_{i=1,2,\ldots,n} \subset \mathcal{X}^*$ by $h_{n,i} = f_i$, $i = 1, 2, 3...$ and define $\{u_n\}$ by

$$u_n(x) = \sum_{i=1}^{n} h_{n,i}(x)x_i$$

Then

$$\lim_{n\to\infty} u_n(x) = \sum_{i=1}^{n} f_i(x)x_i = x.$$

Hence $\{u_n\}$ is a $\mathcal{J}$-frame for $\mathcal{X}$.

Following is the necessary and sufficient condition for the existence of a $\mathcal{J}$-frame in a Banach space $\mathcal{X}$.

Theorem 3.3. *Let $\mathcal{X}$ be a Banach space. Then $\{u_n\} \subset L(\mathcal{X}, \mathcal{X})$ is a $\mathcal{J}$-frame for $\mathcal{X}$ if and only if there exist $\{v_n\} \subset L(\mathcal{X}, \mathcal{X})$ such that $x = \sum_{i=1}^{\infty} v_i(x)$ and $\sup\limits_{1\leq n<\infty} \|\sum_{i=1}^{n} v_i(x)\| \leq \lambda < \infty$.*

Proof. Suppose $\mathcal{X}$ has $\mathcal{J}$-frame $\{u_n\} \subset L(\mathcal{X}, \mathcal{X})$. Assume that $v_1 = u_1,\ v_{2n} = v_{2n+1} = \frac{1}{2}(u_{n+1} - u_n),\ n \in \mathbb{N}$.
Then, we get

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} v_i(x) &= \lim_{n\to\infty} (u_1(x) + \frac{1}{2}(u_2(x) - u_1(x)) + \frac{1}{2}(u_2(x) - u_1(x)) \\
&+ \frac{1}{2}(u_3(x) - u_2(x)) + \frac{1}{2}(u_3(x) - u_2(x)) + ...) \\
&= \lim_{n\to\infty} u_n(x) \\
&= x.
\end{aligned}$$

Also, we have

$$\sup \| \sum_{i=1}^{n} v_i(x) \| = \sup_{1 \leq n < \infty} \|u_n(x)\| < \lambda.$$

Conversely, taking $u_n = \sum_{i=1}^{n} v_i,$ for all n $\in \mathbb{N}$, we have

$$\lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \sum_{i=1}^{n} v_i(x) = x$$

and $\|u_n\| \leq \lambda$. □

In the following result, we give sufficient condition in terms of a $\mathcal{J}$-frame for the existence of bounded approximation property in a Banach space $\mathcal{X}$.

Theorem 3.4. *Let $(\mathcal{X}, \|.\|)$ be a Banach space and let $\{u_n\} \subset L(\mathcal{X}, \mathcal{X})$ be a $\mathcal{J}$-frame for $\mathcal{X}$ satisfying*

$$u_l u_k = u_k u_l = u_{min\{l,k\}}, l \neq k;\ l, k \in \mathbb{N}. \tag{3.3}$$

Then $\mathcal{X}$ has the metric approximation property.

Proof. Define $|||.|||$ on $\mathcal{X}$ by

$$|||x||| = \sup_{1 \leq n \leq \infty} |||u_n(x)|||, x \in \mathcal{X}.$$

Then it is easy to verify that $|||.|||$ is a norm on $\mathcal{X}$. Since $\{u_n\}$ is a $\mathcal{J}$-frame for $\mathcal{X}$,

$$\begin{aligned} \|x\| &= \| \lim_{n\to\infty} u_n(x)\| \\ &\leq |||x||| \\ &\leq \sup_{1 \leq n \leq \infty} \|u_n\| \|x\|,\ x \in \mathcal{X}. \end{aligned}$$

Thus, the norm $|||.|||$ is equivalent to the norm $\|.\|$.
Now define

$$v_n = \frac{1}{2} \sum_{i=1}^{n} u_i,\ n \in \mathbb{N}.$$

Then, $\{v_n\}$ is a sequence in $L(\mathcal{X}, \mathcal{X})$ such that

$$\begin{aligned} \lim_{n\to\infty} v_n(x) &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} u_i(x) \\ &= \lim_{n\to\infty} u_n(x) \\ &= x,\ x \in \mathcal{X}. \end{aligned}$$

Therefore, we have

$$\begin{aligned}
|||v_n||| &= \sup_{x\in\mathcal{X},\|x\|\le 1} |||v_n(x)||| \\
&= \sup_{x\in\mathcal{X},\|u_1(x)\|\le 1,\|u_2(x)\|\le 1,\ldots}\left(\sup_{1\le m<\infty}\left\|\frac{1}{n}\sum_{i=1}^{n}u_m u_i(x)\right\|\right) \\
&= \sup_{x\in\mathcal{X},\|u_1(x)\|\le 1,\|u_2(x)\|\le 1,\ldots}\left(max\left\{\sup_{1\le m\le n}\left\|\frac{1}{n}\sum_{i=1}^{m-1}u_i(x)+u_m(x)\right.\right.\right. \\
&\quad \left.\left.\left.+\sum_{i=m+1}^{n}u_m(x)\right\|,\left\|\frac{1}{n}\sum_{i=1}^{n}u_i(x)\right\|\right\}\right) \\
&\le max\left\{\sup_{1\le m\le n}\left(\frac{\|u_m\|}{n}+\frac{n-1}{n}\right),1\right\} \\
&= 1+\epsilon_0,\ \text{for n } \ge \mathbb{N}_0(\epsilon_0),
\end{aligned}$$

where $\mathbb{N}_0(\epsilon_0)$ is some suitably chosen positive integer. Since $\lim\limits_{n\to\infty} v_n(x)=x$, it follows that $\lim\limits_{n\to\infty}|||v_n|||=1$ and so $\lim\limits_{n\to\infty}\frac{v_n}{|||v_n|||}(x)=x,\ x\in\mathcal{X}$.
Hence $(\mathcal{X},|||.|||)$ has a metric approximate property. □

4. Support cones and frames in Banach spaces

Shah et al. [15] studied cone associated with frames in Banach Spaces. In this section, we have related the concept of retro Banach frame with the geometric notion of a cone in Banach Spaces.
Recall that a closed convex set $\mathcal{C}$ in a Banach space $\mathcal{X}$ is called a *support cone* if for every $s\in\mathcal{C}$ there exist an $f\in\mathcal{X}^*$ such that

$$f(s)=\inf_{x\in\mathcal{C}}f(x)<\sup_{x\in\mathcal{C}}f(x). \tag{4.4}$$

A point $s\in\mathcal{C}$ satisfying (4.4) is called a *proper support point.*

In the following result, we show that the conjugate space of a Banach space $\mathcal{X}$ has a retro Banach frame if and only if $\mathcal{X}$ does not contain a support cone.

Theorem 4.1. *For a Banach space $\mathcal{X}$, the following statements are equivalent:*
(a) $\mathcal{X}^*$ has a retro Banach frame.
(b) $\mathcal{X}$ does not contain a support cone.

Proof. $(a)\Rightarrow(b)$ Let $(\{x_n,T\})(\{x_n\}\subset\mathcal{X},T:{\mathcal{X}^*}_d\longrightarrow\mathcal{X}^*)$ be a retro Banach frame for $\mathcal{X}^*$. Then, there exists constant A,B with $0\le A\le B<\infty$ such that

$$A\|f\|_{\mathcal{X}^*}\le\|\{f(x_n)\}\|_{{\mathcal{X}^*}_d}\le B\|f\|_{\mathcal{X}^*},\ f\in\mathcal{X}^*.$$

Let S be any closed convex subset of $\mathcal{X}$. Then S is separable. Let $\{s_n\}$ be any sequence in S such that $[s_n]=S$ and $\sum_n(\frac{n+1}{n})s_n$ is convergent. Let $s^{'}=\sum_n(\frac{n+1}{n})s_n$. Then, $s^{'}\in S$. Suppose that $\phi\in\mathcal{X}^*$ attains its minimum on S at $s^{'}$. Then $\phi(s_n)=\phi(s^{'})$ for all $n\in\mathbb{N}$. Therefore, $\phi(s)=\phi(s^{'})$ for all $n\in\mathbb{N}$. Hence S is not a support cone of $\mathcal{X}$. Since S is an arbitrary closed convex subset of $\mathcal{X}$, it follows that $\mathcal{X}$ does not contain a support cone.

$(b) \Rightarrow (a)$ Suppose $\mathcal{X}$ does not contain a support cone. Let if possible $\mathcal{X}^*$ does not have a retro Banach frame. Then, $\mathcal{X}$ is non separable because if $\mathcal{X}$ is separable, then there exists a sequence $\{x_n\} \in \mathcal{X}$ such that $[x_n] = \mathcal{X}$. Put $\phi_n = \pi(x_n),\ n \in \mathbb{N}$. Then, $\{\phi_n\} \subset \mathcal{X}^{**}$ is total on $\mathcal{X}^*$. Therefore, by Lemma 2.6, there exist a bounded linear operator $T : \{\{\phi_n(f)\} : f \in \mathcal{X}^*\} \longrightarrow \mathcal{X}^*$ such that $(\{\phi_n\}, T)$ is a Banach frame for $\mathcal{X}^*$. Hence $(\{x_n\}, T)$ is a retro Banach frame for $\mathcal{X}^*$. This is a contradiction. Thus, $\mathcal{X}$ is non separable. Therefore, by Corollary 7 in [21], $\mathcal{X}$ contains a biorthogonal system in $\mathcal{X} \times \mathcal{X}^*$. Hence by Theorem 4 in [3], $\mathcal{X}$ has a support cone. □

Corollary 4.2. If a Banach space $\mathcal{X}$ has a support cone, then $(\mathcal{X}^* \oplus \mathbb{R})$ has no retro Banach frame.

Proof. First we will show that $(\mathcal{X} \oplus \mathbb{R})$ has a support cone. Let S be a support cone in $\mathcal{X}$. Let $K \subset (\mathcal{X} \oplus \mathbb{R})$ be defined by $K = \{t(x^{'}, 1); x^{'} \in S, t \geq 0\}$. Then, K is a closed convex set in $(\mathcal{X} \oplus \mathbb{R})$. Let $t^*(x^*, 1) \in K$ be an arbitrary element. If $t^* = 0$, then $\phi = (0, 1) \in (\mathcal{X}^* \oplus \mathbb{R})$ which supports K at $(0, 0)$. Let $t^* > 0$. Choose $\phi \in \mathcal{X}^*$ such that ϕ support S at x^*. Then $(\phi, -\phi(x^*)) \in (\mathcal{X} \oplus \mathbb{R})$ supports K at $t^*(x^*, 1)$. Hence $(\mathcal{X} \oplus \mathbb{R})$ has a support cone. Therefore, by Theorem 4.1 $(\mathcal{X}^* \oplus \mathbb{R})$ has no retro Banach frame. □

Definition 4.3. A closed convex subset S of a Banach space $\mathcal{X}$ is said to have property (α), if $S = \bigcap_{i \in \mathbb{N}} f_i^{-1}(-\infty, 1]$, where each $f_i \in \mathcal{X}^*$ such that $\|f_i\| = 1$.

Next, we construct a Banach space and prove that the existence of a Banach frame in it is related to some geometric property of the Banach space.

Theorem 4.4. *Let S be a closed convex subset of a Banach space $\mathcal{X}$ containing the origin. Then, S has property (α) if and only if the predual of the space $S^0 = \{f \in \mathcal{X}^* : f(x) \leq 1,\ \text{for all}\ x \in S\}$ has a Banach frame.*

Proof. Let S has the property (α). Write $S = \bigcap_{i \in \mathbb{N}} {f_i}^{-1}(\infty, 1]$, where each $f_i \in \mathcal{X}^*$ such that $\|f_i\| = 1$. Let $U = \overline{conv}^{W^*}(\{f_n\}, \{0\})$. Since $f_n(x) \leq 1$, for all $x \in S$, it follows that $f(x) \leq 1$, for all $x \in S$ and for all $f \in U$. Hence $U \subset S^0$. Now, we will show $U = S^0$. Suppose if $U \neq S^0$, then there exist $f \in S^0 \setminus U$ and $x^{'} \in \mathcal{X}$ such that $f(x^{'}) > 1 > sup_U x^{'}$. Therefore, $f_n(x^{'}) < 1$, for all n and so $x^{'} \in S$. Now $f(x^{'}) > 1$ with $x^{'} \in S$, contradicts that $f \in S^0$. So $U = S^0$ and hence S^0 contains a sequence $\{h_n\}$ such that $[\tilde{h_n}] = S^0$. Therefore, by Lemma 2.6, there exist a bounded linear operator $T : \{\{h_n(y)\} : y \in S^0\} \longrightarrow S^0$ such that $(\{h_n\}, T)$ is a Banach frame for the predual of S^0.

Conversely, let $(\{f_n\}, T)$ be a Banach frame for the predual of the space S^0 and let $\mathcal{H}$ be the predual of S^0. Then, there exists constants A and B with $0 < A \leq \infty$ such that

$$A\|s\|_{\mathcal{H}} \leq \|\{f_n(s)\}\|_{\mathcal{H}_d} \leq B\|s\|_{\mathcal{H}},\ \text{for all}\ s \in \mathcal{H}. \tag{4.5}$$

Clearly, $S \subset \bigcap_{n=1}^{\infty} {f_n}^{-1}(-\infty, 1]$. Also, if $x^{'} \notin S$, then there exist $f \in \mathcal{X}^*$ such that $f(x^{'}) > 1 > sup_S f$. Therefore $f \in S^0$. Further, by using frame inequality (3.2), there is an f_n such that $f_n(x^{'}) > 1$. Hence $S = \bigcap_{n=1}^{\infty} {f_n}^{-1}(-\infty, 1]$. □

Regarding existence of retro Banach frame, one may recall that if a Banach space $\mathcal{X}$ is separable, then $\mathcal{X}^*$ has a retro Banach frame.

In the following result, we improve this result and show that if $\mathcal{X}$ is a Banach space with

Fréchet differentiable norm such that $\mathcal{X}^*$ with weak* topology is hereditarily separable, then $\mathcal{X}^*$ has a retro Banach frame.

Theorem 4.5. *Let $\mathcal{X}$ be a Banach space with a Fréchet differentiable norm. If $\mathcal{X}^*$ is hereditarily separable with respect to weak* topology, then $\mathcal{X}^*$ has retro Banach frame.*

Proof. Let $\mathcal{X}$ be a Banach space with Fréchet differentiable norm. Let $|||\cdot|||$ denote the norm dual to the Fréchet differentiable norm on $\mathcal{X}$. Let $B_{\mathcal{X}^*}$ be the unit ball with respect to the norm $|||\cdot|||$. Let $\{f_n\}_{n=1}^{\infty} \subset B_{\mathcal{X}^*}$ be a Banach frame for $B_{\mathcal{X}}$. Let $S_{\mathcal{X}^*}$ denote the unit sphere of $\mathcal{X}^*$. If $f \in S_{\mathcal{X}^*}$ attains its norm at $x \in S_{\mathcal{X}}$. Then, there exist a net $\{f_{n_\alpha}\}$ of $\{f_n\}$ such that $f_{n_\alpha}(x) \longrightarrow f(x)$. Thus $f_{n_\alpha} \longrightarrow 1$ and so by Lemma 2.7, we have $\|f_{n_\alpha} - f\| \to 0$ as $n \to \infty$. Therefore f is in the norm closure of $\{f_n\}$. Since $\mathcal{X}^*$ is hereditarily separable, by Lemma 2.6 there exists a sequence $\{x_n\} \subset \mathcal{X}$ such that $[x_n] = \mathcal{X}$. Put $\phi_n = \pi(x_n)$ $n \in \mathbb{N}$. Then $\{\phi_n\} \subset \mathcal{X}^{**}$ is total in $\mathcal{X}^*$. Therefore, by Lemma 2.6, there exist a bounded linear operator $T : \{\{\phi_n(f)\} : f \in \mathcal{X}^*\} \longrightarrow \mathcal{X}^*$ such that $(\{\phi_n\}, T)$ is a Banach frame for $\mathcal{X}^*$. Hence $(\{x_n\}, T)$ is a retro Banach frame for $\mathcal{X}^*$. □

References

[1] T. Ando, *On Fundamental Properties of Banach Space with a Cone*, Pacific J. Math. **12**(1962), no.4, 1163-1169.

[2] M. Baronti and P. L. Papini, *Best Approximation and Cones in Banach Spaces*, in Nonlinear Functional Analysis & Its Appl. (NATO Sci. Ser.: C: 173). Dordrecht: Reidel, 1986. pp. 169-176.

[3] J. M. Borwein and J. Vanderwerff, *Banach spaces which Admits support sets*, Proc. Amer. Math. Soc. **124**(1996), 751-756.

[4] P. Casazza, *The art of frame theory*, Taiwan. J. Math. **4**(2000), 129-201.

[5] P. G. Casazza, S. J. Dilworth, E. Odell, Th. Schlumprecht and A Zsak, *Coefficient quantization for frames in Banach spaces*, J. Math. Anal. Appl. **348**(2008), 66-86.

[6] P. Casazza, D. Han and D. Larson, *Frames in Banach spaces*, Contemp. Math. **247**(1999), 149-181.

[7] O. Christensen, *Frames and Bases: An Introductory Course*, Applied and Numerical Harmonic Analysis, Birkhäuser, Basel, 2008.

[8] I. Daubechies, A. Grossmann and Y. Meyer, *Painless non orthogonal expansions*, J. Math. Phys. **27**(1986), 1271-1283.

[9] R. J. Duffin and A. C. Schaeffer, *A class of non-harmonic Fourier series*, Trans. Amer. Math. Soc. **72**(1952), 341-366.

[10] M. Fabian, P. Habala, P. Hajek, V. M. Santalucia, J. Polant and V. zizler, *Functional Analysis and Infinite Dimensional Geometry*, Springer-Verleg, Newyork, Inc., 2001.

[11] H. G. Feichtinger and K. Grochenig, *A unified approach to atomic decopostion via integrable group representations*, Lecture Notes in Mathematics 1302 (Springer. Berlin. 1988), 52-73.

[12] D. Gabor, *Theory of communication*, J. Inst. Electr. Eng. **93**(1946), 429-457.

[13] K. H. Grochenig, *Describing functions: Atomic decompositions versus frames*, Monatsh. fur Mathematik **112**(1991), 1-41.

[14] S. Jahan, V.Kumar and S. K. Kaushik, *On The Existence of Non-Linear frames*, Archivum Mathematicum (BRNO) **53**(2017), 101-109.

[15] S. Jahan, V.Kumar and C. Shekhar, *Cone associated with Frames in Banach Spaces*, (Accepted).

[16] P. K. Jain, S. K. Kaushik and L. K. Vashisht, *On Perturbation of Banach Frames*, Int. J. Wavelets Multiresolut. Inf. Process. **4**(2006), no. 3, 559-565.

[17] P. K. Jain, S. K. Kaushik and L. K. Vashisht, *Banach frames for conjugate Banach spaces*, Z. Anal. Anwend. **23**(2004), 713-720.

[18] S. K. Kaushik, S. K. Sharma and K. T. Poumai, *On Schauder frames in conjugate Banach spaces*, J. Math. **2013**(2013), Article ID 318659, 4 pages.

[19] S. K. Kaushik and S. K. Sharma , *On Approximative Atomic Decompositions in Banach spaces*, Commun. Math. Appl. **3**(2012), no. 3, 293-301.

[20] K. T. Poumai, and S. Jahan, *On K-Atomic Decompositions in Banach Spaces*, Electron. J. Math. Anal. Appl. **6**(2018), no. 1, 183-197.

[21] S. Todorcevic, *Biorthogonal systems and Quotient spaces via Baire category theory*, Math. Ann. **335**(2006), 687-715.

[22] I. Singer, *Bases in Banach Spaces II*, Springer, New York, 1981.

[23] R. Young, *An introduction to non harmonic Fourier series*, Academic Press, New York, 1980.

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