Poincare Journal of Analysis & Applications
Vol. 2018 (1), 35-45
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DUALS OF K-OPERATOR FRAMES IN HILBERT SPACES

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Date of Receiving	:	12.04.2017
Date of Revision	:	11.11.2017
Date of Acceptance	:	10.01.2018

Abstract. We characterize an operator which preserve K-operator frame and generate K-operator frames from old K-operator frames. Also, K-dual of a K-operator frame is defined and gave some of its characterizations.

1. Introduction

Frames are the redundant system which gives basis like expansion of element of underlying spaces. In 1952 Duffin and Schaeffer [6] introduced frame which gives bases like expansion of a vector in Hilbert spaces. Recall that a countable sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ is called frame if there exists positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i,k\in\mathbb{N}} |\langle f, f_k|^2 \leq B\|f\|^2, \ f \in \mathcal{H}.$$

Since then, various generalizations viz. g-frames [13], Fusion frames [1, 3], operator valued frames [8] of frames have been obtained. Găvruţa [9] gave the generalization of frames in the form of K-frame which has been further generalized by [4] and [12] in the setting of operators. K-frames were further studied in [10, 11]. In this paper, we characterize operator which preserve K-operator frames and generates new frames from the old ones. We have also obtained some results related to sum of K-operator frames. Further, we have defined the notion of K-dual of K-operator frames. Finally, necessary and sufficient conditions in the form of various results for K-dual of K-operator frames have been given. This paper is organized as follows: In section 2, we give some basic tools which is needed throughout the paper. In section 3, we prove image of K-operator frame under an operator is a K-operator frame. Also, we prove that sum of K-operator frame and image of K-operator frame under an operator is again a K-operator frame. In section 4, we define K-dual of K-operator frame.

2010 *Mathematics Subject Classification.* 42C15, 42C38.

Key words and phrases. Frames, K operator frames, Bessel sequences.

Communicated by. Laura Gavruta

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2. Basic tools

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers, and $B(\mathcal{H})$ denotes the set of bounded linear operator on separable Hilbert space $\mathcal{H}$.
Li and Cao [2] defined the notion of operator frame for $B(\mathcal{H})$. They gave the following definition.

Definition 2.1. A family of bounded linear operators $\{T_i\}_{i\in\mathbb{N}}$ on Hilbert space $\mathcal{H}$ is said to be an operator frame for $B(\mathcal{H})$, if there exists positive constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{i\in\mathbb{N}} \|T_i x\|^2 \leq B\|x\|^2, \quad \forall\ x \in \mathcal{H} \tag{2.1}$$

where A and B are called a lower bound and an upper bound for the operator frame respectively. An operator frame $\{T_i\}_{i\in\mathbb{N}}$ is said to be tight if $A = B$. It is called Parseval operator frame if $A = B = 1$. If only upper inequality of (2.1) hold, then $\{T_i\}_{i\in\mathbb{N}}$ is called an operator Bessel sequence for $B(\mathcal{H})$.

For a separable Hilbert space $\mathcal{H}$. We define

$$\ell^2(\mathcal{H}) = \{\{x_i\} : x_i \in \mathcal{H},\ \sum_{i\in\mathbb{N}} \|x_i\|^2 < \infty\}.$$

Define an inner product on $\ell^2(\mathcal{H})$ by

$$\langle \{x_i\}, \{y_i\}\rangle = \sum_{i\in\mathbb{N}} \langle x_i, y_i\rangle.$$

Then $\ell^2(\mathcal{H})$ is a Hilbert space with pointwise operations.

Next, we state a result by Douglas which is popularly known as Douglas majorization theorem. This result will be used in the subsequent work.

Theorem 2.2. [7] *Let $\mathcal{H}$ be a Hilbert space and $S, K \in B(\mathcal{H})$. Then the following statements are equivalent:*

(1) $R(K) \subseteq R(S)$.
(2) $KK^* \leq \lambda^2 SS^*$, for some $\lambda > 0$.
(3) $K = SQ$, for some $Q \in B(\mathcal{H})$.

Theorem 2.3. *Let $P,\ Q,\ R \in B(\mathcal{H})$. Then the following are equivalent*

(1) Range(P)$\subset$ Range(Q)+Range(R).
(2) $PP^* \leq \alpha^2(QQ^* + RR^*)$ for some $\alpha > 0$.
(3) $P = AQ + BR$ for some $A,\ B \in B(\mathcal{H})$.

The notion of K-frame for Hilbert space was introduced and studied by Găvruţa [9] who gave the following definition.

Definition 2.4. A sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ is called K-frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|K^* x\| \leq \sum_{k\in\mathbb{N}} |\langle x, f_k\rangle|^2 \leq B\|x\|^2, \quad \text{for all}\ \ x \in \mathcal{H}. \tag{2.2}$$

We call A,B the lower frame bound and the upper frame bound for K-frame $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ respectively. If the only upper inequality of (2.2) is satisfied, then $\{f_k\}_{k\in\mathbb{N}}$ is called Bessel sequence.

Găvruţa [9] also proved the following results.

Theorem 2.5. *Let $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$and $K \in B(\mathcal{H})$. Then, following statements are equivalent:*

(1) $\{f_k\}_{k\in\mathbb{N}}$ is an atomic system for K;
(2) $\{f_k\}_{k\in\mathbb{N}}$ is a K-frame for $\mathcal{H}$;
(3) there exists a Bessel sequence $\{g_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ such that

$$Kx = \sum_{k\in\mathbb{N}} \langle x, g_k\rangle f_k, \quad \forall x \in \mathcal{H}.$$

We call the Bessel sequence $\{g_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ as the K-dual frame of the K-frame $\{f_k\}_{k\in\mathbb{N}}$.

3. K-Operator frames

Definition 3.1. [4] Let $K \in B(\mathcal{H})$. A family of bounded linear operators $\{T_i\}_{i\in\mathbb{N}}$ on Hilbert space $\mathcal{H}$ is said to be a K-operator frame for $B(\mathcal{H})$, if there exists positive constants $A, B > 0$ such that

$$A\|K^*x\|^2 \leq \sum_{i\in\mathbb{N}} \|T_ix\|^2 \leq B\|x\|^2, \quad \forall\ x \in \mathcal{H}, \tag{3.1}$$

where A and B are called a lower bound and an upper bound for the K-operator frame respectively. A K-operator frame $\{T_i\}_{i\in\mathbb{N}}$ is said to be tight if

$$\sum_{i\in\mathbb{N}} \|T_ix\|^2 = A\|K^*x\|^2, \quad \forall\ x \in \mathcal{H}. \tag{3.2}$$

It is called Parseval K-operator frame if $A = 1$ in (3.2). If only upper inequality of (3.1) holds, then $\{T_i\}_{i\in\mathbb{N}}$ is called a K-operator Bessel sequence in $B(\mathcal{H})$. We call $\{T_i\}_{i\in\mathbb{N}}$ an exact K-operator frame for $B(\mathcal{H})$, if it ceases to be a K-operator frame whenever any one of its element is removed. If $K = I$, then K-operator frame is an operator frame.

Let $\{T_i\}_{i\in\mathbb{N}}$ be a K-operator frame for $B(\mathcal{H})$. Define an operator $R : \mathcal{H} \to \ell^2(\mathcal{H})$ by

$$Rx = \{T_ix\}_{i\in\mathbb{N}}, \quad x \in \mathcal{H}.$$

Then, R is a bounded linear operator called analysis operator of the K-operator frame $\{T_i\}_{i\in\mathbb{N}}$. The adjoint of the analysis operator R, $R^*(\{x_i\}) : \ell^2(\mathcal{H}) \to \mathcal{H}$ is defined by

$$R^*(\{x_i\}) = \sum_{i\in\mathbb{N}} T_i^*x_i, \quad \forall\ \ \{x_i\} \in \ell^2(\mathcal{H}).$$

The operator R^* is called the synthesis operator of $\{T_i\}_{i\in\mathbb{N}}$. By composing R and R^*, the frame operator $S : \mathcal{H} \to \mathcal{H}$ for K-operator frame is given by

$$S(x) = R^*Rx = \sum_{i\in\mathbb{N}} T_i^*T_ix.$$

Note that frame operator S, in general need not be invertible.

Theorem 3.2. [4] *For an operator Bessel sequence $\{T_i\}_{i\in\mathbb{N}} \subset B(\mathcal{H})$, the following statements are equivalent:*

(1) $\{T_i\}_{i\in\mathbb{N}}$ is K-operator frame for $B(\mathcal{H})$.
(2) There exists $A > 0$ such that $S \geq AKK^*$, where S is the frame operator for $\{T_i\}_{i\in\mathbb{N}}$.
(3) $K = S^{1/2}Q$, for some $Q \in B(\mathcal{H})$.

The following characterization of K-operator frame is given in [12].

Theorem 3.3. *A sequence $\{T_i\}_{i\in\mathbb{N}} \subset B(\mathcal{H})$ is K-operator frame if and only if Range$(K) \subset$ Range(R^*), where R is analysis operator of K-operator frame.*

4. Main results

The following theorem gives a necessary and sufficient condtion for operator Bessel sequence to be a K-operator frame in term of given operator.

Theorem 4.1. *Suppose that $K \in B(\mathcal{H})$ and $\{T_i\}_{i\in\mathbb{N}}$ is an operator Bessel sequence for $\mathcal{H}$. Let R denotes the analysis operator $\{T_i\}_{i\in\mathbb{N}}$. Then, $\{T_i\}$ is a K-operator frame if and only if there exists a bounded operator $Q \in B(\mathcal{H},\ \ell^2(\mathcal{H}))$ such that $K = R^*Q$.*

Proof. Let $\{T_i\}_{i\in\mathbb{N}}$ is a K-operator frame for $\mathcal{H}$. Then Range$(K) \subset$ Range(R^*). Therefore, by Theorem 2.2 there exists a bounded linear operator $Q \in B(\mathcal{H},\ \ell^2(\mathcal{H}))$ such that $K = R^*Q$.

Conversely, suppose that there exists a bounded operator $Q \in B(\mathcal{H},\ \ell^2(\mathcal{H}))$ such that $K = R^*Q$. Then by Theorem 2.2 Range$(K) \subset$ Range(R^*). Hence, $\{T_i\}$ is a K-operator frame for $\mathcal{H}$. $\square$

Next, we prove that image of K-operator is a K-operator frames under a surjective operator.

Theorem 4.2. *Let $K \in B(\mathcal{H})$ with a dense range and $\{T_i\}_{i\in\mathbb{N}}$ be K-operator frame for $B(\mathcal{H})$. If $Q \in B(\mathcal{H})$ has closed range with $QK = KQ$, then $\{T_iQ^*\}$ is K-operator frame for $B(\mathcal{H})$ if and only if Q is surjective.*

Proof. Suppose that Q is surjective. Since Q has closed range, it has pseudo inverse $Q^\dagger$ such that $QQ^\dagger = I$. Then for each $x \in R(Q)$ we have $K^*x = (T^\dagger)^*T^*K^*x$, so $\|(T^\dagger)^*\|^{-1}\|K^*x\| \leq \|T^*K^*x\|$. Now for each $x \in R(Q)$, we have

$$\begin{aligned}\sum_{i\in\mathbb{N}} \|T_iQ^*x\|^2 &\geq A\|K^*T^*x\|^2 \\ &\geq A\|(T^\dagger)^*\|^{-2}\|K^*x\|^2.\end{aligned}$$

Also, for each $x \in R(Q)$, we have

$$\sum_{i\in\mathbb{N}} \|T_iQ^*x\|^2 \leq B\|Q\|^2\|x\|^2.$$

Hence, $\{T_iQ^*\}$ is a K-operator frame for $B(\mathcal{H})$.

Conversely, suppose $\{T_iQ^*\}$ is K-operator frame for $B(\mathcal{H})$ with frame bounds A and

B. Then for any $f \in \mathcal{H}$, we have

$$A\|K^*x\|^2 \leq \sum_{i\in\mathbb{N}} \|T_iQ^*x\|^2 \leq B\|x\|^2. \tag{4.1}$$

Since $R(K)$ is dense in $\mathcal{H}$, K^* is injective. Thus, using (4.5), Q^* is injective. □

In the next result, we show that image of a K-operator frame under an operator is again a K-operator frame.

Theorem 4.3. *Let $K \in B(\mathcal{H})$ and $\{T_i\}_{i\in\mathbb{N}}$ be K-operator frame for $B(\mathcal{H})$. If $Q \in B(\mathcal{H})$ be an isometry with $K^*Q = QK^*$, then $\{T_iQ\}$ is K-operator frame for $B(\mathcal{H})$.*

Proof. Suppose $\{T_i\}_{i\in\mathbb{N}}$ is K-operator frame for $B(\mathcal{H})$. Then, for each $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{i\in\mathbb{N}} \|T_iQx\|^2 &\geq A\|K^*Qx\|^2 \\ &= A\|QK^*x\|^2 \\ &= A\|K^*x\|^2.\end{aligned}$$

Also,

$$\sum_{i\in\mathbb{N}} \|T_iQx\|^2 \leq B\|Q\|^2\|x\|^2.$$

Hence $\{T_iQ\}$ is a K-operator frame for $B(\mathcal{H})$. □

Next result characterizes a bounded linear operator K in term of frame operator of K-operator frames.

Theorem 4.4. *Let $\{T_i\}_{i\in\mathbb{N}}$ and $\{R_i\}_{i\in\mathbb{N}}$ be K-operator frame for $\mathcal{H}$ with frame operator S_1 and S_2 respectively. Then $K = S_1^{1/2}P + S_2^{1/2}Q$ for some $P,\ Q \in B(\mathcal{H})$.*

Proof. Let $\{T_i\}_{i\in\mathbb{N}}$ and $\{R_i\}_{i\in\mathbb{N}}$ be K-operator frame for $\mathcal{H}$ with frame operator S_1 and S_2 respectively. Then by Theorem 3.2, there exist $\alpha > 0$ such that $S_1 \geq \alpha KK^*$ and $S_2 \geq \alpha KK^*$. Therefore, by Douglas Theorem, we get $R(K) \subset R(S_1^{1/2})$ and $R(K) \subset R(S_2^{1/2})$. Hence $R(K) \subset R(S_1^{1/2}) + R(S_2^{1/2})$. Thus by Theorem 2.3, we obtain $K = S_1^{1/2}P + S_2^{1/2}Q$ for some $P,\ Q \in B(\mathcal{H})$. □

Next, we show that sum of K-operator frame and its image under an operator is again a K-operator frame.

Theorem 4.5. *Let $\{T_i\}_{i\in\mathbb{N}}$ be a K-operator frame for $\mathcal{H}$ with the frame operator S and let Q be a positive operator such that $SQ^* = Q^*S$. Then $\{T_i + T_iQ\}$ is a K-operator frame for $\mathcal{H}$. Moreover, for any natural number n, $\{T_i + T_iQ^n\}$ is a K-operator frame for $\mathcal{H}$.*

Proof. Let $\{T_i\}_{i\in\mathbb{N}}$ be a K-operator frame for $\mathcal{H}$ with the frame operator S. Then, there exist $\lambda > 0$ such that $S \geq \lambda KK^*$. The frame operator for $\{T_i + T_iQ\}$ is given by

$$\begin{aligned}\sum_{i\in\mathbb{N}}(T_i + T_iQ)^*(T_i + T_iQ)(x) &= \sum_{i\in\mathbb{N}} T_i^*(T_i(x) + T_iQ(x)) + Q^*T_i^*(T_i(x) + T_iQ(x)) \\ &= S(I+Q)^*(I+Q)(x)\end{aligned}$$

Since $S(I+Q^*)(I+Q) \geq S \geq \lambda KK^*$, $\{T_i + T_iQ\}$ is a K-operator frame for $\mathcal{H}$. Similarly, for any natural number n, the frame operator for $\{T_i + T_iQ^n\}$ is $S(I+Q^n)^*(I+Q^n) \geq \lambda KK^*$. Hence, $\{T_i + T_iQ^n\}$ is a K-operator frame for $\mathcal{H}$. □

Theorem 4.6. *Let $\{T_i\}_{i\in\mathbb{N}}$ be a K-operator frame for $\mathcal{H}$ with frame operator S and frame bounds A and B. Let $\{I_1,\ I_2\}$ be the partition of I and S_j be frame operator for operator Bessel sequence $\{T_i\}_{i\in I_j}$, $j=1,2$. Then $\{T_i + T_iS_1^a\}_{i\in I_1} \cup \{T_i + T_iS_2^b\}_{i\in I_2}$ is a K-operator frame for $\mathcal{H}$ for any natural numbers a and b.*

Proof. For $a, b \in \mathbb{N}$ and for each $x \in \mathcal{H}$, we have

$$\begin{aligned}\left(\sum_{i\in I_1} \|(T_i + T_iS_1^a)x\|^2\right)^{1/2} &\leq \left(\sum_{i\in I_1} \|T_ix\|^2\right)^{1/2} + \left(\sum_{i\in I_1} \|T_iS_1^a x\|^2\right)^{1/2}\\ &\leq \left(\sum_{i\in \mathbb{N}} \|T_ix\|^2\right)^{1/2} + \left(\sum_{i\in \mathbb{N}} \|T_iS_1^a x\|^2\right)^{1/2}\\ &\leq \sqrt{B}\|x\| + \sqrt{B}\|S_1^a x\|\\ &\leq \sqrt{B}(1 + \|S_1^a\|)\|x\|,\ x \in \mathcal{H}.\end{aligned}$$

So, we obtain

$$\sum_{i\in I_1} \|(T_i + T_iS_1^a)x\|^2 \leq B(1+\|S_1^a\|)^2\|x\|^2,\ x \in \mathcal{H}.$$

Also, we have

$$\begin{aligned}\left(\sum_{i\in I_2} \|(T_i + T_iS_1^b)(x)\|^2\right)^{1/2} &\leq \left(\sum_{i\in I_2} \|T_ix\|^2\right)^{1/2} + \left(\sum_{i\in I_2} \|T_iS_2^b x\|^2\right)^{1/2}\\ &\leq \left(\sum_{i\in \mathbb{N}} \|T_ix\|^2\right)^{1/2} + \left(\sum_{i\in \mathbb{N}} \|T_iS_2^b x\|^2\right)^{1/2}\\ &\leq \sqrt{B}\|x\| + \sqrt{B}\|S_2^b x\|\\ &\leq \sqrt{B}(1 + \|S_2^b\|)\|x\|,\quad x \in \mathcal{H}.\end{aligned}$$

This gives

$$\sum_{i\in I_2} \|(T_i + T_iS_2^b)x\|^2 \leq B(1+\|S_2^b\|)^2\|x\|^2,\ x \in \mathcal{H}.$$

Thus, we conclude that $\{T_i + T_iS_1^a\}_{i\in I_1} \cup \{T_i + T_iS_2^b\}_{i\in I_2}$ is an operator Bessel sequence. Also, the frame operators of $\{T_i + T_iS_1^a\}_{i\in I_1}$ is given by

$$\begin{aligned}&\sum_{i\in I_1}(T_i + T_iS_1^a)^*(T_i + T_iS_1^a)x\\ &= \sum_{i\in I_1} T_i^*T_ix + \sum_{i\in I_1} T_i^*T_iS_1^a x + \sum_{i\in I_1} S_1^aT_i^*T_ix + \sum_{i\in I_1} S_1^aT_i^*T_iS_1^a x\\ &= S_1x + 2S_1S_1^a x + S_1^{2a}S_1x\\ &= S_1(I+S_1^a)^2x,\ x \in \mathcal{H}\end{aligned}$$

and the frame operator of $\{T_i + T_iS_2^b\}_{i\in I_2}$ is given by

$$\begin{aligned}
&\sum_{i\in I_2}(T_i + T_iS_2^b)^*(T_i + T_iS_2^b)x \\
&= \sum_{i\in I_2} T_i^*T_ix + \sum_{i\in I_2} T_i^*T_iS_2^bx + \sum_{i\in I_2} S_2^bT_i^*T_ix + \sum_{i\in I_2} S_2^bT_i^*T_iS_2^bx \\
&= S_2x + 2S_2S_2^bx + S_2^{2b}S_2x \\
&= S_2(I + S_2^b)^2x, \ x \in \mathcal{H}.
\end{aligned}$$

Let S' be the frame operator of $\{T_i + T_iS_1^a\}_{i\in I_1} \cup \{T_i + T_iS_2^b\}_{i\in I_2}$. Since $\{T_i\}$ is a K-operator frame for $B(\mathcal{H})$, there exists $\lambda > 0$ such that $S \geq \lambda KK^*$ and

$$\begin{aligned}
S' &= S_1(I + S_1^a)^2 + S_2(I + S_2^b)^2 \\
&\geq S_1 + S_2 \\
&= S \geq \lambda KK^*.
\end{aligned}$$

Hence $\{T_i + T_iS_1^a\}_{i\in I_1} \cup \{T_i + T_iS_2^b\}_{i\in I_2}$ is a K-operator frame for $B(\mathcal{H})$ for any natural numbers a and b. □

Theorem 4.7. *Let $\{T_i\}_{i\in\mathbb{N}}$ and $\{R_i\}_{i\in\mathbb{N}}$ be operator Bessel sequences in $\mathcal{H}$. If $\{T_i\}_{i\in\mathbb{N}}$ is a K-operator frame and $S_2 + S_3 \geq 0$, then $\{T_i + R_i\}$ is K-operator frame for $B(\mathcal{H})$.*

Proof. Let $\{T_i\}_{i\in\mathbb{N}}$ be K-operator frame for $B(\mathcal{H})$. Then for each $x \in \mathcal{H}$, we have

$$\sum_{i\in\mathbb{N}} \|(T_i + R_i)x\|^2 \leq 2(B_1 + B_2)\|x\|^2.$$

Hence $\{T_i + R - i\}$ is operator Besel sequence in $B(\mathcal{H})$. Now for each $x \in \mathcal{H}$, we have

$$Sx = \sum_{i\in\mathbb{N}}(T_i + R_i)^*(T_i + R_i)x = S_1x + S_2x + S_3x + S_4x,$$

where $S_1x = \sum_{i\in\mathbb{N}} T_i^*T_ix, \ S_2x = \sum_{i\in\mathbb{N}} T_i^*R_ix, \ S_3 = \sum_{i\in\mathbb{N}} R_i^*T_ix, \ S_4x = \sum_{i\in\mathbb{N}} R_i^*R_ix$. Since S_2+S_3 and S_4 are positive, $S_2+S_3+S_4$ is positive operator. Also,$S \geq S_1 \geq AKK^*$. Hence $\{T_i + R_i\}$ is K-operator frame. □

5. Duals of K-operator frame

We begin this section with the following definition of K-dual of K-operator frame.

Definition 5.1. Let $\{T_i\}_{i\in\mathbb{N}}$ be K-operator frame for $\mathcal{H}$. An operator Bessel sequence $\{R_i\}_{i\in\mathbb{N}}$ is called K-dual of K-opertor frame $\{T_i\}_{i\in\mathbb{N}}$ if

$$Kx = \sum_{i\in\mathbb{N}} T_i^*R_ix, \quad x \in \mathcal{H}. \tag{5.1}$$

It is proved in [12] that for every K-operator frame there exists a Bessel sequence which satisfies (5.1). Also, from (5.1) we obtain

$$K^*x = \sum_{i\in\mathbb{N}} R_i^*T_ix, \quad x \in \mathcal{H}.$$

Remark 5.2. If $\{T_i\}_{i\in\mathbb{N}}$ and $\{R_i\}_{i\in\mathbb{N}}$ are operator Bessel sequence satisfying (5.1), then $\{T_i\}_{i\in\mathbb{N}}$ and $\{R_i\}_{i\in\mathbb{N}}$ are K and K^*-operator frame respectively.

Remark 5.3. X. C. Xiao et al. [12] proved that $\{T_i\}_{i\in\mathbb{N}}$ is a K-operator frame if and only if $\{u_{ik} = T_i^* e_k\}$is a K-frame. Therefore, K-dual of K-operator frame can be obtained from K-dual of $\{u_{ik}\}$. Indeed, if we suppose that $\{u'_{ik}\}$ is a K-dual frame for $\{u_{ik}\}$ and if $\{R_i\}_{i\in\mathbb{N}}$ is defined by

$$R_i x = \sum_{k\in\mathbb{N}} \langle x, u'_{ik}\rangle e_{ik}.$$

Then

$$\begin{aligned}\sum_{i\in\mathbb{N}} T_i^* R_i x &= \sum_{i\in\mathbb{N}}\sum_{k\in\mathbb{N}} T_i^* \langle x, u'_{ik}\rangle e_{ik} \\ &= \sum_{i\in\mathbb{N}}\sum_{k\in\mathbb{N}} \langle x, u'_{ik}\rangle u_{ik} = Kf, \quad f \in \mathcal{H}.\end{aligned}$$

Hence $\{R_i\}_{i\in\mathbb{N}}$ is K-dual of K-operator frame $\{T_i\}_{i\in\mathbb{N}}$.

Theorem 5.4. *Let $K \in B(\mathcal{H})$ with closed range. Let $\{T_i\}_{i\in\mathbb{N}}$ be K-operator frame for $B(\mathcal{H})$ with frame operator S and frame bounds A and B respectively. Then $\{T_{i|_{S(R(K))}}(S^{-1})^*\pi_{R(K)}K\}$ is a K-dual of $\{T_i\}_{i\in\mathbb{N}}$.*

Proof. Let $\{T_i\}$ be a K-operator frame for $B(\mathcal{H})$. Since $S : R(K) \to S(R(K))$ is invertible, we have

$$\begin{aligned}Kx &= (S^{-1}S)^*Kx \\ &= S(S^{-1})^*Kx \\ &= S\pi_{S(R(K))}(S^{-1})^*Kx \\ &= \sum_{i\in\mathbb{N}} T_i^*T_i\pi_{S(R(K))}(S^{-1})^*Kx, \text{ for all } x \in \mathcal{H}.\end{aligned}$$

Also, we have

$$\begin{aligned}\sum_{i\in\mathbb{N}} \|T_i\pi_{S(R(K))}(S^{-1})^*Kx\|^2 &= \langle S(S^{-1})^*Kx, (S^{-1})^*Kx\rangle \\ &= \langle Kx, (S^{-1})^*Kx\rangle \\ &\le A^{-1}\|K\|^2\|K^\dagger\|^2\|x\|^2, \ x \in \mathcal{H}.\end{aligned}$$

Hence $\{T_i\pi_{R(K)}(S^{-1})^*K\}$ is a dual of the K-operator frame $\{T_i\}$. □

Remark 5.5. We call the dual of $\{T_i\pi_{S(R(K))}(S^{-1})^*K\}$ to be the canonical dual of the K-operator frame which coincide with the canonical dual of the operator frame if $K = I$.

Theorem 5.6. *Let $\{T_i\}_{i\in\mathbb{N}}$ be a K-operator frame for $B(\mathcal{H})$ with synthesis operator R^*. Then $\{R_i\}_{i\in\mathbb{N}}$ is K-dual of K-operator frame $\{T_i\}_{i\in\mathbb{N}}$ if and only if $R_i^* e_k = Q(e_k\delta_i)$, where $\{\delta_i\}$ is a canonical basis of $\ell^2(\mathbb{N})$ and $Q : \ell^2(\mathcal{H}) \to \mathcal{H}$ is a bounded linear operator such that $QR = K$.*

Proof. Let $\{T_i\}_{i\in\mathbb{N}}$ be K-operator frame for $B(\mathcal{H})$ and let $u_{ik}=T_i^*e_k$. Then for $x\in\mathcal{H}$, we have

$$\begin{aligned}
Kf = QRf &= Q\Bigg(\sum_{i,k\in\mathbb{N}}\langle T_ix, e_{ik}\rangle e_{ik}\delta_i\Bigg)\\
&= \sum_{i\in\mathbb{N}}\sum_{k\in\mathbb{N}}\langle x, u_{ik}\rangle R_i^*e_k\\
&= \sum_{i\in\mathbb{N}}R_i^*\sum_{k\in\mathbb{N}}\langle x, u_{ik}\rangle e_k\\
&= \sum_{i\in\mathbb{N}}R_i^*T_ix.
\end{aligned}$$

Hence $\{R_i\}_{i\in\mathbb{N}}$ is K-dual of K-operator frame $\{T_i\}_{i\in\mathbb{N}}$.
Conversely, suppose $\{R_i\}_{i\in\mathbb{N}}$ is K-dual of K-operator frame $\{T_i\}_{i\in\mathbb{N}}$. Then for each $x\in\mathcal{H}$, we have

$$\sum_{i\in\mathbb{N}}\sum_{k\in\mathbb{N}}\langle x, u_{ik}\rangle R_i^*e_k = \sum_{i\in\mathbb{N}}\sum_{k\in\mathbb{N}}\langle x, u_{ik}\rangle Q(e_k\delta_i)$$

Also, since $\{e_k\}$ is an orthonormal basis of $\ell^2(\mathcal{H})$, we get $R_i^*e_k = Q(e_k\delta_i)$. $\square$

Theorem 5.7. *Let K be a bounded operator on $\mathcal{H}$ with closed range, and $\{T_i\}_{i\in\mathbb{N}}$ be K-operator frame for $B(\mathcal{H})$ with frame bounds A and B. Then, there is one to one corespondence between K-dual of $\{T_i\}_{i\in\mathbb{N}}$ and operator $\psi\in B(\mathcal{H},\ell^2(\mathcal{H}))$ such that $R\psi=0$.*

Proof. Let $\{R_i\}$ be a dual of K-operator frame $\{T_i\}$ with frame bounds C and D and S be its frame operator. Define a mapping $\psi:\mathcal{H}\to\ell^2(\mathcal{H})$ by

$$(\psi x)_i = R_ix - T_i\pi_{S(R(K))}(S^{-1})^*Kx,\ x\in\mathcal{H}.$$

Then, ψ is a bounded linear operator on $\mathcal{H}$. Indeed, for each $x\in\mathcal{H}$, we have

$$\begin{aligned}
\|\psi x\| &= \sum_{i\in\mathbb{N}}\|R_ix - T_i\pi_{S(R(K))}(S^{-1})^*Kx\|^2\\
&= \sum_{i\in\mathbb{N}}\|R_ix\|^2 + \sum_{i\in\mathbb{N}}\|T_i\pi_{S(R(K))}(S^{-1})^*Kx\|^2\\
&+ 2\Bigg(\sum_{i\in\mathbb{N}}\|R_ix\|^2\Bigg)^{1/2}\Bigg(\sum_{i\in\mathbb{N}}\|T_i\pi_{S(R(K))}(S^{-1})^*Kx\|^2\Bigg)^{1/2}\\
&\leq (C\|x\|^2 + A^{-1}\|K\|^2\|K^\dagger\|^2\|x\|^2 + 2\sqrt{C}\sqrt{A^{-1}}\|K\|\|K^\dagger\|\|x\|^2)\\
&= (C + A^{-1}\|K\|^2\|K^\dagger\|^2 + 2\sqrt{C}\sqrt{A^{-1}}\|K\|\|K^\dagger\|)\|x\|^2.
\end{aligned}$$

Also, we have

$$\begin{aligned}R\psi x &= \sum_{i\in\mathbb{N}} T_i^*(\psi x)_i\\ &= \sum_{i\in\mathbb{N}} T_i^*(R_i^* x - T_i\pi_{S(R(K))}(S^{-1})^*Kx)\\ &= Kx - Kx\\ &= 0,\ \text{for all } x\in\mathcal{H}.\end{aligned}$$

Conversely, let $\psi \in B(\mathcal{H}, \ell^2(\mathcal{H}))$ and $R\psi = 0$. Write

$$R_i x = T_i\pi_{S(R(K))}(S^{-1})^*Kx + (\psi x)_i,\ x\in\mathcal{H}.$$

Then, we have

$$\begin{aligned}\sum_{i\in\mathbb{N}}\|R_i x\|^2 &\le \sum_{i\in\mathbb{N}}\|T_i\pi_{S(R(K))}(S^{-1})^*Kx\|^2 + \|\psi x\|_{\ell^2(\mathcal{H})}\\ &\le A^{-1}\|K\|^2\|K^\dagger\|^2\|x\|^2 + \|\psi\|^2\|x\|^2\\ &\le (A^{-1}\|K\|^2\|K^\dagger\|^2 + \|\psi\|^2)\|x\|^2.\end{aligned}$$

Therefore, $\{R_i\}$ is an operator Bessel sequence. Also, we have

$$\begin{aligned}\sum_{i\in\mathbb{N}} T_i^* R_i x &= \sum_{i\in\mathbb{N}} T_i^* T_i\pi_{S(R(K))}(S^{-1})^*Kx + \sum_{i\in\mathbb{N}} T_i^*(\psi x)_i\\ &= Kx + R\psi x = Kx,\ x\in\mathcal{H}.\end{aligned}$$

Hence, $\{R_i\}$ is a K-dual of the K-operator frame $\{T_i\}$. □

Acknowedgements

The authors thank the referee for the critical comments and useful suggestions for the improvement of the paper.

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