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ABOUT SOME CATEGORIES OF SEGAL TOPOLOGICAL ALGEBRAS

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Abstract. We will construct two categories of Segal topological algebras and prove some of their categorical properties. We will show that several properties known for categories (of sets, for example) have analogues in the category $\mathscr{S}(B)$ of Segal topological algebras.

1. Introduction

The study of Segal topological algebras in such generality started in [3]. We will start with recalling the necessary definitions from [3].

By a *topological algebra* we will mean a topological linear space over the field $\mathbb{K}$ (here $\mathbb{K}$ could be either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers), in which is defined a separately continuous (associative, but not necessarily commutative) multiplication.

A topological algebra (A, τ_A) is a left (right or two-sided) *Segal topological algebra* in a topological algebra (B, τ_B) via an algebra homomorphism $f : A \to B$, if

1) $\mathrm{cl}_B(f(A)) = B$;
2) $\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}$;
3) $f(A)$ is a left (respectively, right or two-sided) ideal of B.

In short, we will denote a Segal topological algebra by a triple (A, f, B).

As usual, we denote for a category $\mathscr{C}$ the set of objects of $\mathscr{C}$ by $\mathrm{Ob}(\mathscr{C})$ and for every $A, B \in \mathrm{Ob}(\mathscr{C})$ the set of morphisms from A to B by $\mathrm{Mor}(A, B)$.

We will offer two constructions of the categories of Segal topological algebras, denoted by **Seg** and $\mathscr{S}(B)$, where the objects of the second category are also objects in the first

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category and the morphisms of the second category are also morphisms in the first category.

When one considers the objects of those categories, one can find some similarities with the objects of the category $\mathscr{E}_A(X)$ of A-module bundles and the objects of the category Sh^*_X of sheaves (see [2] for the definitions of the categories $\mathscr{E}_A(X)$ and Sh^*_X). We would still like to remark, that, in reality, the categories **Seg** and $\mathscr{S}(B)$ will be very different from the categories of sheaves or bundles. The objects of the new categories do not have any more the algebraic structures on fibers, which is the case for the objects of the categories of sheaves and bundles. In the categories of sheaves and bundles, one could perform the algebraic operations only inside the fibers of the objects. On the other hand, all objects in the new categories have "total" algebraic structure, enabling to add and multiply any two elements of a fixed object. Having algebraic structure in the fibers is an important property, which enables to show that the categories of sheaves and bundles are additive. Unfortunately, as it is proved in Lemma 1, this will not be the case for the new categories **Seg** and $\mathscr{S}(B)$.

Since the relations between the objects of the category $\mathscr{S}(B)$ are more similar to the relations between the objects of the category $\mathscr{E}_A(X)$ of A-module bundles (for the definition of this category and some structural results of $\mathscr{E}_A(X)$, see [1]), we continue with more thorough study of the category $\mathscr{S}(B)$ in the present paper, leaving the study of the further properties of the category **Seg** for the future investigations.

When dealing with the notions like an equalizer, a coequalizer or a pullback, the author had to write these notions down in the form that corresponds to the structure of the objects of our categories. Therefore, the reader, familiar with the general definitions of those terms from category theory, might feel a bit confused at first glance, when looking at the definitions of those notions in the present case. The author decided not to include the definitions of these terms in the general category theory context in order to avoid repeating the same things in two settings, which might appear totally different. But when one thinks more thoroughly about the concepts behind the definitions and studies the diagrams accompanying the definitions, one can see that the form used here is appropriate for the case of Segal topological algebras, considered as triples.

2. Categories Seg and $\mathscr{S}(B)$

In what follows, we will fix the "sideness" (left, right or two-sided) of Segal topological algebras and will not specify it any more. Everything will work in any of those three classes of Segal topological algebras similarily. Hence, we will emphasize the "sideness" of ideals only if it is really necessary.

In more general case, we will take all Segal topological algebras as objects of the category **Seg**. A morphism between two Segal topological algebras (A, f, B) and (C, g, D) will be a pair (α, β), where $\alpha : A \to C$ and $\beta : B \to D$ are continuous algebra homomorphisms with the condition $g \circ \alpha = \beta \circ f$, producing us a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} \\ C & \xrightarrow{g} & D \end{array}.$$

We will define componentwise composition of morphisms in the category **Seg**, i.e., for any $(A, f, B), (C, g, D), (E, h, F) \in \mathrm{Ob}(\mathbf{Seg})$ and morphisms $(\alpha, \beta) : (A, f, B) \to (C, g, D), (\gamma, \delta) : (C, g, D) \to (E, h, F)$, we define the composition of (γ, δ) and (α, β) as $(\gamma, \delta) \circ (\alpha, \beta) = (\gamma \circ \alpha, \delta \circ \beta)$. Next, we will show that $(\gamma, \delta) \circ (\alpha, \beta)$ is really a morphism between the objects (A, f, B) and (E, h, F).

Suppose that we have $(A, f, B), (C, g, D), (E, h, F) \in \mathrm{Ob}(\mathbf{Seg})$ and morphisms $(\alpha, \beta) : (A, f, B) \to (C, g, D)$ and $(\gamma, \delta) : (C, g, D) \to (E, h, F)$. Then one gets a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ C & \xrightarrow{g} & D \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\ E & \xrightarrow{h} & F \end{array} .$$

By the definition of a morphism in **Seg**, we get the following relations: $g \circ \alpha = \beta \circ f, h \circ \gamma = \delta \circ g$. As the composition of two continuous maps is continuous, the map $\gamma \circ \alpha : A \to E$ is continuous. Now,

$$h \circ (\gamma \circ \alpha) = (h \circ \gamma) \circ \alpha = (\delta \circ g) \circ \alpha = \delta \circ (g \circ \alpha) = \delta \circ (\beta \circ f) = (\delta \circ \beta) \circ f,$$

which means that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \gamma \circ \alpha} & & \downarrow{\scriptstyle \delta \circ \beta} \\ E & \xrightarrow{h} & F \end{array}$$

is commutative and $(\gamma, \delta) \circ (\alpha, \beta) \in \mathrm{Mor}((A, f, B), (E, h, F))$.

It is clear that a (continuous) morphism $(1_A, 1_B)$, defined by $1_A(a) = a$ for every $a \in A$ and $1_B(b) = b$ for every $b \in B$ is an identity morphism for an object (A, f, B). Indeed, take any objects (C, g, D) and (E, h, F) in category **Seg** with morphisms $(\alpha, \beta) : (C, g, D) \to (A, f, B)$ and $(\gamma, \delta) : (A, f, B) \to (E, h, F)$, then we have

$$(1_A, 1_B) \circ (\alpha, \beta) = (1_A \circ \alpha, 1_B \circ \beta) = (\alpha, \beta)$$

and

$$(\gamma, \delta) \circ (1_A, 1_B) = (\gamma \circ 1_A, \delta \circ 1_B) = (\gamma, \delta).$$

Last, we have to show that the composition of the morphisms in our category is associative. Suppose, that $(A, f, B), (C, g, D), (E, h, F), (G, j, H) \in \mathrm{Ob}(\mathbf{Seg})$, $(\alpha, \beta) : (A, f, B) \to (C, g, D)$, $(\gamma, \delta) : (C, g, D) \to (E, h, F)$, $(\epsilon, \zeta) : (E, h, F) \to (G, j, H)$

are the morphisms of the category **Seg** and the following diagram is commutative:

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} \\
C & \xrightarrow{g} & D \\
\big\downarrow{\scriptstyle\gamma} & & \big\downarrow{\scriptstyle\delta} \\
E & \xrightarrow{h} & F \\
\big\downarrow{\scriptstyle\epsilon} & & \big\downarrow{\scriptstyle\zeta} \\
G & \xrightarrow{j} & H
\end{array}.$$

Since $\alpha, \beta, \gamma, \delta, \epsilon$ and ζ are algebra homomorphisms, then $\epsilon \circ (\gamma \circ \alpha) = (\epsilon \circ \gamma) \circ \alpha$, $\zeta \circ (\delta \circ \beta) = (\zeta \circ \delta) \circ \beta$, and we obtain

$$(\epsilon, \zeta) \circ ((\gamma, \delta) \circ (\alpha, \beta)) = (\epsilon, \zeta) \circ (\gamma \circ \alpha, \delta \circ \beta) = (\epsilon \circ (\gamma \circ \alpha), \zeta \circ (\delta \circ \beta)) =$$
$$= ((\epsilon \circ \gamma) \circ \alpha, (\zeta \circ \delta) \circ \beta) = (\epsilon \circ \gamma, \zeta \circ \delta) \circ (\alpha, \beta) = ((\epsilon, \zeta) \circ (\gamma, \delta)) \circ (\alpha, \beta),$$

which gives us that the composition of the morphisms is associative in **Seg**.

Hence, we have checked that **Seg** is actually a category of Segal topological algebras.

Next, we will consider another category of Segal topological algebras, which will be more suitable for determining its properties, as this category is simpler to investigate.

First of all, we will fix a topological algebra (B, τ_B), which we will not change for this category. The objects of the category $\mathscr{S}(B)$ will be all Segal topological algebras in the same topological algebra B, i.e., all Segal algebras in the form of triples $(A, f, B), (C, g, B), \ldots$.

The morphisms between Segal topological algebras (A, f, B) and (C, g, B) will be all continuous algebra homomorphisms $\alpha : A \to C$ with the property $g(\alpha(a)) = f(a)$ for every $a \in A$. When comparing with the category **Seg**, we see that the morphism α in the category $\mathscr{S}(B)$ corresponds to the morphism $(\alpha, 1_B)$ in the category **Seg**. The composition of morphisms in $\mathscr{S}(B)$ will be defined as in the case of pairs of homomorphisms in the category **Seg**, which means that for $(A, f, B), (C, g, B), (D, h, B) \in \mathrm{Ob}(\mathscr{S}(B))$, $\alpha \in \mathrm{Mor}((A, f, B), (C, g, D))$ and $\beta \in \mathrm{Mor}((C, g, D), (D, h, B))$, the composition $\beta \circ \alpha \in \mathrm{Mor}((A, f, B), (D, h, B))$ is defined as the usual composition, i.e., $(\beta \circ \alpha)(a) = \beta(\alpha(a))$ for every $a \in A$. One can check exactly as in the case of the morphisms of the category **Seg** that this composition of morphisms is correctly defined and associative, having identity morphisms in the form of $1_A \in \mathrm{Mor}((A, f, B), (A, f, B))$. Hence, $\mathscr{S}(B)$ is also a category.

Remind that a category $\mathscr{C}$ is called *additive* if $\mathrm{Mor}(A, B)$ is an Abelian group for all $A, B \in \mathrm{Mor}(\mathscr{C})$.

Lemma 2.1. *The categories* **Seg** *and* $\mathscr{S}(B)$ *are not additive.*

Proof. We will give the proof in the case of the category $\mathscr{S}(B)$. The case of **Seg** is similar.

Take any two objects $(A, f, B), (C, g, B)$ of $\mathscr{S}(B)$. Suppose that there exists a zero morphism $\theta \in \mathrm{Mor}((A, f, B), (C, g, B))$ such that $\alpha + \theta = \alpha = \theta + \alpha$ for every morphism $\alpha \in \mathrm{Mor}((A,f,B),(C,g,B))$. Then it is easy to see that $\theta(a) = \theta_C$ for every $a \in A$. But then $g(\theta(A)) = \theta_B$ and $g(\theta(A))$ is not dense in B. Also, $g \circ \theta \neq f$, in general case. Hence, $\theta \notin \mathrm{Mor}((A,f,B),(C,g,B))$, the set $\mathrm{Mor}((A,f,B),(C,g,B))$ does not have a zero element and can not be a monoid, hence it also can not be a group or an abelian group. In the similar way one can check that the set $\mathrm{Mor}((A,f,B),(C,g,D))$ in the category **Seg** can not be a monoid, a group or an abelian group.

With that, we have demonstrated that the categories **Seg** and $\mathscr{S}(B)$ are not additive categories. □

3. Equalizers in the category $\mathscr{S}(B)$

The **equalizer** of morphisms $\alpha, \beta \in \mathrm{Mor}((A,f,B),(C,g,B))$ is a pair $((E,h,B);\epsilon)$ such that

(1) $(E,h,B) \in \mathrm{Ob}(\mathscr{S}(B))$ and $\epsilon \in \mathrm{Mor}((E,h,B),(A,f,B))$ with $\alpha(\epsilon(e)) = \beta(\epsilon(e))$ for every $e \in E$;
(2) for any pair $((D,j,B);\delta)$ with $(D,j,B) \in \mathrm{Ob}(\mathscr{S}(B))$ and $\delta \in \mathrm{Mor}((D,j,B),(A,f,B))$ with $\alpha(\delta(d)) = \beta(\delta(d))$ for every $d \in D$, there exists unique $\gamma \in \mathrm{Mor}((D,j,B),(E,h,B))$ with $\epsilon \circ \gamma = \delta$:

$$\begin{tikzcd}
D \arrow[rrr, "j"] \arrow[dr, dashed, "\gamma"] \arrow[ddr, "\delta"'] & & & B \arrow[dl, dashed, "1_B"'] \arrow[ddl, "1_B"] \\
& E \arrow[r, "h"] \arrow[d, "\epsilon"] & B \arrow[d, "1_B"'] & \\
& A \arrow[r, "f"] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & B \arrow[d, "1_B"] & \\
& C \arrow[r, "g"] & B &
\end{tikzcd}.$$

It is known in the category theory, that in the category of sets, the equalizer of maps $f, g : A \to B$, where A and B are any sets, is a pair $(A_0; 1_{A_0})$ with $A_0 = \{a \in A : f(a) = g(a)\}$. In the category $\mathscr{S}(B)$, the situation is a bit different.

Let $(A,f,B),(C,g,B) \in \mathrm{Ob}(\mathscr{S}(B))$ with $\alpha, \beta \in \mathrm{Mor}((A,f,B),(C,g,B))$. In what follows, we will use the following notations to shorten the text: $A_0 = \{a \in A : \alpha(a) = \beta(a)\}$ and $\tau_{A_0} = \{A_0 \cap U : U \in \tau_A\}$. As α and β are algebra homomorphisms, it is easy to check that for any $x, y \in A_0, \lambda \in \mathbb{K}$ and $a \in A$, we have $\alpha(x+y) = \beta(x+y), \alpha(\lambda x) = \beta(\lambda x)$ and $\alpha(xy) = \beta(xy)$. Hence, $x+y, \lambda x, xy \in A_0$, which means that A_0 is a subalgebra of A. Similarily we can see that $f(A_0)$ is a subalgebra of $f(A)$.

The problem, which occurs while working with the category $\mathscr{S}(B)$, is that the set $f(A_0)$ might not be an ideal of B or might not be dense in B, although $f(A)$ is a dense ideal of B.

Now we are ready to state, what we have observed above.

Lemma 3.1. *Let $(A,f,B),(C,g,B) \in \mathrm{Ob}(\mathscr{S}(\mathrm{B}))$ and $\alpha, \beta \in \mathrm{Mor}((A,f,B),(C,g,B))$. If $\tilde{A}$ is a subalgebra of A_0, equipped with the subspace topology $\tau_{\tilde{A}} = \{U \cap \tilde{A} : U \in \tau_A\}$,* *and $f(\tilde{A})$ is a dense ideal of B, then*

$$(\tilde{A}, f|_{\tilde{A}}, B) \in \mathrm{Ob}(\mathscr{S}(B)) \text{ and } 1_{\tilde{A}} \in \mathrm{Mor}((\tilde{A}, f|_{\tilde{A}}, B), (A, f, B)).$$

Proof. By the definition of $\tilde{A}$ and $\tau_{\tilde{A}}$, the conditions 1) and 3) of the Segal topological algebra are fulfilled. It is easy to see that the condition 2) of the Segal topological algebra is fulfilled, when one considers the map $f|_{\tilde{A}}$ and the subspace topology $\tau_{\tilde{A}}$ on $\tilde{A}$. Hence, $(\tilde{A}, f|_{\tilde{A}}, B) \in \mathrm{Ob}(\mathscr{S}(B))$. As the map $1_{\tilde{A}}$ is a continuous algebra homomorphism and $f \circ 1_{\tilde{A}} = f|_{\tilde{A}}$, then also $1_{\tilde{A}} \in \mathrm{Mor}((\tilde{A}, f|_{\tilde{A}}, B), (A, f, B))$. □

Using Lemma 3.1, we will give some sufficient conditions in order to have an equalizer in the form $((A_0, f|_{A_0}, B); 1_{A_0})$ in the category $\mathscr{S}(B)$.

Lemma 3.2. *Let $(A, f, B), (C, g, B) \in$ $\mathrm{Ob}(\mathscr{S}(\mathrm{B}))$ and $\alpha, \beta \in \mathrm{Mor}((A, f, B), (C, g, B))$. If $f(A_0)$ is a dense ideal of B, then there exists an equalizer $((A_0, f|_{A_0}, B); 1_{A_0})$ of α and β.*

Proof. Suppose that $f(A_0)$ is a dense ideal of B. Then $(A_0, f\,|_{A_0}, B) \in \mathrm{Ob}(\mathscr{S}(B))$ and $1_{A_0} \in \mathrm{Mor}((A_0, f\,|_{A_0}, B), (A, f, B))$, by Lemma 3.1. Also, it is easy to see that $(\alpha \circ 1_{A_0})(a) = \alpha(a) = \beta(a) = (\beta \circ 1_{A_0})(a)$ for every $a \in A_0$. Hence, the first condition of the equalizer is fulfilled.

Suppose, that there exists $(D, j, B) \in \mathrm{Ob}(\mathscr{S}(B))$, $\delta \in \mathrm{Mor}((D, j, B), (A, f, B))$ with $\alpha \circ \delta = \beta \circ \delta$

$$\begin{tikzcd}
D \arrow[rrr, "j"] \arrow[ddr, "\delta"] & & & B \arrow[ddl, "1_B"] \\
& A_0 \arrow[r, "f|_{A_0}"] \arrow[d, "1_{A_0}"] & B \arrow[d, "1_B"'] & \\
& A \arrow[r, "f"] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & B \arrow[d, "1_B"] & \\
& C \arrow[r, "g"] & B &
\end{tikzcd}.$$

Then $\delta(d) \in A$ and $\alpha(\delta(d)) = \beta(\delta(d))$ for every $d \in D$. Thus, $\delta(d) \in A_0$ for every $d \in D$. Now, it is easy to see, that there exists exactly one continuous map $\gamma = \delta : D \to A_0$ with $\delta = 1_{A_0} \circ \gamma$. Hence, the second condition of an equalizer is also fulfilled and $((A_0, f|_{A_0}, B); 1_{A_0})$ is an equalizer of α and β. □

It appears, that whenever the equalizer of α and β exists, the set $f(A_0)$ is dense subset of B.

Lemma 3.3. *Let $(A, f, B), (C, g, B) \in \mathrm{Ob}(\mathscr{S}(B))$ and $\alpha, \beta \in \mathrm{Mor}((A, f, B), (C, g, B))$. If there exist $(E, h, B) \in \mathrm{Ob}(\mathscr{S}(B))$ and $\epsilon \in \mathrm{Mor}((E, h, B), (A, f, B))$ with $\alpha \circ \epsilon = \beta \circ \epsilon$, then $f(A_0)$ is a dense subset of B.*

Proof. As $\epsilon \in \mathrm{Mor}((E, h, B), (A, f, B))$, then $f \circ \epsilon = h$. Hence, $h(E) = (f \circ \epsilon)(E) \subseteq f(A)$. Similarly to the proof of Lemma 3.2 for δ and D, we see that $\epsilon(e) \in A_0$ for every $e \in E$. Therefore, $h(E) = f(\epsilon(E)) \subseteq f(A_0)$. From $(E, h, B) \in \mathrm{Ob}(\mathscr{S}(B))$, we get that $h(E)$ is dense in B. Thus, $f(A_0)$ is also dense in B. □

From the Lemmas 3.2 and 3.3, we can obtain the following Corollaries.

Corollary 3.4. *Let* $(A, f, B), (C, g, B) \in \mathrm{Ob}(\mathscr{S}(B))$ *and* $\alpha, \beta \in \mathrm{Mor}((A, f, B), (C, g, B))$. *Suppose that* $f(A_0)$ *is an ideal of* B. *Then there exists an equalizer of* α *and* β *if and only if* $f(A_0)$ *is dense in* B.

Corollary 3.5. *Let* $(A, f, B), (C, g, B) \in \mathrm{Ob}(\mathscr{S}(B))$ *and* $\alpha, \beta \in \mathrm{Mor}((A, f, B), (C, g, B))$. *If* $f(A) = f(A_0)$, *then* $((A_0, f|_{A_0}, B); 1_{A_0})$ *is an equalizer of* α *and* β.

Unfortunately, we were not able to show that whenever the equalizer of α and β exists, the set $f(A_0)$ had to be an ideal of B. Thus, we need to study the problem of equalizer more deeply. In order to write the texts of the following results shorter, we will introduce the following definition.

Definition 3.6. Let $(A, f, B), (C, g, B)$ be Segal topological algebras and $\alpha, \beta \in \mathrm{Mor}((A, f, B), (C, g, B))$. A subalgebra A_1 of A is called a *Segal equalizer of* α *and* β if and only if $A_1 \subseteq A_0 = \{a \in A : \alpha(a) = \beta(a)\}, f(A_1)$ is a dense ideal of B and for every other subalgebra A_2 of A, such that $A_2 \subseteq A_0$ and $f(A_2)$ is a dense ideal of B, we have $A_2 \subseteq A_1$.

When one considers a Segal equalizer A_1 of α and β equipped with the subset topology $\tau_{A_1} = \{U \cap A_1 : U \in \tau_A\}$, then one obtains, by Lemma 3.1, that $(A_1, f|_{A_1}, B) \in \mathrm{Ob}(\mathscr{S}(B))$ and $1_{A_1} \in \mathrm{Mor}((A_1, f|_{A_1}, B), (A, f, B))$.

Proposition 3.7. Let $(A, f, B), (C, g, B)$ be the objects of $\mathscr{S}(B)$ and $\alpha, \beta \in \mathrm{Mor}((A, f, B), (C, g, B))$. *If there exists a subalgebra* A_1 *of* A *such that* A_1 *is a Segal equalizer of* α *and* β, *then* $((A_1, f|_{A_1}, B), 1_{A_1})$ *is an equalizer of* α *and* β.

Proof. If A_1 is a subalgebra of A such that A_1 is a Segal equalizer of α and β, then we have $(A_1, f|_{A_1}, B) \in \mathrm{Ob}(\mathscr{S}(B))$, $1_{A_1} \in \mathrm{Mor}((A_1, f|_{A_1}, B), (A, f, B))$ and $\alpha(1_{A_1}(a)) = \beta(1_{A_1}(a))$ for every $a \in A$. Hence, the first condition of an equalizer is fulfilled.

Suppose that there exists $(D, j, B) \in \mathrm{Ob}(\mathscr{S}(B))$, $\delta \in \mathrm{Mor}((D, j, B), (A, f, B))$ such that $\alpha(\delta(d)) = \beta(\delta(d))$ for every $d \in D$:

$$\begin{tikzcd}
D \arrow[rrr, "j"] \arrow[dr, dashed, "\gamma"] \arrow[ddr, "\delta"'] & & & B \arrow[dl, dashed, "1_B"'] \arrow[ddl, "1_B"] \\
& A_1 \arrow[r, "f|_{A_1}"] \arrow[d, "1_{A_1}"] & B \arrow[d, "1_B"'] & \\
& A \arrow[r, "f"] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & B \arrow[d, "1_B"] & \\
& C \arrow[r, "g"] & B &
\end{tikzcd} .$$

By the definition of a morphism in our category, we obtain that $j(D)$ is a dense ideal of B and that $f \circ \delta = j$.

Now, $A_2 = \delta(D) \subset A_0$ and $f(A_2) = (f \circ \delta)(D) = j(D)$ is a dense ideal of B. Moreover, A_2 is a subalgebra of A, because δ is an algebra homomorphism. As A_1 is a Segal equalizer of α and β, we see that $A_2 \subseteq A_1$.

Define a map $\gamma : D \to A_1$ by $\gamma(d) = \delta(d)$ for every $d \in D$. Then γ is continuous and $1_{A_1} \circ \gamma = \gamma = \delta$. Moreover, since $f \circ \delta = j$, we obtain that

$$f \circ \gamma = f|_{A_1} \circ \gamma = f|_{A_1} \circ (1_{A_1} \circ \gamma) = f \circ (1_{A_1} \circ \gamma) = j.$$

Hence, $\gamma \in \mathrm{Mor}((D, j, B), (A_1, f|_{A_1}, B))$.

Suppose that there exists some $\epsilon \in \mathrm{Mor}((D, j, B), (A_1, f|_{A_1}, B))$ such that $1_{A_1} \circ \epsilon = \delta$. Then $\epsilon(d) = 1_{A_1}(\epsilon(d)) = \delta(d) = \gamma(d)$ for every $d \in D$, which means that $\epsilon = \gamma$ and the map $\gamma : D \to A_1$, such that $1_{A_1} \circ \gamma = \delta$, is unique. Hence, $((A_1, f|_{A_1}, B), 1_{A_1})$ is an equalizer of α and β. $\square$

Proposition 3.8. Let $(A, f, B), (C, g, B)$ be the objects of $\mathscr{S}(B)$ and $\alpha, \beta \in \mathrm{Mor}((A, f, B), (C, g, B))$. *If there exists an equalizer $((E, h, B), \epsilon)$ of α and β, then there exists a subalgebra $A_1 = \epsilon(E)$ of A such that A_1 is a Segal equalizer of α and β.*

Proof. Suppose that there exists an equalizer $((E, h, B), \epsilon)$ of α and β. Then $(\alpha \circ \epsilon)(e) = (\beta \circ \epsilon)(e)$ for every $e \in E$, which means that $A_1 = \epsilon(E) \subset A_0$.

As $f \circ \epsilon = h$, then $f(A_1) = f(\epsilon(E)) = h(E)$ is a dense ideal of B, because $(E, h, B) \in \mathrm{Ob}(\mathscr{S}(B))$.

For a better understanding of the present proof, we include also the following diagram:

$$\begin{tikzcd}
A_2 \arrow[rrr, "f|_{A_2}"] \arrow[dr, dashed, "\gamma"] \arrow[ddr, "1_{A_2}"'] & & & B \arrow[dl, dashed, "1_B"'] \arrow[ddl, "1_B"] \\
& E \arrow[r, "h"] \arrow[d, "\epsilon"] & B \arrow[d, "1_B"'] & \\
& A \arrow[r, "f"] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & B \arrow[d, "1_B"] & \\
& C \arrow[r, "g"] & B &
\end{tikzcd}.$$

Suppose that there exists a subalgebra A_2 of A_0 such that $f(A_2)$ is a dense ideal of B. Then $(A_2, f|_{A_2}, B) \in \mathrm{Ob}(\mathscr{S}(B))$ and $1_{A_2} \in \mathrm{Mor}((A_2, f|_{A_2}, B), (A, f, B))$, by Lemma 3.1. Moreover, $(\alpha \circ 1_{A_2})(a) = (\beta \circ 1_{A_2})(a)$ for every $a \in A_2$. Hence, $\alpha \circ 1_{A_2} = \beta \circ 1_{A_2}$.

Since $((E, h, B), \epsilon)$ is an equalizer of α and β, there exists $\gamma \in \mathrm{Mor}((A_2, f|_{A_2}, B), (E, h, B))$ such that $\epsilon \circ \gamma = 1_{A_2}$. Take any $a \in A_2$. Then $a = 1_{A_2}(a) = (\epsilon \circ \gamma)(a) \in \epsilon(E) = A_1$. Hence, $A_2 \subseteq A_1$ and we see that A_1 is a Segal equalizer of α and β. $\square$

From Propositions 3.7 and 3.8, we obtain a criterion for the existence of an equalizer in the category $\mathscr{S}(B)$.

Theorem 3.9. *Let $(A, f, B), (C, g, B) \in \mathrm{Ob}(\mathscr{S}(B))$ and $\alpha, \beta \in \mathrm{Mor}((A, f, B), (C, g, B))$. Then there exists an equalizer of α and β if and only if there exists a subalgebra A_1 of A such that A_1 is a Segal equalizer of α and β.*

4. Coequalizers in the category $\mathscr{S}(B)$

The **coequalizer** of morphisms $\alpha, \beta \in \mathrm{Mor}((A, f, B), (C, g, B))$ is a pair $((Q, k, B); \lambda)$ such that

(1) $(Q, k, B) \in \mathrm{Ob}(\mathscr{S}(B))$ and $\lambda \in \mathrm{Mor}((C, g, B), (Q, k, B))$ with $\lambda(\alpha(a)) = \lambda(\beta(a))$ for every $a \in A$;
(2) for any pair $((R, l, B); \mu)$, where $(R, l, B) \in \mathrm{Ob}(\mathscr{S}(B))$ and $\mu \in \mathrm{Mor}((C, g, B), (R, l, B))$ with $\mu(\alpha(a)) = \mu(\beta(a))$ for every $a \in A$, there exists unique $\nu \in \mathrm{Mor}((Q, k, B), (R, l, B))$ with $\nu \circ \lambda = \mu$.

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & B \arrow[d, "1_B"] & \\
& C \arrow[r, "g"] \arrow[d, "\lambda"] \arrow[ddl, "\mu"'] & B \arrow[d, "1_B"'] \arrow[ddr, "1_B"] & \\
& Q \arrow[r, "k"] \arrow[dl, dashed, "\nu"] & B \arrow[dr, dashed, "1_B"'] & \\
R \arrow[rrr, "l"] & & & B
\end{tikzcd}.$$

In what follows, we will need to use the smallest two-sided ideal I of C generated by the set $M = \{\alpha(a) - \beta(a) : a \in A\}$. It can be shown that

$$I = \Big\{ \sum_{k=1}^{n} (c_k m_k d_k + f_k m_k + m_k g_k + \lambda_k m_k) : n \in \mathbb{Z}^+, c_k, d_k, f_k, g_k \in C, m_k \in M, \lambda_k \in \mathbb{K} \Big\}.$$

On I we will consider the subspace topology τ_I generated by the topology τ_A of A, i.e., $\tau_I = \{U \cap I : U \in \tau_A\}$. In the theory of topological algebras, it is known that a quotient space C/I of a topological algebra C by its two-sided ideal I is also a topological algebra, when equipped with the quotient topology.

Theorem 4.1. *Let* $(A, f, B), (C, g, B) \in \mathrm{Ob}(\mathscr{S}(B))$ *and* I *be the smallest two-sided ideal of* C*, generated by the set* $M = \{\alpha(a) - \beta(a) : a \in A\}$. *Then the coequalizer of morphisms* $\alpha, \beta \in \mathrm{Mor}((A, f, B), (C, g, B))$ *is the pair* $((C/I, \tilde{g}, B); p)$*, where* $\tilde{g} : C/I \to B$ *is defined by* $\tilde{g}([c]) = g(c)$ *for each* $[c] \in C/I$*,* $p : C \to C/I$ *is the canonical projection and* C/I *is equipped with the quotient topology* $\tau_{C/I} = \{V \subseteq C/I : p^{-1}(V) \in \tau_C\}$.

Proof. Since $\alpha, \beta \in \mathrm{Mor}((A, f, B), (C, g, B))$, then $g(\alpha(a)) = f(a) = g(\beta(a))$ for every $a \in A$.

Take any $i \in I$. Then there exist $n_i \in \mathbb{Z}^+, m_1, \ldots, m_{n_i} \in M, \lambda_1, \ldots, \lambda_{n_i} \in \mathbb{K}$ and $c_1, \ldots, c_{n_i}, d_1, \ldots, d_{n_i}, f_1, \ldots, f_{n_i}, g_1, \ldots, g_{n_i} \in C$ such that

$$i = \sum_{k=1}^{n_i} (c_k m_k d_k + f_k m_k + m_k g_k + \lambda_k m_k).$$

Now, for every $m_k \in M$, there exists $a_k \in A$ such that $m_k = \alpha(a_k) - \beta(a_k)$. Hence, $g(m_k) = (g \circ \alpha)(a_k) - (g \circ \beta)(a_k) = f(a_k) - f(a_k) = \theta_B$ for every $k \in \{1, \ldots, n_i\}$. Thus, $g(i) = \theta_B$. As it holds for every $i \in I$, then $g(I) = \{\theta_B\}$.

First, we will show that $(C/I, \tilde{g}, B) \in \mathrm{Ob}(\mathscr{S}(B))$.

We know that C/I, equipped with the quotient topology, is a topological algebra.

Let $c, d \in C$ such that $[c] = [d]$. Then $c - d \in I$ and $g(c - d) = \theta_B$. Thus, $\tilde{g}([c]) = g(c) = g(d) = \tilde{g}([d])$ and the map $\tilde{g}$ is correctly defined. Moreover, $\tilde{g}$ is an algebra homomorphism, because g was an algebra homomorphism.

Notice, that $\tilde{g}(C/I) = g(C)$ is a dense ideal of B and that $\tilde{g} \circ p = g$, because $(\tilde{g} \circ p)(c) = g(c)$ for every $c \in C$. Hence, $C/I \neq [\theta_C]$, which means that $I \neq C$.

Take any $U \in \tau_B$. Then $g^{-1}(U) \in \tau_C$, because $(C, g, B) \in \mathrm{Ob}(\mathscr{S}(B))$. Now,

$$\tilde{g}^{-1}(U) = \{[c] : \tilde{g}([c]) \in U\} = \{p(c) : g(c) \in U\} =$$

$$= \{p(c) : c \in g^{-1}(U)\} = p(\{c : c \in g^{-1}(U)\}) = p(g^{-1}(U)) \in \tau_{C/I},$$

because the projection p is open and $g^{-1}(U) \in \tau_C$. Therefore, $\tau_{C/I} \supseteq \{\tilde{g}^{-1}(U) : U \in \tau_B\}$ and $(C/I, \tilde{g}, B)$ is an object of the category $\mathscr{S}(B)$.

It is known that p, as a canonical projection map, is a continuous algebra homomorphism. Moreover, it is clear that $p(\alpha(a)) = p(\beta(a))$ for every $a \in A$. Hence, the first condition of the coequalizer is fulfilled.

Suppose that there are $(R, l, B) \in \mathrm{Ob}(\mathscr{S}(B))$ and $\mu \in \mathrm{Mor}((C, g, B), (R, l, B))$ with $\mu(\alpha(a)) = \mu(\beta(a))$ for every $a \in A$:

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & B \arrow[d, "1_B"] & \\
 & C \arrow[r, "g"] \arrow[d, "p"] \arrow[ddl, "\mu"'] & B \arrow[d, "1_B"'] \arrow[ddr, "1_B"] & \\
 & C/I \arrow[r, "\tilde{g}"] \arrow[dl, dashed, "\tilde{\mu}"] & B \arrow[dr, dashed, "1_B"'] & \\
R \arrow[rrr, "l"] & & & B
\end{tikzcd}.$$

This gives us $\mu(\alpha(a) - \beta(a)) = \theta_R$ for every $a \in A$. Hence, $\mu(I) = \{\theta_R\}$. Defining a map $\tilde{\mu} : C/I \to R$ by $\tilde{\mu}([c]) := \mu(c)$, we see that $\tilde{\mu}$ is continuous and $\tilde{\mu} \circ p = \mu$, because μ was a continuous map and $\mu(c_1) = \mu(c_2)$ for all $c_1, c_2 \in C$ with $[c_1] = [c_2]$, i.e., with $c_1 - c_2 \in I$. It is also clear that $\tilde{\mu}$ is the unique map with the property $\tilde{\mu} \circ p = \mu$. Hence, the second condition of a coequalizer is also fulfilled and the pair $((C/I, \tilde{g}, B); p)$ is the coequalizer of morphisms α and β. $\square$

5. Pullbacks in the category $\mathscr{S}(B)$

Let $(A, f, B), (C, g, B), (D, h, B) \in \mathrm{Ob}(\mathscr{S}(B)), \alpha \in \mathrm{Mor}((A, f, B), (D, h, B))$ and $\beta \in \mathrm{Mor}((C, g, B), (D, h, B))$. An object (P, j, B) of the category $\mathscr{S}(B)$, together with morphisms $\gamma \in \mathrm{Mor}((P, j, B), (A, f, B))$ and $\delta \in \mathrm{Mor}((P, j, B), (C, g, B))$, is called a **pullback** of morphisms α and β, if

(1) $\alpha \circ \gamma = \beta \circ \delta$;
(2) for every $(R, k, B) \in \mathrm{Ob}(\mathscr{S}(B))$ and $\lambda \in \mathrm{Mor}((R, k, B), (A, f, B))$, $\mu \in \mathrm{Mor}((R, k, B), (C, g, B))$ such that $\alpha \circ \lambda = \beta \circ \mu$, there exists unique

morphism $\nu \in \operatorname{Mor}((R,k,B),(P,j,B))$ with $\gamma \circ \nu = \lambda$ and $\delta \circ \nu = \mu$

$$\begin{tikzcd}
P \arrow[rrr, "j"] \arrow[dr, "\gamma"] \arrow[ddrr, phantom] \arrow[ddd, "\delta"', bend left=0, shift left] & & & B \arrow[dl, "1_B"'] \arrow[dddl, "1_B"] \\
& A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "1_B"'] & \\
& D \arrow[r, "h"] & B & \\
& C \arrow[r, "g"] \arrow[u, "\beta"] & B \arrow[u, "1_B"] & \\
R \arrow[rrr, "k"] \arrow[uuuu, dashed, "\nu"] \arrow[uuur, dotted, "\lambda"] \arrow[ur, dotted, "\mu"] & & & B \arrow[uuuu, dashed, "1_B"'] \arrow[uuul, dotted, "1_B"'] \arrow[ul, dotted, "1_B"']
\end{tikzcd} .$$

In what follows, we will need to use some subalgebra of the direct product of topological algebras. Therefore, we will remind some facts about the algebraic operations on the direct product, the topologies of direct product and its subalgebra.

Let (X,τ_X) and (Y,τ_Y) be topological algebras over the field $\mathbb{K}$. Their direct product $X \times Y = \{(x,y) : x \in X, y \in Y\}$ is an algebra, if one defines the algebraic operations coordinate-wise, i.e.,

$$\begin{aligned}
(x_1,y_1)+(x_2,y_2) &= (x_1+x_2, y_1+y_2),\\
\lambda(x_1,y_1) &= (\lambda x_1, \lambda y_1),\\
(x_1,y_1)(x_2,y_2) &= (x_1x_2, y_1y_2)
\end{aligned}$$

for all $\lambda \in \mathbb{K}, (x_1,y_1),(x_2,y_2) \in X \times Y$. The topology $\tau_{X\times Y}$ on $X \times Y$ is the product topology, i.e., its base is the collection

$$\mathscr{B}_{X\times Y} = \{U \times V : U \in \tau_X, V \in \tau_Y\}.$$

Moreover, if Z is a subalgebra of $X \times Y$, then we consider on Z the subspace topology $\tau_Z = \{U \cap Z : U \in \tau_{X\times Y}\}$, which makes (Z,τ_Z) a topological algebra. In what follows, we will define the algebraic operations on the direct product of two topological algebras (X,τ_X) and (Y,τ_Y) coordinate-wise and will mean by the "subspace topology of the product topology of (X,τ_X) and (Y,τ_Y)" of the subalgebra Z of $X \times Y$ the construction, which gives us τ_Z from τ_X and τ_Y.

Proposition 5.1. Let $(A,f,B),(C,g,B),(D,h,B)$ be the objects of $\mathscr{S}(B)$, $\alpha \in \operatorname{Mor}((A,f,B),(D,h,B))$ *and* $\beta \in \operatorname{Mor}((C,g,B),(D,h,B))$. *Suppose that the ideal* $B_0 = f(A) \cap g(C)$ *of* B *is dense in* B *and the set*

$$P_b = \{(a,c) \in A \times C : \alpha(a) = \beta(c), h(\alpha(a)) = b\}^{1}$$

is nonempty for every $b \in B_0$. *Then*

1) *the triple* (P,j,B), *with* $P = \{(a,c) \in A \times C : \alpha(a) = \beta(c), h(\alpha(a)) \in B_0\}$, $j((a,c)) = f(a) = g(c)$ *and the topology of* P *is the subspace topology of the product topology of* (A,τ_A) *and* (C,τ_C), *is also an object of the category* $\mathscr{S}(B)$;

[1] Notice, that $h(\beta(b)) = h(\alpha(a))$.

2) *the projections* $p_A : P \to A$ *and* $p_C : P \to C$*, defined by* $p_A((a,c)) = a$ *and* $p_C((a,c)) = c$ *for every* $(a,c) \in P$*, are such that* $p_A \in \mathrm{Mor}((P,j,B),(A,f,B))$ *and* $p_C \in \mathrm{Mor}((P,j,B),(C,g,B))$*;*

3) (P,j,B)*, together with the projections* p_A *and* p_C*, is the pullback of morphisms* α *and* β*.*

Proof. 1) Since $(A,f,B),(C,g,B)$ and (D,h,B) are objects of the category $\mathscr{S}(B)$ and α and β are morphisms in the same category, then $h\circ\alpha = f, h\circ\beta = g$ and the topologies τ_A, τ_C and τ_D satisfy the condition 2) of the definition of a Segal topological algebra.

Take any $(a_1,c_1),(a_2,c_2) \in P$ and $\lambda \in \mathbb{K}$. Then $\alpha(a_1) = \beta(c_1), \alpha(a_2) = \beta(c_2)$ and $h(\alpha(a_1)), h(\alpha(a_2)) \in B_0$. As α and β are algebra homomoprhisms and B_0 is an ideal in B, then $\alpha(a_1+a_2) = \beta(c_1+c_2)$, $\alpha(a_1a_2) = \beta(c_1c_2), \alpha(\lambda a_1) = \beta(\lambda c_1)$ and $h(\alpha(a_1+a_2)), h(\alpha(a_1a_2)), h(\alpha(\lambda a_1)) \in B_0$. Hence, $(a_1,c_1)+(a_2,c_2) \in P$, $(a_1,c_1)(a_2,c_2) \in P$ and $\lambda(a_1,c_1) \in P$, which means that P is a subalgebra of $A\times C$. As P is equipped with the subspace topology τ_P of the product topology of (A,τ_A) and (C,τ_C), then (P,τ_P) is a topological algebra.

Take any $(a,c) \in P$. Then

$$f(a) = (h\circ\alpha)(a) = h(\alpha(a)) = h(\beta(c)) = (h\circ\beta)(c) = g(c)$$

and j is correctly defined.

Take any $b \in B_0$. By the assumption, we know that $P_b \neq \emptyset$. Notice, that

$$P = \bigcup_{b\in B_0} P_b.$$

Therefore, $j(P) = B_0$ is a dense ideal of B. By the construction of the topology on P, it follows that also the second condition of a Segal topological algebra is fulfilled. Hence, (P,j,B) is an object of the category $\mathscr{S}(B)$.

2) As projections, p_A and p_C are continuous algebra homomorphisms. Notice, that

$$(f\circ p_A)((a,c)) = f(a) = j((a,c)) = g(c) = (g\circ p_C)((a,c)) \text{ and}$$
$$(\alpha\circ p_A)((a,c)) = \alpha(a) = \beta(c) = (\beta\circ p_C)((a,c)) \text{ for every } (a,c)\in P.$$

Thus, $f\circ p_A = j = g\circ p_C$ and $\alpha\circ p_A = \beta\circ p_C$. Hence, p_A and p_B are morphisms in the category $\mathscr{S}(B)$, satisfying also the first condition of a pullback.

3) Suppose that $(R,k,B) \in \mathrm{Ob}(\mathscr{S}(B))$ and $\lambda \in \mathrm{Mor}((R,k,B),(A,f,B))$, $\mu \in \mathrm{Mor}((R,k,B),(C,g,B))$ are such that $\alpha\circ\lambda = \beta\circ\mu$.

$$\begin{tikzcd}
P \arrow[rrr, "j"] \arrow[dr, "p_A"] \arrow[dddr, "p_C"'] & & & B \arrow[dl, "1_B"'] \arrow[dddl, "1_B"] \\
& A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "1_B"'] & \\
& D \arrow[r, "h"] & B & \\
& C \arrow[r, "g"] \arrow[u, "\beta"] & B \arrow[u, "1_B"'] & \\
R \arrow[rrr, "k"] \arrow[uuuu, dashed, "\nu"] \arrow[uuur, dotted, "\lambda"] \arrow[ur, dotted, "\mu"] & & & B \arrow[uuuu, dashed, "1_B"'] \arrow[uuul, dotted, "1_B"'] \arrow[ul, dotted, "1_B"']
\end{tikzcd}.$$

Then $f \circ \lambda = k = g \circ \mu$. Take any $r \in R$ and denote $a_r = \lambda(r), c_r = \mu(r)$. Now, $\alpha(a_r) = (\alpha \circ \lambda)(r) = (\beta \circ \mu)(r) = \beta(c_r)$ and $f(a_r) = h(\alpha(a_r)) = h(\beta(c_r)) = g(c_r)$, which means that $h(\alpha(a_r)) \in f(A) \cap g(C) = B_0$ and $(a_r, c_r) \in P$. Define a map $\nu : R \to P$ by $\nu(r) = (\lambda(r), \mu(r))$. Then ν is a continuous algebra homomorphism, because λ and μ were continuous algebra homomorphisms. Moreover, $p_A \circ \nu = \lambda$ and $p_C \circ \nu = \mu$. Notice, that $j \circ \nu = (f \circ p_A) \circ \nu = f \circ (p_A \circ \nu) = f \circ \lambda = k$. Hence, $\nu \in \mathrm{Mor}((R,k,B),(P,j,B))$.

Take any $\kappa \in \mathrm{Mor}((R,k,B),(P,j,B))$ for which $\lambda = p_A \circ \kappa$ and $\mu = p_C \circ \kappa$. Then $\kappa(r) = (\lambda(r), \mu(r))$ for every $r \in R$, which means that $\kappa = \nu$. Thus, $\nu \in \mathrm{Mor}((R,k,B),(P,j,B))$, satisfying $p_A \circ \nu = \lambda$ and $p_C \circ \nu = \mu$, is uniquely defined by λ and μ. Hence, (P,j,B) is a pullback of morphisms α and β. □

Proposition 5.2. Let $(A,f,B),(C,g,B),(D,h,B)$ be the objects of $\mathscr{S}(B)$, $\alpha \in \mathrm{Mor}((A,f,B),(D,h,B))$ *and* $\beta \in \mathrm{Mor}((C,g,B),(D,h,B))$. *Suppose that there exists an object* (R,k,B) *of the category* $\mathscr{S}(B)$*, together with morphisms* $\gamma \in \mathrm{Mor}((R,k,B),(A,f,B))$ *and* $\delta \in \mathrm{Mor}((R,k,B),(C,g,B))$*, which is the pullback of morphisms* α *and* β*. Then the ideal* $B_0 = f(A) \cap g(C)$ *of* B *is dense in* B *and the set* $P_b = \{(a,c) \in A \times C : \alpha(a) = \beta(c), h(\alpha(a)) = b\}$ *is nonempty for every* $b \in B_0$.

Proof. As (R,k,B) is the pullback of α and β, then $B_1 := k(R)$ is a dense ideal of B, $\alpha \circ \gamma = \beta \circ \delta$ and $f \circ \gamma = k = g \circ \delta$.

Now, $B_1 = k(R) = (f \circ \gamma)(R) \subseteq f(A)$ and $B_1 = k(R) = (g \circ \delta)(R) \subseteq g(C)$. Hence, $B_1 \subseteq f(A) \cap g(C) = B_0$. As B_1 was dense in B, B_0 must also be dense in B.

Consider the set $Q = \{(a,c) : f(a) = g(c)\}$ and define a map $l : Q \to B$ by $l((a,c)) = f(a) = g(c)$ for every $(a,c) \in Q$. Notice, that then $l(Q) \subseteq B_0$. Next, we will show that $(Q,l,B) \in \mathrm{Ob}(\mathscr{S}(B))$.

Take any $(a_1,c_1),(a_2,c_2) \in Q$ and $\lambda \in \mathbb{K}$. Then $f(a_1) = g(c_1)$ and $f(a_2) = g(c_2)$. Since f and g are algebra homomorphisms, then

$$f(a_1 + a_2) = f(a_1) + f(a_2) = g(c_1) + g(c_2) = g(c_1 + c_2),$$

$$f(a_1 a_2) = f(a_1) f(a_2) = g(c_1) g(c_2) = g(c_1 c_2),$$

$$f(\lambda a_1) = \lambda f(a_1) = \lambda g(c_1) = g(\lambda c_1),$$

which means that $(a_1,c_1) + (a_2,c_2), (a_1,c_1)(a_2,c_2), \lambda(a_1,c_1) \in Q$. Thus, Q is a subalgebra of $A \times C$. When equipped with the subspace topology τ_Q of the product topology of (A,τ_A) and (C,τ_C), the set Q becomes a topological algebra.

Since f and g were algebra homomorphisms, then it is easy to see that l is an algebra homomorphism. Moreover, for any $b \in B_0$, there exists $a_b \in A$ and $c_b \in C$ such that $f(a_b) = b = g(c_b)$. Hence, $(a_b,c_b) \in Q$ and $l((a_b,c_b)) = b$. Therefore, $l(Q) = B_0$ is a dense ideal of B.

It remains to check that the topology τ_Q satisfies the condition 2) of the Segal topological algebra. Since $(A,f,B),(C,g,B) \in \mathrm{Ob}(\mathscr{S}(B))$, then $\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}$ and $\tau_C \supseteq \{g^{-1}(U) : U \in \tau_B\}$. Hence, the product topology $\tau_{A \times C} \supseteq \{f^{-1}(U) \times g^{-1}(V) : U,V \in \tau_B\}$ and $\tau_Q \supseteq \{l^{-1}(U) = f^{-1}(U) \times g^{-1}(U) : U \in \tau_B\}$. Therefore, $(Q,l,B) \in \mathrm{Ob}(\mathscr{S}(B))$.

Define the maps $p_A : Q \to A$ and $p_C : Q \to C$ by $p_A((a,c)) = a$ and $p_C((a,c)) = c$ for every $(a,c) \in Q$. Then it is easy to see that p_A an p_C, as the projections, are continuous algebra homomorphisms and $\alpha \circ p_A = \beta \circ p_C$.

As (R, k, B) is the pullback of α and β, there must exist $\nu \in \operatorname{Mor}((Q, l, B), (R, k, B))$ such that $\gamma \circ \nu = p_A$, $\delta \circ \nu = p_C$ and $k \circ \nu = l$. Hence, $B_0 = l(Q) = k(\nu(Q)) \subseteq k(R) = B_1$, which implies $B_0 = B_1$.

Take any $b \in B_0 = B_1$. Then there exists $r_b \in R$ such that $k(r_b) = b$. But now there exist $a_b = \gamma(r_b) \in A, c_b = \delta(r_b) \in C$ such that

$$(h \circ \alpha)(a_b) = f(a_b) = (f \circ \gamma)(r_b) = k(r_b) = b$$

$$\begin{tikzcd}
R \arrow[rrr, "k"] \arrow[dr, "\gamma"] \arrow[ddr, "\delta"'] & & & B \arrow[dl, "1_B"'] \arrow[ddl, "1_B"] \\
& A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "1_B"'] & \\
& D \arrow[r, "h"] & B & \\
& C \arrow[r, "g"] \arrow[u, "\beta"'] & B \arrow[u, "1_B"] & \\
Q \arrow[rrr, "l"] \arrow[uuuu, dashed, "\nu"] \arrow[uuur, dotted, "p_A"] \arrow[ur, dotted, "p_C"] & & & B \arrow[uuuu, dashed, "1_B"'] \arrow[uuul, dotted, "1_B"'] \arrow[ul, dotted, "1_B"']
\end{tikzcd}.$$

We also have $\alpha(a_b) = (\alpha \circ \gamma)(r_b) = (\beta \circ \delta)(r_b) = \beta(c_b)$. Hence, $(a_b, c_b) \in P_b$ and the set P_b is not empty for any $b \in B_0$. $\square$

From the Proposition 5.2 and the part 3 of Proposition 5.1, we conclude the following Theorem.

Theorem 5.3. *Let $(A, f, B), (C, g, B), (D, h, B)$ be the objects of $\mathscr{S}(B)$, $\alpha \in \operatorname{Mor}((A, f, B), (D, h, B))$ and $\beta \in \operatorname{Mor}((C, g, B), (D, h, B))$. Then there exists the pullback of morphisms α and β if and only if the ideal $B_0 = f(A) \cap g(C)$ of B is dense in B and the set $P_b = \{(a, c) \in A \times C : \alpha(a) = \beta(c), h(\alpha(a)) = b\}$ is nonempty for every $b \in B_0$.*

Remark 5.4. There remains an open question: Whether the pushouts exist in the category $\mathscr{S}(B)$? If yes, then under which extra hypothesis, if any?

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