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PROPERTIES OF THE REGULAR SPECTRUM OF ELEMENTS IN TOPOLOGICAL ALGEBRAS

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Abstract. Relations between the usual spectrum of elements and the regular (or extended) spectrum of elements of topological algebras are described. Conditions when the spectral radius, the regular spectral radius and the radius of boundedness of elements in topological (not necessarily unital and locally convex) algebras coincide are given.

Let A be a *topological algebra* over $\mathbb{C}$ with separately continuous multiplication (in short, a topological algebra). In particular, when the multiplication, as a map $A \times A \to A$, is continuous, we speak about a *topological algebra with jointly continuous multiplication.*

Let Inv$A$ denote the set of all invertible elements in A and Qinv$A$ the set of all quasi-invertible elements in A (that is, of elements $a \in A$, for which there is an element a_q^{-1} (the *quasi-inverse* of a) such that $a + a_q^{-1} = aa_q^{-1} = a_q^{-1}a$). A topological algebra A is called a *Q-algebra*, if the set Qinv$A$ (for unital algebras Inv$A$) is open in A. Moreover, a topological algebra A is *locally complete* (in the locally convex case, G. R. Allan used in [4], p. 401, the term *pseudo-complete*), if every subalgebra of A, generated by a closed, bounded, idempotent and absolutely pseudoconvex subset U, is complete in the normed topology (let us remind that U is *idempotent*, if $UU \subset U$, *absolutely k-convex*, if

$$U = \Gamma_k(U) = \Big\{ \sum_{v=1}^{n} \alpha_v u_v : n \in \mathbb{N}, u_1, \ldots, u_n \in U, \alpha_1, \ldots, \alpha_n \in \mathbb{C}, \sum_{v=1}^{n} |\alpha_v|^k \leqslant 1 \Big\}$$

and *absolutely pseudoconvex*, if U is absolutely k-convex for some $k \in (0,1]$). In addition, A is a *locally convex* (*locally k-convex* or *locally m-convex*) *algebra* if A has a base of neighbourhoods of zero, consisting of absolutely convex (respectively, absolutely

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k-convex or idempotent or absolutely convex) sets. Moreover, A is a *topological algebra with idempotently pseudoconvex von Neumann bornology*[1], if for every idempotent and bounded subset U of A there is a number $k \in (0,1]$ such that $\Gamma_k(U)$ is bounded in A. It is easy to show that the von Neumann bornology of every locally k-convex algebra is idempotently pseudoconvex. Let

$$S(a,\lambda) = \left\{ \left(\frac{a}{\lambda}\right)^n : n \in \mathbb{N} \right\} \tag{1}$$

for each $a \in A$ and $\lambda \in \mathbb{C} \setminus \{0\}$. An element $a \in A$ is *pseudoconvexly bounded*, if there exist $\lambda > 0$ and $k \in (0,1]$ such that $\Gamma_k(S(a,\lambda))$ is bounded in A. We denote by A_{pcb} the collection of all pseudoconvexly bounded elements in A and by $\mathscr{B}_k$ the set of all idempotent and absolutely k-convex bounded subsets of A.

1. Introduction

1. **Bounded elements in a topological algebra**. Let A be a unital locally convex algebra over the field $\mathbb{C}$. In 1954 L. Waelbroeck introduced in [7] (see also [6]) the notion of a regular element of an algebra. He said that $a \in A$ is *regular*, if there is a neighbourhood O of ∞ such that the resolvent map R_a of a, defined by

$$R_a(\lambda) = (a - \lambda e_A)^{-1}$$

for each $\lambda \notin \mathrm{sp}_A(a)$ (the spectrum of a), has bounded limit in O, if $|\lambda| \to \infty$.

After that, in 1956, S. Warner said in [9] that an element a is *idempotently bounded* in a locally m-convex algebra A, if there is a $\lambda > 0$ such that the set

$$I(\{\lambda a\}) = \bigcup_{n \in \mathbb{N}} \{\lambda a\}^n$$

is bounded.

Next, in 1965, G. R. Allan called in [4] an element $a \in A$ *bounded*, if the set $S(a,\lambda)$ is bounded in A for some $\lambda > 0$. Hence, every idempotently bounded element of a locally m-convex algebra is bounded and vice versa. Moreover (see, for example, [8], Proposition 11), these three notions, given by Waelbroeck, Warner and Allan, are the same in case of a unital b-algebra (that is, of a unital locally convex algebra, the von Neumann bornology of which has a base of completant, idempotent and absolutely convex sets). We use the term "bounded element a" in case when the set $S(a,\lambda)$ is bounded for some $\lambda > 0$ in an arbitrary topological (that is, not necessarily unital nor locally convex) algebra A with separately continuous multiplication. We denote the set of all bounded elements in A, as it was used in [4], by A_0.

2. **Regular spectrum of elements in a topological algebra**. Let A be a unital algebra. The *spectrum* $\mathrm{sp}_A(a)$ of $a \in A$ is defined by

$$\mathrm{sp}_A(a) = \{\lambda \in \mathbb{C} : a - \lambda e_A \notin \mathrm{Inv}A\}$$

and in case, when A is not necessarily a unital algebra, by

$$\mathrm{sp}_A(a) = \{\lambda \in \mathbb{C} \setminus \{0\} : \frac{a}{\lambda} \notin \mathrm{Qinv}A\} \cup \{0\}.$$

[1] The von Neumann bornology on A consists of all bounded sets in A.

In 1954, L. Waelbroeck introduced in [6] the notion of a "regular spectrum of an element" in a unital commutative locally convex algebra A. He said that the set $\mathrm{sp}_A^r(a)$, defined by

$$\mathrm{sp}_A^r(a) = \{\lambda \in \mathbb{C} : a - \lambda e_A \text{ has not a regular inverse in } A\} \cup S$$

(where $S = \{\emptyset\}$, if a is regular, and $S = \{\infty\}$, if a is not regular), is the *regular spectrum* of $a \in A$.

G. R. Allan defined in [4] the *extended spectrum* of a in a unital locally convex algebra similarly, using only the term "bounded element a" instead of the term ,,regular element a", that is, in case of a unital locally convex algebra A,

$$\mathrm{sp}_A^r(a) = \mathrm{sp}_A(a) \cup \{\lambda \in \mathbb{C} : a - \lambda e_A \in \mathrm{Inv}A \text{ but } (a - \lambda e_A)^{-1} \notin A_0\} \cup S, \tag{2}$$

where $S = \{\emptyset\}$, if $a \in A_0$, and $S = \{\infty\}$, if $a \notin A_0$. Since W. Żelazko used in [10], p. 130, the name ,"extended spectrum of an element" in other sense, we shall use later on the name "regular spectrum of an element" in order to describe the present situation.

Next, we give this notion (as in [3]) in case of a (not necessarily unital and not necessarily locally convex) topological algebra A. We say that the *regular spectrum* of $a \in A$ is the set

$$\mathrm{sp}_A^r(a) = \mathrm{sp}_A(a) \cup \{\lambda \in \mathbb{C} \setminus \{0\} : \frac{a}{\lambda} \in \mathrm{Qinv}A \text{ but } \left(\frac{a}{\lambda}\right)_q^{-1} \notin A_0\} \cup S, \tag{3}$$

where $S = \{\emptyset\}$, if $a \in A_0$, and $S = \{\infty\}$, if $a \notin A_0$. That is, $\mathrm{sp}_A^r(a) \setminus S$ of $a \in A$ consists of all $\lambda \in \mathbb{C}$ such that either $\frac{a}{\lambda}$ with $\lambda \neq 0$ has not quasi-inverse in A or $\frac{a}{\lambda}$ has in A the quasi-inverse, but it does not belong to A_0. We put

$$D_a(\lambda) = \left(\frac{a}{\lambda}\right)_q^{-1}$$

for any $a \in A$ and $\lambda \notin \mathrm{sp}_A^r(a)$.

It has been shown in [3] that the sets in the right of (3) and (2) coincide, if A is a unital topological algebra.

3. Let A be a topological algebra. By $\mathrm{r}_A(a)$ and $\mathrm{r}_A^r(a)$ we denote the *spectral radius* and the *regular spectral radius* of $a \in A$, defined by

$$\mathrm{r}_A(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}_A(a)\} \text{ and } \mathrm{r}_A^r(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}_A^r(a)\}.$$

Then $\mathrm{r}_A(a) \leqslant \mathrm{r}_A^r(a)$ and $\mathrm{r}_A(a) = \mathrm{r}_A^r(a)$, if $\mathrm{Qinv}A \subset A_0$ (in unital case, if $\mathrm{Inv}A \subset A_0$). Moreover, by $\beta(a)$ we denote the *radius of boundedness* of $a \in A$, defined by

$$\beta(a) = \inf\{\lambda > 0 : S(a, \lambda) \text{ is bounded in } A\} = \inf\{\lambda > 0 : \left(\frac{a}{\lambda}\right)^n \text{ vanishes in } A\}$$

for each $a \in A$.

4. In what follows, we give relations between the usual spectrum of elements and the regular spectrum of elements of topological algebras as well as conditions, when the spectral radius, the regular spectral radius and the radius of boundedness of elements in topological algebras coincide.

2. Relations between the usual and the regular spectra

The next theorem generalizes a result of Allan in [4] to the case of non locally convex algebras. We put $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ and denote by $\mathrm{cl}_{\mathbb{C}_\infty}(U)$ the closure of U in the topology of $\mathbb{C}_\infty$.

Theorem 2.1. *Let A be a topological algebra and $a \in A$. If A satisfies the conditions*
(a) $a \in A_0$
and
(b) $D_a(\lambda) \in A_0$ *(in unital case, $R_a(\lambda) \in A_0$) for any $\lambda \in \mathbb{C} \setminus \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a))$, then*

$$\mathrm{sp}_A(a) \subseteq \mathrm{sp}_A^r(a) \subseteq \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a)). \tag{4}$$

Moreover, if the right side inclusion of (4) *holds, then A has the property* (b) *and the property* (a) *if the spectrum $\mathrm{sp}_A(a)$ of a is bounded.*

Proof. The left side inclusion of (4) holds by the definition of the regular spectrum. To prove the right side inclusion of (4), it is enough to consider only the case when $\mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a)) \neq \mathbb{C}_\infty$. Then there is a number $\lambda_0 \in \mathbb{C}_\infty \setminus \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a))$. If $\lambda_0 \neq \infty$, then $D_a(\lambda_0) \in A_0$ (in unital case, $R_a(\lambda) \in A_0$), by the condition (b). Therefore, $\lambda_0 \notin \mathrm{sp}_A^r(a)$ ($\infty \notin \mathrm{sp}_A^r(a)$, by the condition (a)). Hence, $\mathbb{C}_\infty \setminus \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a)) \subseteq \mathbb{C}_\infty \setminus \mathrm{sp}_A^r(a)$. Consequently, $\mathrm{sp}_A^r(a) \subseteq \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a))$.

Let now the right side inclusion of (4) be valid and $\lambda \in \mathbb{C} \setminus \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a))$. Then, from $\lambda \notin \mathrm{sp}_A(a)$ follows that $D_a(\lambda)$ (in unital case, $R_a(\lambda)$) exists in A, and, from $\lambda \notin \mathrm{sp}_A^r(a)$, that $D_a(\lambda) \in A_0$ (respectively, $R_a(\lambda) \in A_0$). If now the spectrum $\mathrm{sp}_A(a)$ is bounded, then $\infty \notin \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a))$. Hence, $\infty \notin \mathrm{sp}_A^r(a)$ or $a \in A_0$. □

Corollary 2.2. (M. Abel [3] and G. R. Allan [4]) *Let A be a locally convex algebra with jointly continuous multiplication and with continuous quasi-inversion $a \to a_q^{-1}$ (in unital case, with continuous inversion $a \to a^{-1}$). Then the inclusions* (4) *hold for each $a \in A$.*

Proof. It is shown in the proof of [3, Proposition 3.6], that $a \in A_0$ and $R_a(\lambda) \in A_0$ for each $\lambda \in \mathbb{C} \setminus \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a))$ in case of a unital locally convex algebra A with jointly continuous multiplication and with continuous inversion $a \to a^{-1}$. Hence (4) holds by Theorem 2.1.

If A is not unital, then, instead of A, we consider the unitization $A_1 = A \times \mathbb{C}$ of A, in which the multiplication is jointly continuous because the multiplication in A is jointly continuous (see [5, Proposition 2.2.9]). Moreover (see, for example, [5], p. 156), the inversion is continuous in A_1 by the continuity of the quasi-inversion in A. Hence, by [3, Proposition 2.3] and the first part of the proof, we have that

$$\mathrm{sp}_A(a) \subseteq \mathrm{sp}_A^r(a) = \mathrm{sp}_{A_1}^r((a,0)) \subseteq \mathrm{cl}_{\mathbb{C}_\infty}[\mathrm{sp}_{A_1}((a,0))] = \mathrm{cl}_{\mathbb{C}_\infty}[\mathrm{sp}_A(a)]$$

for each $a \in A$. □

Next we use the following result from [3, Corollary 4.5]:

Proposition 2.3. Let A be a locally complete Hausdorff algebra with idempotently pseudoconvex von Neumann bornology. Then the regular spectrum $\mathrm{sp}_A^r(a)$ is closed in $\mathbb{C}_\infty$ and not empty for every $a \in A$.

Theorem 2.4. *Let A be a locally complete Hausdorff algebra with idempotently pseudoconvex von Neumann bornology. If A has the properties* (a) *and* (b) *of Theorem* 2.1, then

$$\mathrm{sp}_A^r(a) = \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a))$$

for each $a \in A$.

Proof. This statement is true, by Theorem 2.1 and Proposition 2.3. □

Corollary 2.5. *Let A be a locally complete locally convex Hausdorff algebra with jointly continuous multiplication and continuous quasi-inversion. Then*

$$\mathrm{sp}_A^r(a) = \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a))$$

for each $a \in A$.

Proof. This statement is true by Corollary 2.2 and Theorem 2.4, because the von Neuman bornology of A is idempotently pseudoconvex in case of a locally convex algebra. □

Corollary 2.6. *Let A be a locally complete Hausdorff algebra with idempotently pseudoconvex von Neumann bornology. If A has the properties* (a) *and* (b) *of Theorem* 2.1*, then $a \in A_0$ if and only if $\mathrm{sp}_A(a)$ is bounded.*

Proof. If $a \in A_0$, then $\infty \notin \mathrm{sp}_A^r(a)$ by the definition of the regular spectrum. Therefore, $\infty \notin \mathrm{sp}_A(a)$. Consequently, $\mathrm{sp}_A(a)$ is bounded.

Let now $\mathrm{sp}_A(a)$ be bounded. Then $\infty \notin \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a))$. Hence, $\infty \notin \mathrm{sp}_A^r(a)$, by Corollary 2.5. So, $a \in A_0$, by the definition of the regular spectrum. □

Corollary 2.7. (M. Abel [3] and G. R. Allan [4]) *Let A be a locally complete locally convex Hausdorff algebra with jointly continuous multiplication and continuous quasi-inversion. Then $a \in A_0$ if and only if $\mathrm{sp}_A(a)$ is bounded.*

3. Relations between the spectral radius, the regular spectral radius and the radius of boundedness of elements

To generalize the result [2, Theorem 4.5] to the case of nonunital and non locally convex topological algebras, we need the following.

Proposition 3.1. Let A be a locally complete Hausdorff algebra with idempotently pseudoconvex von Neumann bornology and let a be an arbitrary element in $A \setminus \{\theta_A\}$. Then

(a) for any $\lambda_0 \in \mathbb{C} \setminus \mathrm{sp}_A^r(a)$, there exist a number $k(\lambda_0) \in (0, 1]$, an open neighbourhood $O(\lambda_0) \subset \mathbb{C} \setminus \mathrm{sp}_A^r(a)$ of λ_0 and a set $B(\lambda_0) \in \mathscr{B}_{k(\lambda_0)}$ such that $D_a(\lambda) \in A_{B(\lambda_0)}$ for each $\lambda \in O(\lambda_0)$.

(b) if $\infty \notin \mathrm{sp}_A^r(a)$, then there exist $k \in (0, 1]$, an open neighbourhood O of ∞ and a set $C \in \mathscr{B}_k$ such that $D_a(\lambda) \in A_C$, whenever

$$\lambda \in O \subset \mathbb{C}_\infty \setminus \mathrm{sp}_A^r(a).$$

Proof. See the proof of [3, Proposition 4.4]. □

Proposition 3.2. Let A be a locally complete Hausdorff algebra with idempotently pseudoconvex von Neumann bornology, $a \in A_0$ and K be a compact subset in $\mathbb{C}_\infty \setminus \mathrm{sp}_A^r(a)$. Then there exist a number $k \in (0,1]$ and in A a set $B \in \mathscr{B}_k$ such that $D_a(\lambda) \in A_B$ for each $\lambda \in K$.

Proof. If $a = \theta_A$, then $\theta_A = D_{\theta_A}(\lambda) \in A_B$ for every $\lambda \neq 0$ and $B \in \mathscr{B}_{pcb}$. Let now $a \neq \theta_A$ and C be the maximal commutative subalgebra of A such that $a \in C$. Then $\mathrm{Qinv}C = C \cap \mathrm{Qinv}A$, $\mathrm{sp}_C^r(a) = \mathrm{sp}_{\mathrm{A}}^{\mathrm{r}}(\mathrm{a})$ (by [3, Lemma 3.7]) and $D_a(\lambda) \in C$ for each $\lambda \in \mathbb{C} \backslash \mathrm{sp}_A^r(a)$. Now, for any $\lambda_0 \in \mathbb{C}_\infty \setminus \mathrm{sp}_{\mathrm{A}}^{\mathrm{r}}(\mathrm{a})$ there exist a number $k(\lambda_0) \in (0,1]$, an open neighbourhood $O(\lambda_0) \subset \mathbb{C}_\infty \setminus \mathrm{sp}_A^r(a)$ of λ_0 and in A a set $B(\lambda_0) \in \mathscr{B}_{k(\lambda_0)}$ such that $D_a(\lambda) \in A_{B(\lambda_0)}$ for each $\lambda \in O(\lambda_0)$, by Proposition 3.1. Because K is a compact set in $\mathbb{C}_\infty \setminus \mathrm{sp}_A^r(a)$, there exist $n \in \mathbb{N}$ and points $\lambda_1, \ldots, \lambda_n \in K$ such that the neighbourhoods $O(\lambda_1), \ldots, O(\lambda_{n-1})$ and $O(\lambda_n)$ cover the set K. Moreover, for every $l \in \{1, .., n\}$ there exists $k_l \in (0,1]$ such that $B(\lambda_l) \in \mathscr{B}_{k_l}$ and $D_a(\lambda) \in A_{B(\lambda_l)}$ for each $\lambda \in O(\lambda_l)$ as above. Since every maximal commutative subalgebra is closed in A, every closed, bounded, idempotent and absolutely k-convex subset in C with $k \in (0,1]$ is closed, bounded, idempotent and absolutely k-convex subset in A as well, and C is also locally complete. Thus, there exist $k \in (0,1]$ and in C a set $B \in \mathscr{B}_k$ such that

$$\bigcup_{l=1}^{n} (B(\lambda_1) \cap C) \subset B$$

by [2, Lemma 4.2]. Therefore,

$$\bigcup_{v=1}^{n} A_{B(\lambda_v) \cap C} \subset A_B.$$

Consequently, $D_a(\lambda) \in A_B$ for each $\lambda \in K$, because for every fixed $\lambda \in K$ there exists a $v \in \{1, \ldots, n\}$ such that $\lambda \in O(\lambda_v)$ and

$$D_a(\lambda) \in A_{B(\lambda_v)} \cap C \subset A_{B(\lambda_v) \cap C} \subset A_B. \qquad \square$$

Let next $\mathscr{B}_p$ denote the collection of all closed, bounded, idempotent and absolutely pseudoconvex subsets in A and

$$\mathscr{B}(a) = \{B \in \mathscr{B}_p : a \in A_B\}$$

for each $a \in A_0$.

Proposition 3.3. Let A be a topological Hausdorff algebra with idempotently pseudoconvex von Neumann bornology. Then

1) $A_0 = A_{pcb}$

and

2) $\mathrm{r}_A^r(a) = \inf\{\mathrm{r}_{A_B}(a) : B \in \mathscr{B}(a)\}$ for each $a \in A_0$.

Proof. 1) Let A be as stated. Then $A_{pb} \subset A_0$. Let now $a \in A_0$. Now there is a $\rho > 0$ such that $S(a, \rho)$ is a bounded and idempotent subset in A. Hence, there exists a $k \in (0,1]$ such that $\Gamma_k(S(a, \rho))$ is bounded in A. So, a is a pseudoconvexly bounded element in A. Consequently, $A_0 = A_{pcb}$.

2) If $a \in A_0$, then there is a $\rho > 0$ such that $S(a, \rho)$ is an idempotent and bounded set in A. Since the von Neumann bornology of A is idempotently

pseudoconvex, then there is a $k \in (0,1]$ such that $\Gamma_k(S(a,\rho))$ is a bounded and absolutely k-convex set in A. Moreover, as the closure of bounded (idempotent and absolutely k-convex) set is bounded (respectively, idempotent and absolutely k-convex) set in A, then $B = \mathrm{cl}_A(\Gamma_k(S(a,\rho))) \in \mathscr{B}_k$, the subalgebra A_B of A, generated by B, is a k-normed algebra with respect to the k-homogeneous submultiplicative norm p_B defined by

$$p_B(a) = \inf\{|\lambda|^k : a \in \lambda B\}$$

for each $a \in A_B$ (see, for example, [1, Proposition 2.2]), because A is a Hausdorff space, and the topology on A_B, defined by p_B, is not weaker than the subset topology on A_B. It is known (see [1, Proposition 2.3]) that

$$A_{pb} = \bigcup_{B \in \mathscr{B}_p} A_B.$$

Since $A_B \subset A$, then $\mathrm{sp}^r_A(a) \subset \mathrm{sp}^r_{A_B}(a)$ for each $a \in A_B$. Therefore, $\mathrm{r}^r_A(a) \leqslant \mathrm{r}_{A_B}(a)$ for every $B \in \mathscr{B}(a)$, because every element in A_B is bounded. Thus,

$$\mathrm{r}^r_A(a) \leqslant \inf\{\mathrm{r}_{A_B}(a) : B \in \mathscr{B}(a)\}.$$

To prove the opposite inequality, let $\mu > \mathrm{r}^r_A(a)$ and

$$K_\mu = \{\lambda \in \mathbb{C}_\infty : |\lambda| \geqslant \mu\}.$$

Then $K_\mu \subset \mathbb{C}_\infty \setminus \mathrm{sp}^r_A(a)$ is a compact set. Hence, by Proposition 3.2, there are a number $k_\lambda \in (0,1]$ and a set $B_\lambda \in \mathscr{B}_{k_\lambda}$ such that $D_a(\lambda) \in A_{B_\lambda}$ for each $\lambda \in K_\mu$. Let C be the maximal commutative subalgebra of A such that $a \in C$. Since $B_\lambda \cap C$ and $B \cap C$ are idempotent and bounded subsets in C, there exist a number $k \in (0,1]$ and a set $E \in \mathscr{B}_k$ such that $E \subset C$ and $(B \cap C) \cup (B_\lambda \cap C) \subset E$ by [2, Lemma 4.2]. Hence, $A_B \cap C \subset A_{B \cap C} \subset A_E$ and $D_a(\lambda) \in A_E$ for each $\lambda \in K_\mu$. Therefore, $E \in \mathscr{B}(a)$ and $\mu \geqslant \mathrm{r}_{A_E}(a)$. Taking this into account, from

$$\inf\{\mathrm{r}_{A_B}(a) : B \in \mathscr{B}(a)\} \leqslant \mathrm{r}_{A_E}(a) \leqslant \mu$$

for all $\mu > \mathrm{r}^r_A(a)$ follows that

$$\inf\{\mathrm{r}_{A_B}(a) : B \in \mathscr{B}(a)\} \leqslant \mathrm{r}^{\mathrm{r}}_{\mathrm{A}}(\mathrm{a}).$$

Thus, $r^r_A(a) = \inf\{\mathrm{r}_{A_B} : B \in \mathscr{B}(a)\}$ for each $a \in A_0$. □

Lemma 3.4. *Let A be a complete k-normed algebra over $\mathbb{C}$ for some $k \in (0,1]$. Then $\beta_A(a) = \mathrm{r}_A(a)$ for each $a \in A$.*

Proof. For the commutative case, the statement is true by [1, Proposition 4.1]. In non-commutative case, we fix $a \in A$ and consider, instead of A, the maximal commutative subalgebra C of A which contains a. Since C is closed in A, then C is a commutative complete k-normed algebra and

$$\beta_A(a) = \beta_C(a) = \mathrm{r}_C(a) = \mathrm{r}_A(a),$$

by the proof in commutative case, and the fact that $\mathrm{sp}_A(a) = \mathrm{sp}_C(a)$ for each $a \in A$ (see, for example, [3, Lemma 3.7]). □

Theorem 3.5. *Let A be a locally complete Hausdorff algebra with idempotently pseudoconvex von Neumann bornology. Then $\beta_A(a) = \mathrm{r}^r_A(a)$ for each $a \in A$. In particular case, when* $\mathrm{Qinv}A \subset A_0$*, then $\beta_A(a) = \mathrm{r}_A(a)$.*

Proof. Let A be as stated, $a \in A$. If $\mathrm{r}^r_A(a) = \infty$, then $\beta_A(a) \leqslant \mathrm{r}^r_A(a)$. Therefore, we can assume that $\mathrm{r}^r_A(a) < \infty$. Then $a \in A_0$. Since every $B \in \mathscr{B}(a)$ defines a $k_B \in (0,1]$ such that A_B is a complete k_B-normed subalgebra in A, as above, then

$$\beta_A(a) \leqslant \inf\{\beta_{A_B}(a) : B \in \mathscr{B}(a)\} = \inf\{\mathrm{r}_{A_B}(a) : B \in \mathscr{B}(a)\} = \mathrm{r}^r_A(a)$$

by Lemma 3.4 and Proposition 3.3.

The proof of the inequality $\mathrm{r}^r_A(a) \leqslant \beta_A(a)$ is similar to the proof of [2, Theorem 4.5]. □

Corollary 3.6. *Let A be a locally complete and locally convex Hausdorff algebra. Then $\beta_A(a) = \mathrm{r}^r_A(a)$ ($\beta_A(a) = \mathrm{r}_A(a)$ if $\mathrm{Qinv}A \subset A_0$) for each $a \in A$.*

Proof. The statement of Corollary 3.6 is true by Theorem 3.5, because the von Neumann bornology of a locally convex algebra is idempotently pseudoconvex. □

Remark. For unital algebras, Propositions 3.2 and 3.3 and Theorem 3.5 have been proved in [2] and in locally convex case in [4].

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