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ON SOME CHARACTERIZATIONS OF Q-ALGEBRAS FOR UNITAL LOCALLY PSEUDOCONVEX ALGEBRAS

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Abstract. In [10] there were studied some equivalent conditions for locally m-pseudoconvex algebra to be a Q-algebra. The proofs were given using the submultiplicative property of the elements of the family of pseudoseminorms that define the topology of the algebra. In this paper it is shown that the submultiplicativity condition of pseudoseminorms is not necessary for some of the equivalent conditions.

1. Introduction

We consider a topological algebra E over $\mathbb{C}$ with separately continuous multiplication. A topological algebra E is called *Q-algebra* when the set $\mathrm{Qinv}(E)$ of quasi-invertible elements of E is open. A unital algebra E is called a *Q-algebra* when the set $\mathrm{Inv}(E)$ of invertible elements of E is open. It is well known that Banach algebras are Q-algebras but they are not the only ones. This kind of algebras have been studied by several researchers ([1],[3],[5]), some of them have given equivalent conditions for topological algebras to be Q-algebras. In [8] has been given some equivalent conditions for unital complex normed algebras. Later, in [5], [10] and [11] were given analogous conditions for non unital seminormed, locally m-convex and unital locally m-pseudoconvex algebras, respectively. In all these papers, it is assumed that the norms, seminorms or pseudoseminorms, that give the topology for studied topological algebras, have the submultiplicative condition.

In this paper will be studied some characterizations of unital locally pseudoconvex Q-algebras. Notice, that, in this case, the submultiplicative condition for the pseudoseminorms is not needed.

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2. Preliminaries

1. Let E be a unital topological algebra over the field of complex numbers $\mathbb{C}$, with separately continuous multiplication (in short, a *topological algebra*).

A subset of a linear space X over a field $\mathbb{K}$ ($\mathbb{K}$ denotes $\mathbb{C}$ or $\mathbb{R}$) is called *ρ-convex*, with $0 < \rho \leqslant 1$, if $x, y \in S, \alpha, \beta \in \mathbb{R}, \alpha, \beta \geqslant 0$ and $\alpha^\rho + \beta^\rho = 1$, implies that $\alpha x + \beta y \in S$. Notice that when $\rho = 1$, then the notions of ρ-convex and convex coincide. A subset of X is called *pseudoconvex*, if it is ρ-convex for some ρ ($0 < \rho \leqslant 1$). A topological linear space is called *locally pseudoconvex*, if it has a basis of pseudoconvex neighborhoods of zero $\{U_\alpha : \alpha \in \Lambda\}$, where each U_α is ρ_α-convex ($0 < \rho_\alpha \leqslant 1$). If $\rho_\alpha = \rho$ for every $\alpha \in \Lambda$, then X is called *locally ρ-convex* and if $\rho = 1$, then X is called *locally convex*. Let ρ be a real number such that $0 < \rho \leqslant 1$. A *non-homogeneous seminorm* (called ρ-seminorm or pseudoseminorm in [3]) is a real-valued function $p = p(x)$ on X which satisfies:

(i) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$;
(ii) $p(\lambda x) = |\lambda|^\rho p(x)$ for all $x \in X,\ \lambda \in \mathbb{K}$.

The number ρ is called the homogeneity index of p. If we consider a family of non-homogeneous seminorms $\{p_\alpha : \alpha \in \Lambda\}$ on a linear space X and $O_{\alpha,r} = \{x \in X : p_\alpha(x) < r\}$, where $r \in \mathbb{R}, r > 0$, then the family of finite intersections of $O_{\alpha,r}$ (varying α, r) gives a basis of pseudoconvex neighborhoods of zero for X. Conversely, if X is a locally pseudoconvex space with $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ a basis of pseudoconvex neighborhoods of zero (as every topological linear space has a basis of balanced and absorbing neighborhoods of zero, we can assume that every $U_\alpha \in \mathscr{U}$ is balanced and absorbing) then the Minkowski functionals $p_{U_\alpha}(x) = \inf_\beta\{\beta > 0 : x \in \beta^{\frac{1}{\rho_\alpha}} U_\alpha\}$, associated to each $U_\alpha \in \mathscr{U}$, determine a family $\{p_\alpha := p_{U_\alpha} : \alpha \in \Lambda\}$ of ρ_α-seminorms (see [3, p.179, Proposition 4.1.10]).

If the underlying topological linear space of a topological algebra E is locally pseudoconvex, then E is called a *locally pseudoconvex algebra* and its topology can be defined by a family $\mathscr{P} = \{p_\alpha : \alpha \in \Lambda\}$ of pseudoseminorms. When every p_α in the family $\mathscr{P}$ satisfies the submultiplicativity condition

$$(1)\ \ p_\alpha(xy) \leqslant p_\alpha(x)p_\alpha(y) \ \text{ for each } \ \alpha \in \Lambda,$$

then E is called a *locally m-pseudoconvex*.

When each $p_\alpha \in \mathscr{P}$ is a seminorm (the homogeneity index of every $p_\alpha \in \mathscr{P}$ is equal to 1), then E is called a *locally convex algebra* and if also every element of the family $\mathscr{P}$ satisfies the submultiplicativity condition (1), E is called *locally m-convex algebra*.

A locally pseudoconvex algebra E is called *locally A-pseudoconvex*, if for every $x \in E$ there exist $M(x, \alpha) = M_{x,\alpha} > 0$ and $N(x, \alpha) = N_{x,\alpha} > 0$ such that,

$$p_\alpha(xy) \leqslant M_{x,\alpha}^{\rho_\alpha} p_\alpha(y) \ \text{ and } \ p_\alpha(yx) \leqslant N_{x,\alpha}^{\rho_\alpha} p_\alpha(y) \ \text{ for every } \ y \in E.$$

If $\rho_\alpha = \rho$ for every $\alpha \in \Lambda$, then the algebra is called locally A-*(ρ-convex)* and if $\rho_\alpha = 1$ for every $\alpha \in \Lambda$, the algebra is called locally A-convex.

If E is an algebra over $\mathbb{C}$, the *spectrum* $\sigma(x)$ of x is given by

$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda^{-1}x \notin \mathrm{Qinv}(E)\} \cup \{0\}$$

(when E is a unital algebra, then by

$$\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda e \notin \mathrm{Inv}(E)\})$$

and the *spectral radius* of x is defined by

$$r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

3. Some characterizations of locally pseudoconvex Q-algebras

It is known that in the case of locally convex algebras, the following theorem holds (see [4, Theorem 2.7]).

Theorem 3.1. *Let E be a locally convex algebra. Then, the following are equivalent:*

(1) *E is Q-algebra;*
(2) *$r(x) \leqslant g_V(x)$, for every $x \in E$, where V is a basic balanced neighborhood of zero in E and g_V the gauge function*[1] *of V;*
(3) *$\{x \in E : g_V(x) < 1\} \subset \mathrm{Qinv}E$, V as in* (2)*;*
(4) *$\{x \in E : g_V(x) < 1\} \subset S_E$, where $S_E = \{x \in E : r(x) \leqslant 1\}$ and V as in* (2)*;*
(5) *the Jacobson radical*[2] *$J(E)$ of E is closed and $E/J(E)$ is a Q-algebra.*

We notice that some of these equivalences are also true in general for any topological algebra, the property Q given by (2) (see [12]) and also the equivalences between (1), (3) and (4) (see [7], Lemma I.6.4 and Lemma II.4.2) are true for any topological algebra.

Remark 3.2. We can apply results that are known for Q-algebras without unit to the case of a topological algebra E with unit e, since by [7, p. 48] if E has a unit, then

$$x \circ y = \theta \Leftrightarrow (e - x)(e - y) = e \text{ for every } x, y \in E.$$

Then, $\mathrm{Inv}(E) = \{e\} - \mathrm{Qinv}(E)$. From that, $\mathrm{Inv}(E)$ is open if and only if $\mathrm{Qinv}(E)$ is open.

In this section, we study some conditions that are equivalent to the fact that an algebra is a Q-algebra for unital locally pseudoconvex algebras. Notice, that even if we know that some results are equivalent for any topological algebra, it is interesting to study similar properties in some particular cases that help us to understand better the structure of such algebras.

Theorem 3.3. *Let*[3] *$(E, \{p_\alpha : \alpha \in \Lambda\})$ be a (complex) locally pseudoconvex algebra with unit e. Consider the following conditions:*

1) *E is a Q-algebra;*
2) *$\exists\ \alpha \in \Lambda$ and ϵ with $0 < \epsilon < 1$ such that $p_\alpha(e - x) < \epsilon \Rightarrow x \in \mathrm{Inv}(E)$;*
3) *$\exists\ \alpha \in \Lambda$ and ϵ with $0 < \epsilon < 1$ such that $p_\alpha(x) < \epsilon \Rightarrow \sum_{n=0}^{\infty} x^n$ converges in E;*

[1] Here, $g_V = \inf_{\alpha}\{\alpha > 0 : x \in \alpha V\}$.

[2] That is, the intersection of all maximal right (or left) ideals of E.

[3] We assume that our family of pseudoseminorms is saturated. Otherwise, we can saturate the family (see [3, p.191]).

4) $\exists\ \alpha \in \Lambda$ *such that* $\sup\limits_{p_\alpha(x)\leqslant 1} r(x) < \infty$*;*

5) $\exists\ \alpha \in \Lambda$ *and* ϵ *with* $0 < \epsilon \leqslant 1$ *such that* $r(x)^{\rho_\alpha} \leqslant \frac{1}{\epsilon} p_\alpha(x)$ *for all* $x \in E$*.*

Then, 3) *implies* 2) *and* 1), 2), 4) *and* 5) *are equivalent.*

Proof. 3) $\Rightarrow$ 2) Let $\alpha \in \Lambda$ and $0 < \epsilon < 1$ be as in 3) and $x \in E$ be such that $p_\alpha(e-x) < \epsilon$. Then, by 3)

$$\sum_{n=0}^{\infty} (e-x)^n$$

converges in E. Hence, $(e-x)^n \to \theta$, where θ is the zero element of E.

Now, since

$$(e-(e-x))\left(\sum_{k=0}^{n}(e-x)^k\right) = \left(\sum_{k=0}^{n}(e-x)^k\right)(e-(e-x)) = e-(e-x)^{n+1} \to e,$$

then $\left(\sum_{n=0}^{\infty}(e-x)^n\right)x = x\left(\sum_{n=0}^{\infty}(e-x)^n\right) = e$, which means that $x \in \mathrm{Inv}(E)$.

2) $\Rightarrow$ 1) Let $\alpha \in \Lambda$ be as in 2) and $0 < \epsilon < 1$. Then we have that

$$\{x : p_\alpha(e-x) < \epsilon\} \subset \mathrm{Inv}(E).$$

This is, there exists a neighborhood of e, contained in $\mathrm{Inv}(E)$, so E is a Q-algebra (see [7, p. 43, Lemma 6.4]).

1) $\Rightarrow$ 5) We assumed that $\mathscr{P} = \{p_\alpha : \alpha \in \Lambda\}$ is saturated, then

$$\mathscr{B} = \{O_{\alpha,\epsilon} : \alpha \in \Lambda, \epsilon > 0\},$$

where $O_{\alpha,\epsilon} = \{x \in E : p_\alpha(x) < \epsilon\}$ is a base of neighborhoods of zero in E (see [3, p. 191, Lemma 4.3.8]) and every $O_{\alpha,\epsilon} \in \mathscr{B}$ is absorbing[4], ρ_α-convex and balanced[5] by Proposition 4.1.13, Remark 4.1.14 and Lemma 4.1.4 in [3]. Since by 1) E is a Q-algebra, then $\mathrm{Inv}(E)$ is open. Thus, there exists $O = O_{\alpha,\epsilon} \in \mathscr{B}$ such that $e + O \subseteq \mathrm{Inv}(E)$. We can assume that $\epsilon < 1$.

Now, we consider any $x \in E$. Since O is absorbing, then there exists $\lambda_x > 0$ such that $\kappa x \in O$ for every κ with $0 < |\kappa| \leqslant \lambda_x$. We assume without loss of generality that $\lambda_x < 1$. If $\mu \geqslant \frac{1}{\lambda_x^{\rho_\alpha}}$, then $0 < \frac{1}{\mu^{\frac{1}{\rho_\alpha}}} \leqslant \lambda_x$. Thus, $\frac{1}{\mu^{\frac{1}{\rho_\alpha}}} x \in O$.

So, there exists $\mu > 0$ such that $x \in \mu^{\frac{1}{\rho_\alpha}} O$. Consider any $\varphi > 0$ such that $x \in \varphi^{\frac{1}{\rho_\alpha}} O$, then

$$\frac{1}{\varphi^{\frac{1}{\rho_\alpha}}} x \in O$$

and

$$x - \varphi^{\frac{1}{\rho_\alpha}} e = -\varphi^{\frac{1}{\rho_\alpha}}\left(e - \frac{1}{\varphi^{\frac{1}{\rho_\alpha}}} x\right) \in \mathrm{Inv}(E).$$

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[4] A subset S of a linear space X is called absorbing, if for each $x \in X$ there is a real number $\epsilon_x > 0$ such that $\lambda x \in S$ for every λ with $0 < |\lambda| \leqslant \epsilon_x$.

[5] A subset S of a linear space X is called balanced, if $x \in S, |\lambda| \leqslant 1 \Rightarrow \lambda x \in S$.

Hence, $\varphi^{\frac{1}{\rho_\alpha}} \notin \sigma(x)$. Moreover, if we suppose that there exists $\lambda > \varphi^{\frac{1}{\rho_\alpha}}$ such that $\lambda \in \sigma(x)$, then $x - \lambda e \notin \mathrm{Inv}(E)$, but since

$$x \in \varphi^{\frac{1}{\rho_\alpha}} O = \frac{\varphi^{\frac{1}{\rho_\alpha}}}{\lambda} \lambda O \subset \lambda O,$$

then $\frac{x}{\lambda} \in O \subset \mathrm{Inv}(E)$, and hence

$$x - \lambda e = -\lambda \left(e - \frac{1}{\lambda} x \right) \in \mathrm{Inv}(E).$$

It is $\lambda \notin \sigma(x)$. So, $r(x) \leqslant \varphi^{\frac{1}{\rho_\alpha}}$ for any $\varphi > 0$ such that $x \in \varphi^{\frac{1}{\rho_\alpha}} O$. Thus,

$$r(x)^{\rho_\alpha} \leqslant \inf_{\varphi} \{\varphi > 0 : x \in \varphi^{\frac{1}{\rho_\alpha}} O\} = p_O(x). \tag{3.1}$$

By the definition of O, we have that

$$p_O(x) = \inf_{\varphi} \{\varphi > 0 : p_\alpha(x) < \varphi \epsilon\} = \inf_{\varphi} \{\varphi > 0 : p_\alpha \left(\frac{1}{\epsilon^{\frac{1}{\rho_\alpha}}} x \right) < \varphi\}. \tag{3.2}$$

Moreover, we affirm that

$$\inf_{\varphi} \left\{ \varphi > 0 : p_\alpha \left(\frac{1}{\epsilon^{\frac{1}{\rho_\alpha}}} x \right) < \varphi \right\} = p_\alpha \left(\frac{1}{\epsilon^{\frac{1}{\rho_\alpha}}} x \right). \tag{3.3}$$

Indeed, for every $w \in E$, we have that $p_\alpha(w) \leqslant \inf_{\varphi} \{\varphi > 0 : p_\alpha(w) < \varphi\}$, because $\{\varphi > 0 : p_\alpha(w) < \varphi\} = (p_\alpha(w), \infty)$.

If we suppose that $p_\alpha(w) < \inf_{\varphi} \{\varphi > 0 : p_\alpha(w) < \varphi\}$, then we have that

$$\delta := \inf_{\varphi} \{\varphi > 0 : p_\alpha(w) < \varphi\} - p_\alpha(w) > 0$$

implies $p_\alpha(w) + \frac{\delta}{2} > p_\alpha(w)$ and, hence, $p_\alpha(w) + \frac{\delta}{2} \in \{\varphi > 0 : p_\alpha(w) < \varphi\}$. Thus, $\inf_{\varphi} \{\varphi > 0 : p_\alpha(w) < \varphi\} \leqslant p_\alpha(w) + \frac{\delta}{2}$, which is a contradiction, since $\inf_{\varphi} \{\varphi > 0 : p_\alpha(w) < \varphi\} > p_\alpha(w) + \frac{\delta}{2}$. So, $p_\alpha(w) = \inf_{\varphi} \{\varphi > 0 : p_\alpha(w) < \varphi\}$ for every $w \in E$, in particular, for $\frac{1}{\epsilon^{\frac{1}{\rho_\alpha}}} x$. Now, by (3.1), (3.2) and (3.3), we can conclude that

$$r(x)^{\rho_\alpha} \leqslant p_\alpha \left(\frac{1}{\epsilon^{\frac{1}{\rho_\alpha}}} x \right) = \frac{1}{\epsilon} p_\alpha(x).$$

5) $\Rightarrow$ 2) We consider $\alpha \in \Lambda$ and ϵ as in 5). Take any $x \in E$ such that $p_\alpha(e - x) < \epsilon$ and set $y = e - x$. Then $\frac{1}{\epsilon} p_\alpha(y) < 1$. From 5) follows that $r(y)^{\rho_\alpha} \leqslant \frac{1}{\epsilon} p_\alpha(y)$. Thus, we have that $r(y) < 1$ (otherwise, since $0 < \rho_\alpha \leqslant 1$ and $r(y) \geqslant 1$, then $r(y)^{\rho_\alpha} \geqslant 1$). Hence, $1 \notin \sigma(y)$ or, equivalently, $y - e \in \mathrm{Inv}(E)$, which implies that $x \in \mathrm{Inv}(E)$.

5) $\Rightarrow$ 4) We consider α as in 5). Then $r(x)^{\rho_\alpha} \leqslant \frac{1}{\epsilon} p_\alpha(x)$ for $0 < \epsilon \leqslant 1$ and every $x \in E$. Now, for $x \in E$, such that $p_\alpha(x) \leqslant 1$, we have that $r(x)^{\rho_\alpha} \leqslant \frac{1}{\epsilon}$ and hence $r(x) \leqslant \frac{1}{\epsilon^{\rho_\alpha}}$. Thus, $\sup\limits_{p_\alpha(x) \leqslant 1} r(x) \leqslant \frac{1}{\epsilon^{\rho_\alpha}} < \infty$.

4) $\Rightarrow$ 5) We will use a similar idea to the idea that was used in [10]. For the sake of completeness, it is included here.

We consider $\alpha \in \Lambda$ as in 4) such that

$$0 \leqslant M_\alpha = \sup_{p_\alpha(x) \leqslant 1} r(x) < \infty.$$

If $p_\alpha(x) = 0$, then $0 = p_\alpha(mx)$ for every $m > 0$. By [11, Proposition 2.2.1 (a)] $mr(x) = r(mx)$ and since $mr(x) = r(mx) \leqslant M_\alpha$, then $r(x) \leqslant \frac{M_\alpha}{m}$ for every $m > 0$. This means that $r(x) = 0$ and 5) holds. Now, if $p_\alpha(x) \neq 0$, then

$$p_\alpha \left(\frac{x}{p_\alpha(x)^{\frac{1}{\rho_\alpha}}} \right) = 1$$

implies that

$$\frac{1}{p_\alpha(x)} r(x)^{\rho_\alpha} = r \left(\frac{x}{p_\alpha(x)^{\frac{1}{\rho_\alpha}}} \right)^{\rho_\alpha} \leqslant M_\alpha^{\rho_\alpha}.$$

Thus, $r(x)^{\rho_\alpha} \leqslant M_\alpha^{\rho_\alpha} p_\alpha(x)$. If $M_\alpha^{\rho_\alpha} > 1$, we take $\epsilon := \frac{1}{M_\alpha^{\rho_\alpha}}$ and we conclude that $0 < \epsilon < 1$ and $r(x)^{\rho_\alpha} \leqslant \frac{1}{\epsilon} p_\alpha(x)$. Now, if $M_\alpha^{\rho_\alpha} < 1$, then we take $\delta := 1 - M_\alpha^{\rho_\alpha} > 0$ and $\epsilon := \frac{1}{M_\alpha^{\rho_\alpha} + 2\delta}$. Since $M_\alpha^{\rho_\alpha} + 2\delta > 1$, then $0 < \epsilon < 1$ and for such ϵ we have that $r(x)^{\rho_\alpha} \leqslant M_\alpha^{\rho_\alpha} p_\alpha(x) < (M_\alpha^{\rho_\alpha} + 2\delta) p_\alpha(x) = \frac{1}{\epsilon} p_\alpha(x)$. If $M_\alpha^{\rho_\alpha} = 1$, then $\epsilon = 1$. $\square$

For a locally m-pseudoconvex algebra $(E, \{p_\alpha : \alpha \in \Lambda\})$, it is known (see [10, Theorem 3.1]) that E is a Q-algebra if and only if there exists $\alpha \in \Lambda$ such that

$$r(x)^{\rho_\alpha} = \lim_n \ p_\alpha(x^n)^{\frac{1}{n}} = \inf_n \ p_\alpha(x^n)^{\frac{1}{n}}.$$

It is also known that every seminorm in the family $\{p_\alpha : \alpha \in \Lambda\}$ satisfies the condition

$$p_\alpha(xy) \leqslant p_\alpha(x) p_\alpha(y) \ \text{ for every } \ x, y \in E. \tag{3.4}$$

Then, (as an immediate consequence of [3, Lemma 3.3.6]) for each $\alpha \in \Lambda$, there exists $\lim_n \ p_\alpha(x^n)^{\frac{1}{n}}$ and

$$\lim_n \ p_\alpha(x^n)^{\frac{1}{n}} = \inf_n \ p_\alpha(x^n)^{\frac{1}{n}} \tag{3.5}$$

for every $x \in E$. From this result, as a remark, we notice that in the paper [10, Theorem 3.1], the proof of 6) implies 4) could be modificated for a shorter one.

But when we consider locally pseudoconvex algebras (the pseudo-seminorms that give the topology of the algebra do not necessarily satisfy the inequality (3.4)), we can not be sure that such limit exists for such algebras and if it exists it is not necessarily true the equality (3.5), for every $x \in E$. In [2, Example 2.7], is given a locally A-convex algebra, E, where $\lim_n \ p_\alpha(x^n)^{\frac{1}{n}}$ exists and there exists $x \in E$ such that $\lim_n \ p_\alpha(x^n)^{\frac{1}{n}} > \inf_n \ p_\alpha(x^n)^{\frac{1}{n}}$.

Nevertheless, if $(E, \{p_\alpha : \alpha \in \Lambda\})$ is a locally A-pseudoconvex algebra, then

$$(p_\alpha(x^n))^{\frac{1}{n}} \leqslant M_{x,\alpha}^{\frac{(n-1)\rho_\alpha}{n}} p_\alpha(x)^{\frac{1}{n}} \ \text{ for every } \ \alpha \in \Lambda.$$

Hence, there exists $\limsup_{n\to\infty} \ p_\alpha(x^n)^{\frac{1}{n}}$, for every $\alpha \in \Lambda$ and we would like to know, whether E is a Q-algebra if and only if there exists $\alpha \in \Lambda$ such that

$$r(x)^{\rho_\alpha} = \limsup_{n\to\infty} \ p_\alpha(x^n)^{\frac{1}{n}}.$$

The next theorem answers partially such question.

Let E be a complex algebra and $0 < \rho \leqslant 1$. A ρ-seminorm[6] on E is called an *A-ρ-seminorm*, if for every $x \in E$, there exist constants $M(x) > 0$ and $N(x) > 0$ such that $p(xy) \leqslant M(x)p(y)$ and $p(yx) \leqslant N(x)p(y)$ for every $y \in E$. If p is an A-ρ-seminorm on E, we say that (E, p) is an *A-(ρ-seminormed) algebra*.

We notice that an *A-(ρ-seminormed) algebra* is a special case of locally A-pseudoconvex algebra.

Remember that if E is an unital topological algebra, a net $(x_\lambda)_{\lambda\in\Lambda}$ is called *invertibly convergent* if there exists $x \in E$ such that $(xx_\lambda)_{\lambda\in\Lambda}$ and $(x_\lambda x)_{\lambda\in\Lambda}$ converge to e. A unital topological algebra is called *invertibly complete*[7] if every Cauchy invertibly convergent sequence is convergent.

Theorem 3.4. *Let (E, p) be an A-*(ρ-seminormed) algebra with unit. Then 1) implies 2).

1) E is a Q-algebra
2) E is invertibly complete and
$$r(x)^\rho \leqslant \limsup_{n\to\infty} \ p(x^n)^{\frac{1}{n}}$$
for every $x \in E$.

Moreover, if $\rho = 1$, then 1) implies 2′).

2′) E is invertibly complete and
$$r(x) = \limsup_{n\to\infty} \ p(x^n)^{\frac{1}{n}}$$
for every $x \in E$.

Proof. Since E is a Q-algebra by 1), then E is invertibly complete (see [7, p.45, Theorem 6.4]). Now, by Corollary 3.7 in [6], we have that
$$r(x)^\rho \leqslant \limsup_{n\to\infty} \ p(x^n)^{\frac{1}{n}}$$
for every $x \in E$. If $\rho = 1$, then by 1) implies 2) we have that
$$r(x) \leqslant \limsup_{n\to\infty} \ p(x^n)^{\frac{1}{n}} \tag{3.6}$$
Moreover, by Theorem 1 in [9], there exists a submultiplicative seminorm q on E such that $p(x) \leqslant q(x)$ for every $x \in E$. Since E is a Q-algebra by 1), then we have that $r(x) = \lim_{n\to\infty} \ q(x^n)^{\frac{1}{n}}$ by [11, p. 212, Theorem 2.2.5]. So,
$$\limsup_{n\to\infty} \ p(x^n)^{\frac{1}{n}} \leqslant \limsup_{n\to\infty} \ q(x^n)^{\frac{1}{n}} = \lim_{n\to\infty} q(x^n)^{\frac{1}{n}} = r(x).$$
We conclude by (3.6), that $r(x) = \limsup_{n\to\infty} \ p(x^n)^{\frac{1}{n}}$. □

[6] This is a pseudoseminorm with homogeneity index ρ.

[7] In [9, Definition 1.5], it is called advertibly complete.

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