Poincare Journal of Analysis & Applications
Vol. 2019 (1), Special Issue (ICAM, Delhi), 31-51

FRAME BASED RECONSTRUCTION OF SIGNALS IN MULTIRATE IMPLEMENTATION

KHOLE TIMOTHY POUMAI, S. K. KAUSHIK AND POONAM MANTRY†

Date of Receiving	:	01.05.2018
Date of Revision	:	06.04.2019
Date of Acceptance	:	16.04.2019

Abstract. We define multidimensional discrete time Zak transform (MDTZT) and discuss various properties of MDTZT. We show that MDTZT is relevant to discrete sampling of multivariate discrete time signals. Also, we study multivariate discrete time Weyl Heisenberg (MDTWH) systems for oversampling and critical sampling schemes. By utilizing Zak transform, we give the formulations of the existence of MDTWH frames, tight MDTWH frames and dual of MDTWH frames in the context of oversampling. Moreover, we show that MDTZT provides a good method to evaluate the coefficients of the MDTWH system expansion in the case of oversampling. Also, we discuss the sampling results of the MDTWH system for critical sampling. Finally, we give the applications of MDTWH frames to reconstruction of the multivariate Weyl Heisenberg system in the periodic space $\mathrm{L}^2([0,1)^d)$.

1. Introduction

Let $\mathscr{H}$ be a real (or complex) separable Hilbert space with inner product $\langle .,. \rangle$. A family $\{f_n\}_{n\in\mathbb{Z}} \subseteq \mathscr{H}$ is a *frame* (or *Hilbert frame*) for $\mathscr{H}$, if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{n=-\infty}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \text{ for all } f \in \mathscr{H}. \tag{1.1}$$

The scalars A and B are called the *lower* and *upper frame bounds* of the frame, respectively. They are not unique. If $A = B$, then $\{f_n\}$ is called an *A-tight frame* and if $A = B = 1$, then $\{f_n\}$ is called a *Parseval frame.* The inequality in (1.1) is called the

2010 *Mathematics Subject Classification.* 42C15, 42C30, 42C05, 46B15.
Key words and phrases. Zak transform, Weyl Heisenberg frames, multidimensional sampling.
The authors pay their sincere thanks to the referee for his critical remarks and suggestions which have improved the paper significantly.
Communicated by: Askari Hemmat
†Corresponding author.

frame inequality of the frame.
The operator $\mathscr{T} : \ell^2(\mathbb{Z}) \to \mathscr{H}$, defined by

$$\mathscr{T}(\{c_n\}) = \sum_{n\in\mathbb{Z}} c_n f_n, \ \{c_n\} \in \ell^2(\mathbb{Z}),$$

is called the *pre-frame operator* (or *synthesis operator*) and its adjoint operator $\mathscr{T}^* : \mathscr{H} \to \ell^2(\mathbb{Z})$, is called the *analysis operator* and is given by

$$\mathscr{T}^*(f) = \{\langle f, f_n\rangle\}, \text{ for all } f \in \mathscr{H}.$$

Composing $\mathscr{T}$ and $\mathscr{T}^*$, we obtain the *frame operator* $S = \mathscr{T}\mathscr{T}^* : \mathscr{H} \to \mathscr{H}$, given by

$$S(f) = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k, \text{ for all } f \in \mathscr{H}.$$

The frame operator S is a positive, bounded, self-adjoint and invertible operator on $\mathscr{H}$.

A sequence $\{f_n\}_{n\in\mathbb{Z}} \subseteq \mathscr{H}$ is a *Riesz basis* for $\mathscr{H}$ if $\{f_n\}_{n\in\mathbb{Z}}$ is complete in $\mathscr{H}$ and there exist constants $A, B > 0$ such that

$$A \sum_{n=-\infty}^{\infty} |a_n|^2 \le \| \sum_{n=-\infty}^{\infty} a_n f_n\|^2 \le B \sum_{n=-\infty}^{\infty} |a_n|^2, \text{ for all } \{a_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

A family $\{f_n\}_{n\in\mathbb{Z}} \subseteq \mathscr{H}$ is a *Riesz basis* for $\mathscr{H}$ if and only if $\{f_n\}_{n\in\mathbb{Z}}$ is a frame for $\mathscr{H}$ and $\langle f_n, S^{-1} f_m\rangle = \delta_{nm}, n, m \in \mathbb{Z}$. A family $\{f_n\}_{n\in\mathbb{Z}} \subseteq \mathscr{H}$ is a *Bessel sequence* for $\mathscr{H}$ if there exists a constant $B > 0$ such that

$$\sum_{n=-\infty}^{\infty} |\langle f, f_n\rangle|^2 \le B\|f\|^2, \text{ for all } f \in \mathscr{H}.$$

Let $\{f_n\}_{n\in\mathbb{Z}} \subseteq \mathscr{H}$ be a Bessel sequence with synthesis operator $\mathscr{T}$. Then, we have the following (see [5,13])

(a) $\{f_n\}_{n\in\mathbb{Z}} \subseteq \mathscr{H}$ is a frame if and only if $\mathscr{T}$ maps $\ell^2(\mathbb{Z})$ onto $\mathscr{H}$ such that $\mathscr{T}(e_n) = f_n$, for all $n \in \mathbb{Z}$.
(b) $\{f_n\}_{n\in\mathbb{Z}} \subseteq \mathscr{H}$ is a Riesz basis if and only if $\mathscr{T}$ maps $\ell^2(\mathbb{Z})$ bijectively onto $\mathscr{H}$ such that $\mathscr{T}(e_n) = f_n$, for all $n \in \mathbb{Z}$.
(c) $\{f_n\}_{n\in\mathbb{Z}} \subseteq \mathscr{H}$ is a Parseval frame if and only if $\mathscr{T}\mathscr{T}^* = I_{\mathscr{H}}$ such that $\mathscr{T}(e_n) = f_n$, for all $n \in \mathbb{Z}$.
(d) $\{f_n\}_{n\in\mathbb{Z}} \subseteq \mathscr{H}$ is an orthonormal basis if and only if $\mathscr{T}$ is a unitary map from $\ell^2(\mathbb{Z})$ onto $\mathscr{H}$ such that $\mathscr{T}(e_n) = f_n$, for all $n \in \mathbb{Z}$.

For more details related to frames and Riesz bases, one may refer to [5, 9, 13].
The discrete time Fourier transform [10,19] on $\ell^2(\mathbb{Z})$ is a map $\mathscr{F}$ or $\hat{}: \ell^2(\mathbb{Z}) \to \mathrm{L}^2([0,1))$, defined by

$$\mathscr{F}z(\theta) = \sum_{n\in\mathbb{Z}} z(n)e^{-2\pi i n\theta}, \text{ for } z \in \ell^2(\mathbb{Z}) \text{ and } \theta \in [0,1).$$

The inverse discrete time Fourier transform on $\mathrm{L}^2([0,1))$ is a map $\mathscr{F}^{-1} : \mathrm{L}^2([0,1)) \to \ell^2(\mathbb{Z})$, defined by

$$\mathscr{F}^{-1}f(n) = \langle f, e^{-2\pi i n\theta}\rangle = \int_0^1 f(\theta)\overline{e^{-2\pi i n\theta}} d\theta,$$

where $f \in \mathrm{L}^2([0,1))$ and $\theta \in [0,1)$.

Remark 1.1. The map $\mathscr{F}$, defined above, is one-to-one and onto with inverse $\mathscr{F}^{-1}$. So, for $z, w \in \ell^2(\mathbb{Z})$, we have

$$z(n) = \mathscr{F}^{-1}\mathscr{F}z(n) = \int_0^1 \mathscr{F}z(\theta)\overline{e^{-2\pi in\theta}}d\theta.$$

Also, we have $\langle z, w\rangle = \langle \widehat{z}, \widehat{w}\rangle$ and $\|z\|^2 = \|\widehat{z}\|^2$ (see [10, 19]).

Let $L \in \mathbb{N}$. Then a discrete time Zak transform (DTZT) [2, 4, 14] is the map $\mathscr{Z} : \ell^2(\mathbb{Z}) \to \mathscr{L}^2(\mathbb{Z} \times \mathbb{R})$, defined by

$$\mathscr{Z}z(n, \theta) = \sum_{k\in\mathbb{Z}} z(n + kL)e^{-2\pi ik\theta},$$

where $z \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$. $\mathscr{Z}z$ is quasiperiodic in the following sense: for $(n, \theta) \in \mathbb{Z} \times \mathbb{R}$, we have

$$\mathscr{Z}z(n + lL, \theta) = e^{2\pi il\theta}\mathscr{Z}z(n, \theta) \text{ and } \mathscr{Z}z(n, \theta + l) = \mathscr{Z}z(n, \theta),\ l \in \mathbb{Z}.$$

Therefore, $\mathscr{Z}z$ is completely determined by its values on the set

$$\mathbb{L} = \mathbb{Z}_L \times [0, 1) = \{0, 1, 2, 3, ..., L - 1\} \times [0, 1).$$

Let $L^2(\mathbb{L})$ be the Hilbert space of all mappings $\Phi : \mathbb{L} \to \mathbb{C}$ for which

$$\|\Phi\|_2^2 = \sum_{n=0}^{L-1}\int_0^1 |\Phi(n, \theta)|^2 d\theta < \infty$$

and inner product of $\Phi, \Psi \in L^2(\mathbb{L})$ is given by

$$\langle\Phi, \Psi\rangle_2 = \sum_{n=0}^{L-1}\int_0^1 \Phi(n, \theta)\overline{\Psi(n, \theta)}d\theta.$$

Lemma 1.2. [4, 14] *The discrete time Zak transform $\mathscr{Z}$ is a unitary operator from $\ell^2(\mathbb{Z})$ onto $L^2(\mathbb{L})$.*

Let $\mathscr{E}_n \in \mathrm{L}^2([0, 1))$ be defined by $\mathscr{E}_n(\theta) = e^{-2\pi in\theta}$, for $n \in \mathbb{Z}$ and $\theta \in [0, 1)$. We know that $\{\mathscr{E}_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $\mathrm{L}^2([0, 1))$. One can observe that the discrete time Fourier transform $\mathscr{F}$ on $\ell^2(\mathbb{Z})$ is the *synthesis operator* of the orthonormal basis $\{\mathscr{E}_n\}_{n\in\mathbb{Z}}$ such that $\mathscr{F}(e_n) = \mathscr{E}_n$, for all $n \in \mathbb{Z}$. Similarly, let $\mathscr{E}_{m,k} \in L^2(\mathbb{L})$ be defined by $\mathscr{E}_{m,k}(n, \theta) = \frac{1}{\sqrt{L}}e^{2\pi i\frac{m}{L}n}e^{-2\pi ik\theta}$, for $(n, \theta) \in \mathbb{L}$, $m \in \mathbb{Z}_L$, and $k \in \mathbb{Z}$. We know that $\{\mathscr{E}_{m,k}\}_{m\in\mathbb{Z}_L,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{L})$. Also, if we write $x_{m,k} = \frac{1}{\sqrt{L}}E_{\frac{m}{L}}T_{kL}\chi_{\mathbb{Z}_L}$, for $m \in \mathbb{Z}_L$ and $k \in \mathbb{Z}$, then $\{x_{m,k}\}_{m\in\mathbb{Z}_L,k\in\mathbb{Z}}$ is an orthonormal basis of $\ell^2(\mathbb{Z})$. Further, we observe that the discrete time Zak transform $\mathscr{Z}$ on $\ell^2(\mathbb{Z})$ is the *synthesis operator* of the orthonormal basis $\{\mathscr{E}_{m,k}\}$ such that $\mathscr{Z}(x_{m,k}) = \mathscr{E}_{m,k}$, for all $n \in \mathbb{Z}$.

Let $d \in \mathbb{N}$, a function $f \in L^2(\mathbb{R}^d)$ is a d-dimensional signal. The d-dimensional Fourier transform $\mathscr{F}^d$ (see [8]) is a map on $L^2(\mathbb{R}^d)$, defined by

$$\mathscr{F}^d f(\bar{\theta}) = \int_{\mathbb{R}^d} f(\bar{x})e^{2\pi i\bar{x}\cdot\bar{\theta}}d\bar{x},\ \text{for } f \in L^2(\mathbb{R}^d),\ \bar{\theta} \in \mathbb{R}^d.$$

In multidimensional signal processing [8, 18, 20, 21], the multidimensional sampling and underlying continuous space $L^2(\mathbb{R}^d)$ play a bigger role than in one dimensional signal processing. The fact is, some digital images and many digital videos are undersampled in one dimension case. When dealing with the motion in video, at least half-pixel accuracy is needed. Also, variety of sampling can be carried out in multidimensional space that does not exist in one dimensional space. In multidimensional sampling, the sampled version of d-dimensional signal $f \in L^2(\mathbb{R}^d)$ is a sequence of complex numbers $\{z(\bar{n})\}_{\bar{n}\in\mathbb{Z}^d}$, defined by

$$z(\bar{n}) = f(\mathbf{D}\bar{n}),\ \text{for}\ \bar{n} \in \mathbb{Z}^d,$$

where $\mathbf{D}$ is a real $d \times d$ nonsingular matrix and $\{\bar{t} : \bar{t} = \mathbf{D}\bar{n}, \bar{n} \in \mathbb{Z}^d\}$ is the set of all the sample points. In particular, rectangular sampling is defined to be sampling scheme for which $\mathbf{D}$ is a diagonal matrix. In multirate digital signal processing [19, 20], the sampling rate of a signal is changed in order to increase the efficiency specially in image and video signal processing operations. Indeed, it is the case of resizing the images or videos for the purpose of display on a digital image or video monitor. Decimation or downsampling of d-dimensional sequence z is defined as

$$y(\bar{n}) = z(\mathbf{C}\bar{n}),\ \text{for}\ \bar{n} \in \mathbb{Z}^d,$$

where $\mathbf{C}$ is a $d \times d$ nonsingular matrix of integers. Since $\mathbf{C}$ is a $d \times d$ nonsingular matrix of integers, so it generates lattice Lat*(C)* [21]. Now, expansion or upsampling of a d-dimensional sequence x is defined as

$$u(\bar{n}) = \begin{cases} x(\mathbf{C}^{-1}\bar{n}),\ \text{if}\ \bar{n} \in \text{Lat}(\mathbf{C}) \\ 0,\ \text{if}\ \bar{n} \notin \text{Lat}(\mathbf{C}). \end{cases}$$

In this paper, with motivations from multirate digital signal processing, we introduce multivariate discrete time Weyl Heisenberg (MDTWH) system in multidimensional sequence space $\ell^2(\mathbb{Z}^d)$. In section 2, we define multidimensional discrete time Zak transform (MDTZT), which plays a significant role in the expansion of multivariate discrete time Weyl Heisenberg (MDTWH) frames. In fact, MDTZT allows the formulation of theorems on the existence of MDTWH frames. We also discuss various properties of MDTZT. For instance, we show that MDTZT is quasi periodic and a unitary operator. We give various relationships between multidimensional discrete time Fourier transform and MDTZT. We also prove that the product of two MDTZTs can be represented by a multidimensional Fourier series. In section 3, we discuss various discrete sampling results of MDTWH systems on oversampled scheme and critical sampling scheme. We show that the frame operator of MDTWH frame is associated with the sampling operator and interpolation operator. Although, the applications of orthonormal bases are quite appealing in signal processing and other fields as they can be implemented in the fast discrete algorithms. However, they have a localization problem due to Balian-Low Theorem. Moreover, if the signal contains both impulsive and harmonic components, then a single orthonormal basis cannot efficiently represent both of them. In that direction, frames play a better role in representing the signals. Using MDTZT, we give various characterizations of MDTWH systems. We also give a necessary and sufficient condition for the existence of MDTWH frames and establish the existence of the dual MDTWH frames which immediately provides us the condition to study the existence of tight MDTWH frames. Also, we discuss the existence of Riesz MDTWH basis and orthonormal basis as a particular case. Finally, we discuss the applications of MDTWH frames to formulate the construction of multivariate Weyl Heisenberg frames in the periodic space $\mathrm{L}^2([0,1)^d)$.

2. Multidimensional discrete time Zak transform in $\ell^2(\mathbb{Z}^d)$

Zak defined and studied Zak transform (ZT), which is also known as Gel'fand mapping or Weil-Brezin mapping. Zak transform [12, 15, 16, 22] is an important tool in the theory of Weyl Heisenberg (WH) frames and Gabor system expansions. Theoretically, ZT provides the formulation of theorems on the existence of WH frames. Practically, ZT is widely used for the numerical calculations of WH frame bounds, in the theory of coherent states, wavelet transforms and filter bank theory. Later, Janssen [16] discussed ZT which is a signal transform relevant to time-continuous signals sampled at a uniform rate and an arbitrary clock phase. Further, Heil [14] defined discrete time Zak transform (DTZT) and gave analysis of discrete time Weyl Heisenberg frames in $\ell^2(\mathbb{Z})$. But many digital images and digital videos are under sampled in one dimensional space, so there is a need of multidimensional space. For that purpose, we define multidimensional discrete time Zak transform (MDTZT) and we study various properties of MDTZT. Firstly, let us recall the translation operator and modulation operator [8, 18, 20, 21] in multidimensional sequence space $\ell^2(\mathbb{Z}^d)$.

Modulation operator: for $\bar{a} \in \mathbb{R}^d$, define $E_{\bar{a}} :\ \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ by

$$E_{\bar{a}} z(\bar{n}) = e^{2\pi i \bar{a}\cdot\bar{n}} z(\bar{n}),\ \bar{n} \in \mathbb{Z}^d.$$

Translation operator: for $\bar{k} \in \mathbb{Z}^d$, define $T_{\bar{k}} :\ \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ by

$$T_{\bar{k}} z(\bar{n}) = z(\bar{n} - \bar{k}),\ \bar{n} \in \mathbb{Z}^d.$$

We now define, for $d \in \mathbb{N}$ and $N_1, N_2, ..., N_d \in \mathbb{N}$, matrices $\mathscr{N}$ and $\tilde{\mathscr{N}}$ by

$$\mathscr{N} = \begin{pmatrix} N_1 & 0 & ... & 0 \\ 0 & N_2 & ... & 0 \\ . & . & ... & 0 \\ . & . & . & 0 \\ 0 & 0 & 0 & N_d \end{pmatrix},\ \tilde{\mathscr{N}} = \begin{pmatrix} 1/N_1 & 0 & ... & 0 \\ 0 & 1/N_2 & ... & 0 \\ . & . & ... & 0 \\ . & . & . & 0 \\ 0 & 0 & 0 & 1/N_d \end{pmatrix}$$

Denote $\mathbb{Z}^d_{\mathscr{N}} = \{\bar{n} = (n_1, n_2, ..., n_d) | n_j = 0, 1, 2, ..., N_j - 1,\ 1 \leq j \leq d\}$ and $I^d = \underbrace{[0,1) \times [0,1) .... \times [0,1)}_{d\ times}$. Let $\mathbb{L}^d = \mathbb{Z}^d_{\mathscr{N}} \times I^d$ and consider the Hilbert space

$$L^2(\mathbb{L}^d) = \left\{ \phi : \mathbb{L}^d \longrightarrow \mathbb{C} | \sum_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} \int_{I^d} |\phi(\bar{n}, \bar{\theta})|^2 d\bar{\theta} < \infty \right\}$$

For $\phi, \psi \in L^2(\mathbb{L}^d)$, we define inner product as

$$\langle \phi, \psi \rangle = \sum_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} \int_{I^d} \phi(\bar{n}, \bar{\theta}) \psi\overline{(\bar{n}, \bar{\theta})} d\bar{\theta}$$

and norm as

$$\|\phi\|^2 = \sum_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} \int_{I^d} |\phi(\bar{n}, \bar{\theta})|^2 d\bar{\theta}.$$

We now generalize the discrete time Zak transform in the sequence space $\ell^2(\mathbb{Z}^d)$.

Definition 2.1. The multidimensional discrete time Zak transform (MDTZT) is a map $\mathscr{Z}^d : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{L}^d)$, defined by

$$\mathscr{Z}^d z(\bar{n},\ \bar{\theta}) = \sum_{\bar{k}\in\mathbb{Z}^d} z(\bar{n} + \mathscr{N}\bar{k})e^{-2\pi i\bar{k}\cdot\bar{\theta}},$$

where $(\bar{n}, \bar{\theta}) \in \mathbb{L}^d$ and $z \in \ell^2(\mathbb{Z}^d)$.

Remark 2.2. The multidimensional Zak transform is quasi periodic, i.e, for $\bar{l} \in \mathbb{Z}^d$,
(a) $\mathscr{Z}^d z(\bar{n}, \bar{\theta} + \bar{l}) = \mathscr{Z}^d z(\bar{n}, \bar{\theta})$;
(b) $\mathscr{Z}^d z(\bar{n} + \mathscr{N}\bar{l}, \bar{\theta}) = e^{2\pi i\bar{l}\cdot\bar{\theta}}\mathscr{Z}^d z(\bar{n}, \bar{\theta})$.

It can be easily seen that $\mathscr{Z}^d$ is completely determined by its values on the set $\mathbb{L}^d$. We verify that $\mathscr{Z}^d$ is well defined. Let $\bar{k} \in \mathbb{Z}^d$ and

$$F_{\bar{k}}(\bar{n},\ \bar{\theta}) = z(\bar{n} + \mathscr{N}\bar{k})e^{-2\pi i\bar{k}\cdot\bar{\theta}}.$$

Then $F_{\bar{k}} \in L^2(\mathbb{L}^d)$. Since

$$\sum_{\bar{l}\in\mathbb{Z}^d} a(\bar{l}) = \sum_{\bar{k}\in\mathbb{Z}^d}\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}} a(\bar{m} + \mathscr{N}\bar{k})$$

for $a \in \ell^2(\mathbb{Z}^d)$, we compute

$$\begin{aligned}\sum_{\bar{k}\in\mathbb{Z}^d} \|F_{\bar{k}}\|^2_{L^2(\mathbb{L}^d)} &= \sum_{\bar{k}\in\mathbb{Z}^d}\sum_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} \int_{I^d} |z(\bar{n} + \mathscr{N}\bar{k})e^{-2\pi i\bar{k}\cdot\bar{\theta}}|^2 d\bar{\theta} \\ &= \sum_{\bar{k}\in\mathbb{Z}^d}\sum_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} |z(\bar{n} + \mathscr{N}\bar{k})|^2 \\ &= \sum_{\bar{l}\in\mathbb{Z}^d} |z(\bar{l})|^2 = \|z\|^2_{\ell^2(\mathbb{Z}^d)}\end{aligned}$$

and for $\bar{j} \neq \bar{k}$, we have

$$\langle F_{\bar{k}}, F_{\bar{j}}\rangle_{L^2(\mathbb{L}^d)} = \sum_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} z(\bar{n} + \mathscr{N}\bar{k})\overline{z(\bar{n} + \mathscr{N}\bar{j})} \int_{I^d} e^{-2\pi i(\bar{k}-\bar{j})\cdot\bar{\theta}} d\bar{\theta}.$$

Now

$$\begin{aligned}\int_{I^d} e^{-2\pi i(\bar{k}-\bar{j}).\bar{\theta}} d\bar{\theta} \quad &= \quad \left(\int_I e^{-2\pi i(k_1-j_1)\theta_1} d\theta_1\right)\left(\int_I e^{-2\pi i(k_2-j_2)\theta_2} d\theta_2\right) \\ &\qquad ...\left(\int_I e^{-2\pi i(k_d-j_d)\theta_d} d\theta_d\right).\end{aligned}$$

So, for $j_i \neq k_i$, we obtain $\int_I e^{-2\pi i(k_i-j_i)\cdot\theta_i} d\theta_i = 0$. Thus, we have $\langle F_{\bar{k}}, F_{\bar{j}}\rangle_{L^2(\mathbb{L}^d)} = 0$. Since

$$\|\mathscr{Z}^d z\|^2 = \langle \mathscr{Z}^d z, \mathscr{Z}^d z\rangle = \left\langle \sum_{\bar{k}\in\mathbb{Z}^d} F_{\bar{k}}, \sum_{\bar{k}\in\mathbb{Z}^d} F_{\bar{k}} \right\rangle = \sum_{\bar{k}\in\mathbb{Z}^d} \|F_{\bar{k}}\|^2 = \|z\|^2_{\ell^2(\mathbb{Z}^d)},$$

it follows that $\mathscr{Z}^d z(\bar{n},\bar{\theta}) = \sum_{\bar{k}\in\mathbb{Z}^d} z(\bar{n}+\mathscr{N}\bar{k})e^{-2\pi i\bar{k}\cdot\bar{\theta}}$ is well defined.

In the following, we define orthonormal bases in multidimensional spaces $\ell^2(\mathbb{Z}^d)$ and $L^2(\mathbb{L}^d)$. We give a simple proof using Poisson sum formula and approximate identity.

Proposition 2.3. Let $\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}$ and $\bar{k}\in\mathbb{Z}^d$.

(i) If $x_{\bar{m},\bar{k}} = \frac{1}{(det(\mathscr{N}))^{\frac{1}{2}}} E_{\tilde{\mathscr{N}}\bar{m}} T_{\mathscr{N}\bar{k}}\chi_{\mathbb{Z}^d_{\mathscr{N}}}$, then $\{x_{\bar{m},\bar{k}}\}_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}},\bar{k}\in\mathbb{Z}^d}$ is an orthonormal basis of $\ell^2(\mathbb{Z}^d)$.

(ii) If $\mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{k}}(\bar{n},\bar{\theta}) = \frac{1}{(det(\mathscr{N}))^{\frac{1}{2}}} e^{2\pi i\tilde{\mathscr{N}}\bar{m}\cdot\bar{n}}e^{-2\pi i\bar{k}\cdot\bar{\theta}}$, then $\{\mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{k}}\}_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}},\bar{k}\in\mathbb{Z}^d}$ is an orthonormal basis for $L^2(\mathbb{L}^d)$.

Proof. (i) Let $\bar{s}\in\mathbb{Z}^d$ and

$$\mathbb{Z}^d_{\mathscr{N}\bar{s}} = \{(n_1,n_2,...,n_d) : N_js_j \le n_j \le N_js_j+N_j-1,\ \text{for } j=1,2,..,d\}.$$

For any $\bar{k}\in\mathbb{Z}^d$, we have $T_{\mathscr{N}\bar{k}}\chi_{\mathbb{Z}^d_{\mathscr{N}}}(\bar{n}) = \chi_{\mathbb{Z}^d_{\mathscr{N}\bar{k}}}(\bar{n})$, for all $\bar{n}\in\mathbb{Z}^d$. For $\bar{n}\in\mathbb{Z}^d$, we get
$\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle z,x_{\bar{m},\bar{k}}\rangle x_{\bar{m},\bar{k}}(\bar{n}) = \frac{1}{det(\mathscr{N})}\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle z,E_{\tilde{\mathscr{N}}\bar{m}}T_{\mathscr{N}\bar{k}}\chi_{\mathbb{Z}^d_{\mathscr{N}}}\rangle e^{2\pi i\tilde{\mathscr{N}}\bar{m}\cdot\bar{n}}\chi_{\mathbb{Z}^d_{\mathscr{N}\bar{k}}}(\bar{n})$.
By division algorithm, we can write $\bar{n} = \mathscr{N}\bar{k}'+\bar{r}$, where $\bar{r}\in\mathbb{Z}^d_{\mathscr{N}}$. Notice that if $\bar{k}\neq\bar{k}'$, then $\chi_{\mathbb{Z}^d_{\mathscr{N}\bar{k}}}(\bar{n})=0$. So, we have

$$\begin{aligned}\frac{1}{det(\mathscr{N})}&\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle z,E_{\tilde{\mathscr{N}}\bar{m}}T_{\mathscr{N}\bar{k}}\chi_{\mathbb{Z}^d_{\mathscr{N}}}\rangle e^{2\pi i\tilde{\mathscr{N}}\bar{m}\cdot\bar{n}}\chi_{\mathbb{Z}^d_{\mathscr{N}\bar{k}}}(\bar{n})\\ &= \frac{1}{det(\mathscr{N})}\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}}\langle z,E_{\tilde{\mathscr{N}}\bar{m}}T_{\mathscr{N}\bar{k}'}\chi_{\mathbb{Z}^d_{\mathscr{N}}}\rangle e^{2\pi i\tilde{\mathscr{N}}\bar{m}\cdot\bar{r}}.\end{aligned}$$

Also, we compute

$$\langle z,E_{\tilde{\mathscr{N}}\bar{m}}T_{\mathscr{N}\bar{k}'}\chi_{\mathbb{Z}^d_{\mathscr{N}}}\rangle = \sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{N}\bar{k}'}} z(\bar{p})e^{-2\pi i\tilde{\mathscr{N}}\bar{m}\cdot\bar{p}} = \sum_{\bar{j}\in\mathbb{Z}^d_{\mathscr{N}}} z(\bar{j}+\bar{n}-\bar{r})e^{-2\pi i\tilde{\mathscr{N}}\bar{m}\cdot\bar{j}},$$

by taking $\bar{j} = \bar{p}-\mathscr{N}\bar{k}'$. Thus, we obtain

$$\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle z,x_{\bar{m},\bar{k}}\rangle x_{\bar{m},\bar{k}}(\bar{n}) = \frac{1}{det(\mathscr{N})}\sum_{\bar{j}\in\mathbb{Z}^d_{\mathscr{N}}} z(\bar{j}+\bar{n}-\bar{r})\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}} e^{-2\pi i\tilde{\mathscr{N}}\bar{m}\cdot(\bar{j}-\bar{r})} = z(\bar{n}).$$

(ii) Let $f\in L^2(\mathbb{L}^d)$. Then

$$\langle f,\mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{k}}\rangle = \frac{1}{\sqrt{det(\mathscr{N})}}\sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{N}}}\int_{I^d} f(\bar{p},\bar{\phi})e^{-2\pi i\tilde{\mathscr{N}}\bar{m}\cdot\bar{p}}e^{2\pi i\bar{k}\cdot\bar{\phi}}d\bar{\phi}.$$

For $(\bar{n},\bar{\theta})\in\mathbb{L}^d$, we have

$$\begin{aligned}
&\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle f,\mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{k}}\rangle\mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{k}}(\bar{n},\bar{\theta})\\
&\quad=\frac{1}{\sqrt{det(\mathscr{N})}}\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle f,\mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{k}}\rangle e^{2\pi i\tilde{\mathscr{N}}\bar{m}\cdot\bar{n}}e^{-2\pi i\bar{k}\cdot\bar{\theta}}\\
&\quad=\frac{1}{det(\mathscr{N})}\sum_{\bar{k}\in\mathbb{Z}^d}\Big(\sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{N}}}\int_{I^d}f(\bar{p},\bar{\phi})e^{2\pi i\bar{k}.\bar{\phi}}d\bar{\phi}\Big)e^{-2\pi i\bar{k}\cdot\bar{\theta}}\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}}e^{-2\pi i\tilde{\mathscr{N}}\bar{m}\cdot(\bar{p}-\bar{n})}
\end{aligned}$$

$$=\sum_{\bar{k}\in\mathbb{Z}^d}\int_{I^d}f(\bar{n},\bar{\phi})e^{-2\pi i\bar{k}\cdot(\bar{\theta}-\bar{\phi})}d\bar{\phi}.$$

Therefore, using Poisson sum formula [22], we obtain

$$\sum_{\bar{k}\in\mathbb{Z}^d}\int_{I^d}f(\bar{n},\bar{\phi})e^{-2\pi i\bar{k}\cdot(\bar{\theta}-\bar{\phi})}d\bar{\phi}=\sum_{\bar{k}\in\mathbb{Z}^d}\int_{I^d}f(\bar{n},\bar{\phi})\delta(\bar{\theta}-\bar{\phi}-\bar{k})d\bar{\phi}.$$

Substituting $\bar{\phi}+\bar{k}=\bar{\omega}$ and using approximate identity, we get

$$\begin{aligned}
\sum_{\bar{k}\in\mathbb{Z}^d}\int_{I^d}f(\bar{n},\bar{\phi})\delta(\bar{\theta}-\bar{\phi}-\bar{k})d\bar{\phi} &= \sum_{\bar{k}\in\mathbb{Z}^d}\int_{k_1}^{k_1+1}\int_{k_2}^{k_2+1}\dots\int_{k_d}^{k_d+1}f(\bar{n},\bar{\omega}-\bar{k})\delta(\bar{\theta}-\bar{\omega})d\bar{\omega}\\
&= \int_{\mathbb{R}^d}f(\bar{n},\bar{\omega})\delta(\bar{\theta}-\bar{\omega})d\bar{\omega}\\
&= f(\bar{n},\bar{\theta}).
\end{aligned}$$

Also, it can be easily verified that $\langle\mathscr{E}_{\tilde{\mathscr{N}}\bar{n},\bar{k}},\mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{l}}\rangle=\delta_{\bar{n}\bar{m}}\delta_{\bar{k}\bar{l}}$. □

Using the facts of above discussions, we are now ready to show that MDTZT $\mathscr{Z}^d$ is a unitary operator.

Theorem 2.4. *The multidimensional discrete time Zak transform $\mathscr{Z}^d$ is a unitary operator from $\ell^2(\mathbb{Z}^d)$ onto $L^2(\mathbb{L}^d)$.*

Proof. Note that

$$\begin{aligned}
\mathscr{Z}^d(x_{\bar{m},\bar{k}}(\bar{n},\bar{\theta}))&=\sum_{\bar{p}\in\mathbb{Z}^d}x_{\bar{m},\bar{k}}(\bar{n}+\mathscr{N}\bar{p})e^{-2\pi i\bar{p}\cdot\bar{\theta}}\\
&=\sum_{\bar{p}\in\mathbb{Z}^d}\frac{1}{(det(\mathscr{N}))^{\frac{1}{2}}}E_{\tilde{\mathscr{N}}\bar{m}}T_{\mathscr{N}\bar{k}}\chi_{\mathbb{Z}^d_N}(\bar{n}+\mathscr{N}\bar{p})e^{-2\pi i\bar{p}\cdot\bar{\theta}}\\
&=\sum_{\bar{p}\in\mathbb{Z}^d}\frac{1}{(det(\mathscr{N}))^{\frac{1}{2}}}E_{\tilde{\mathscr{N}}\bar{m}}\chi_{\mathbb{Z}^d_N}(\bar{n}+\mathscr{N}\bar{p}-\mathscr{N}\bar{k})e^{-2\pi i\bar{p}\cdot\bar{\theta}}
\end{aligned}$$

Since $\bar{n} \in \mathbb{Z}^d_{\mathscr{N}}$, $\bar{n} + \mathscr{N}\bar{p} - \mathscr{N}\bar{k} \in \mathbb{Z}^d_{\mathscr{N}}$ only for $\bar{p} = \bar{k}$. Thus, we obtain

$$\begin{aligned}\mathscr{Z}^d(x_{\bar{m}\bar{k}}(\bar{n}, \bar{\theta})) &= \frac{1}{(det(\mathscr{N}))^{\frac{1}{2}}} E_{\tilde{\mathscr{N}}\bar{m}} \chi_{\mathbb{Z}^d_{\mathscr{N}}}(\bar{n}) e^{-2\pi i \bar{k}\cdot\bar{\theta}} \\ &= \frac{1}{(det(\mathscr{N}))^{\frac{1}{2}}} e^{2\pi i \tilde{\mathscr{N}}\bar{m}\cdot\bar{n}} e^{-2\pi i \bar{k}\cdot\bar{\theta}} \\ &= \mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{k}}(\bar{n}, \bar{\theta}).\end{aligned}$$

Therefore, $\mathscr{Z}^d$ maps the orthonormal basis $\{x_{\bar{m},\bar{k}}\}_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}, \bar{k}\in\mathbb{Z}^d}$ of $\ell^2(\mathbb{Z}^d)$ to the orthonormal basis $\{\mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{k}}\}_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}, \bar{k}\in\mathbb{Z}^d}$ of $L^2(\mathbb{L}^d)$. Hence, $\mathscr{Z}^d$ is unitary. □

Next, we give relationships between the multidimensional discrete time Fourier transform and MDTZT.

Proposition 2.5. Let $x, y \in \ell^2(\mathbb{Z}^d)$. Then

(1) $\mathscr{Z}^d x(\bar{n}, \bar{\theta}) = \frac{1}{det(\mathscr{N})} e^{2\pi i \bar{\theta}\cdot\mathscr{N}\bar{n}} \sum\limits_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}} \mathscr{F}^d x(\tilde{\mathscr{N}}(\bar{\theta} + \bar{m})) e^{2\pi i \bar{n}\cdot\tilde{\mathscr{N}}\bar{m}}$;

(2) $\sum\limits_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} \mathscr{Z}^d x(\bar{n}, \mathscr{N}\bar{\theta}) e^{-2\pi i \bar{\theta}\cdot\bar{n}} = \mathscr{F}^d x(\bar{\theta})$;

(3) $\sum\limits_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} \mathscr{Z}^d x(\bar{n}, \bar{\theta}) \overline{\mathscr{Z}^d y(\bar{n}, \bar{\theta})} = \frac{1}{det(\mathscr{N})} \sum\limits_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}} \mathscr{F}^d x(\tilde{\mathscr{N}}(\bar{\theta} + \bar{m})) \overline{\mathscr{F}^d y(\tilde{\mathscr{N}}(\bar{\theta} + \bar{m}))}$.

From Remark 2.2, it is observed that the product of two MDT Zak transforms is $\mathscr{N}\bar{l}$-periodic with respect to component $\bar{n}$ and $\bar{l}$-periodic with frequency component $\bar{\theta}$, for $\bar{l} \in \mathbb{Z}^d$. This yields that the product of two MDT Zak transforms can be represented by a multidimensional Fourier series expansion.

Theorem 2.6. *Let $x, y \in \ell^2(\mathbb{Z}^d)$, and $(\bar{n}, \bar{\theta}) \in \mathbb{L}$. Then*

$$\mathscr{Z}^d x(\bar{n}, \bar{\theta}) \overline{\mathscr{Z}^d y(\bar{n}, \bar{\theta})} = \frac{1}{det(\mathscr{N})} \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}} \sum_{\bar{k}\in\mathbb{Z}^d} \langle x, E_{\tilde{\mathscr{N}}\bar{m}} T_{\mathscr{N}\bar{k}} y\rangle e^{2\pi i \tilde{\mathscr{N}}\bar{m}\cdot\bar{n}} e^{-2\pi i \bar{k}\cdot\bar{\theta}}.$$

Proof. We know that $\{\mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{k}}\}$ is an orthonormal basis for $L^2(\mathbb{L}^d)$ and $\mathscr{Z}^d x \cdot \overline{\mathscr{Z}^d y} \in L^2(\mathbb{L}^d)$. So,we have

$$\mathscr{Z}^d x \cdot \overline{\mathscr{Z}^d y} = \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}} \sum_{\bar{k}\in\mathbb{Z}^d} \langle \mathscr{Z}^d x \cdot \overline{\mathscr{Z}^d y}, \mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{k}}\rangle \mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{k}}.$$

Note that

$$\begin{aligned}\langle \mathscr{Z}^d x \cdot \overline{\mathscr{Z}^d y}, \mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{k}}\rangle &= \sum_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} \int_{I^d} \mathscr{Z}^d x(\bar{n},\bar{\theta})\overline{\mathscr{Z}^d y(\bar{n},\bar{\theta})}\ \overline{\mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{k}}((\bar{n},\bar{\theta}))}d\bar{\theta} \\ &= \frac{1}{(det(\mathscr{N}))^{1/2}} \sum_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} \sum_{\bar{l}\in\mathbb{Z}^d} \sum_{\bar{j}\in\mathbb{Z}^d} x(\bar{n}+\mathscr{N}\bar{l})\overline{y(\bar{n}+\mathscr{N}\bar{j})}e^{-2\pi i\tilde{\mathscr{N}}\bar{m}\cdot\bar{n}} \\ &\quad\times \int_{I^d} e^{-2\pi i\bar{l}\cdot\bar{\theta}}\ \overline{e^{-2\pi i(\bar{j}+\bar{k})\cdot\bar{\theta}}}d\bar{\theta} \\ &= \frac{1}{(det(\mathscr{N}))^{1/2}} \sum_{\bar{j}\in\mathbb{Z}^d} \sum_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} x(\bar{n}+\mathscr{N}(\bar{j}+\bar{k}))\overline{y(\bar{n}+\mathscr{N}\bar{j})}e^{-2\pi i\tilde{\mathscr{N}}\bar{m}\cdot\bar{n}}.\end{aligned}$$

Take $\bar{s} = \bar{n} + (\bar{j}+\bar{k})\mathscr{N}$ and let

$$\mathbb{Z}^d_{\bar{j}\bar{k}\mathscr{N}} = \{(s_1, s_2, ..., s_d) : j_q + k_i \le s_q \le j_q + q_i + N_q - 1 \text{ and } 1 \le q \le d\}.$$

Then we obtain

$$\begin{aligned}\langle \mathscr{Z}^d x \cdot \overline{\mathscr{Z}^d y}, \mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{k}}\rangle &= \frac{1}{(det(\mathscr{N}))^{1/2}} \sum_{\bar{j}\in\mathbb{Z}^d} \sum_{\bar{s}\in\mathbb{Z}^d_{\bar{j}\bar{k}\mathscr{N}}} x(\bar{s})\overline{y(\bar{s}-\mathscr{N}\bar{k})}e^{-2\pi i\tilde{\mathscr{N}}\bar{m}\cdot\bar{s}} \\ &= \frac{1}{(det(\mathscr{N}))^{1/2}} \sum_{\bar{l}\in\mathbb{Z}^d} x(\bar{l})\overline{E_{\tilde{\mathscr{N}}\bar{m}}T_{\mathscr{N}\bar{k}}y(\bar{l})} \\ &= \frac{1}{(det(\mathscr{N}))^{1/2}}\langle x, E_{\tilde{\mathscr{N}}\bar{m}}T_{\mathscr{N}\bar{k}}y\rangle.\end{aligned}$$

□

Let $x, y \in \ell^2(\mathbb{Z}^d)$ and $\bar{m} \in \mathbb{Z}^d$. *Convolution* of x and y is given by

$$x * y(\bar{m}) = \sum_{\bar{k}\in\mathbb{Z}^d} x(\bar{m}-\bar{k})y(\bar{k}) = \sum_{\bar{k}\in\mathbb{Z}^d} x(\bar{k})y(\bar{m}-\bar{k}).$$

The reflection of $z \in \ell^2(\mathbb{Z}^d)$ is $\tilde{z}(\bar{n}) = \overline{z(-\bar{n})}$ for $\bar{n} \in \mathbb{Z}^d$. Convolution plays an important role in the sampling theory. The convolution with the impluse response is also termed as filtering. The following properties related to MDTZT of convolution, reflection and modulation can be easily verified.

Remark 2.7. Let $x, y \in \ell^2(\mathbb{Z}^d)$ and $\bar{n} \in \mathbb{Z}^d_{\mathscr{N}}, \bar{\theta} \in I^d$, $\bar{\theta}_0 \in \mathbb{R}^d$. Then

(a) $\mathscr{Z}^d(x*y)(\bar{n},\bar{\theta}) = \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}} \mathscr{Z}^d x(\bar{n}-\bar{m},\bar{\theta})\mathscr{Z}^d y(\bar{m},\bar{\theta}) = \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}} \mathscr{Z}^d x(\bar{m},\bar{\theta})\mathscr{Z}^d y(\bar{n}-\bar{m},\bar{\theta})$;

(b) $\mathscr{Z}^d\tilde{x}(\bar{n},\bar{\theta}) = \overline{\mathscr{Z}^d x(-\bar{n},\bar{\theta})}$;

(c) $\mathscr{Z}^d(E_{\bar{\theta}_0}x)(\bar{n},\bar{\theta}) = e^{2\pi i\bar{\theta}_0\cdot\bar{n}}\mathscr{Z}^d x(\bar{n},\bar{\theta}-\mathscr{N}\bar{\theta}_0)$.

3. Multivariate Weyl Heisenberg frames in $\ell^2(\mathbb{Z}^d)$

Gabor system or Weyl Heisenberg system is a sequence in $L^2(\mathbb{R})$ of the form

$$\mathscr{G}(g, a, b) = \{e^{2\pi ibmx}g(x-ak)\}_{m,k\in\mathbb{Z}},$$

where $g \in \mathrm{L}^2(\mathbb{R})$ and $a, b > 0$ are fixed. We call g as the generator or the atom. If $\mathscr{G}(g, a, b)$ is a frame for $L^2(\mathbb{R})$, then we call $\mathscr{G}(g, a, b)$ as a Gabor frame. If $\mathscr{G}(g, a, b)$ is a Riesz Basis for $L^2(\mathbb{R})$, then we call $\mathscr{G}(g, a, b)$ as a Gabor Riesz basis. Gabor systems are named after Dennis Gabor [11] who defined Gabor system $\mathscr{G}(\phi,1,1)$, generated by the Gaussian function $\phi(x)=e^{-\pi x^2}$. Gabor conjectured incorrectly that every element $f\in L^2(\mathbb{R})$ can be represented by

$$f=\sum_{m,k\in\mathbb{Z}} c_{mk}(f)E_mT_k\phi, \tag{3.1}$$

for some scalars $c_{mk}(f)$. Gabor expected that $\mathscr{G}(\phi,1,1)$ would be a Schauder basis or a Riesz basis for $L^2(\mathbb{R})$. However, $\mathscr{G}(\phi,1,1)$ is neither of them, since it is over complete. Janssen proved the Gabor's conjecture: each $f\in L^2(\mathbb{R})$ has an expansion of the form in equation (3.1) but he also showed that the series converges only in the sense of tempered distributions not in the L^2 norm. From *Balian-Low Theorem* [1,17], we know that a badly behaved atom g can generate Gabor Riesz basis. However, redundant Gabor frames with better generators do exist and they enable us with useful tools for many applications. Further, C. Heil [14] defined discrete time Weyl Heisenberg (DTWH) system $\{E_{\frac{m}{M}}T_{kL}\nu\}$ in $\ell^2(\mathbb{Z})$, where $M,L\in\mathbb{N}$ are fixed and $m=1,2,...,M$, $k\in\mathbb{Z}$, $\nu\in\ell^2(\mathbb{Z})$. If $\{E_{\frac{m}{M}}T_{kL}\nu\}$ is a frame (or Riesz basis) for $\ell^2(\mathbb{Z})$, then we call $\{E_{\frac{m}{M}}T_{kL}\nu\}$ as a DTWH frame (or DTWH Riesz basis) [6, 14]. C. Heil used DT Zak transform to study DTWH systems in $\ell^2(\mathbb{Z})$. However, there is a limitation because the survey was confined to the case where $M=L$. In this case, if DTWH system is a frame, then it is automatically a Riesz basis. Later, Bolcskei and Hlawatsch [4] studied the theory of DTWH frames extensively. In fact, they established the necessary and sufficient condition for the existence of DTWH frames using Zibulski-Zeevi's method [22]. These DTWH systems have an important role in the discrete sampling theory especially in multirate digital signal processing. However, when we deal with digital images and digital videos, higher dimensional spaces are required for sampling in order to avoid the under sampling problems as in one dimensional spaces. In this direction, we attempt to fill the gap between the two theories and subsequently define the notion of *multivariate discrete time Weyl Heisenberg* (MDTWH) systems in multidimensional sequence space $\ell^2(\mathbb{Z}^d)$. However, in this paper we will confine to rectangular sampling scheme. We discuss the existence of MDTWH frames with oversampling scheme in the reconstruction of multidimensional signals. We will also show the explicit relation of MDTWH frame operator with sampling operator and interpolation operator. Further, we study the existence of the dual of MDTWH frames and tight MDTWH frames using MDT Zak transform. At the end, we will discuss the existence of MDTWH Riesz basis and orthonormal basis as a particular case of MDTWH frames with critical sampling scheme. Now, we define MDTWH frames as follows.

Let $d\in\mathbb{N}$, $M_1,M_2,...,M_d\in\mathbb{N}$ and $\mathscr{M}$ be $d\times d$ diagonal matrix given by

$$\mathscr{M}=\begin{pmatrix} M_1 & 0 & ... & 0\\ 0 & M_2 & ... & 0\\ . & . & ... & 0\\ . & . & . & 0\\ 0 & 0 & 0 & M_d\end{pmatrix},\ \tilde{\mathscr{M}}=\begin{pmatrix} 1/M_1 & 0 & ... & 0\\ 0 & 1/M_2 & ... & 0\\ . & . & ... & 0\\ . & . & . & 0\\ 0 & 0 & 0 & 1/M_d\end{pmatrix}.$$

Let $\mathbb{Z}^d_{\mathscr{M}}=\{\bar{n}=(n_1,n_2,.....,n_d)|n_j=0,1,2,....,M_j-1,\ 1\le j\le d\}$.

Definition 3.1. *Multivariate discrete time Weyl Heisenberg frame (MDTWH frame).* Let $\bar{m} \in \mathbb{Z}^d_{\mathscr{M}}$ and $\nu \in \ell^2(\mathbb{Z}^d)$. Define

$$\nu_{\bar{m},\bar{k}} = E_{\tilde{\mathscr{M}}\bar{m}} T_{\mathscr{N}\bar{k}} \nu, \ \bar{k} \in \mathbb{Z}^d.$$

Then $\{\nu_{\bar{m},\bar{k}}\}$ is said to be a MDTWH frame for $\ell^2(\mathbb{Z}^d)$ if there exist constants $A, \ B \ > 0$ such that, for all $z \in \ell^2(\mathbb{Z}^d)$,

$$A\|z\|^2 \leq \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} \sum_{\bar{k}\in\mathbb{Z}^d} |\langle z, \nu_{\bar{m},\bar{k}}\rangle|^2 \leq B\|z\|^2. \tag{3.2}$$

MDTWH system $\{\nu_{\bar{m},\bar{k}}\}$ is MDTWH Parseval frame if $A = B = 1$ and MDTWH system $\{\nu_{\bar{m},\bar{k}}\}$ is MDTWH Bessel sequence if it satisfies the upper inequality in (3.2).

Definition 3.2. Let $w \in \ell^2(\mathbb{Z}^d)$ and $w_{\bar{m},\bar{k}} = E_{\tilde{\mathscr{M}}\bar{m}} T_{\mathscr{N}\bar{k}} w$, for $\bar{m} \in \mathbb{Z}^d_{\mathscr{M}}$ and $\bar{k} \in \mathbb{Z}^d$. Let $\{\nu_{\bar{m},\bar{k}}\}$ and $\{w_{\bar{m},\bar{k}}\}$ be MDTWH Bessel sequences in $\ell^2(\mathbb{Z}^d)$. A pair $(\{\nu_{\bar{m},\bar{k}}\}, \{w_{\bar{m},\bar{k}}\})$ is a dual multivariate discrete time Weyl Heisenberg tight frame (DMDTWHT frame) if

$$z = \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} \sum_{\bar{k}\in\mathbb{Z}^d} \langle z, w_{\bar{m},\bar{k}}\rangle \nu_{\bar{m},\bar{k}}, \text{ for all } z \in \ell^2(\mathbb{Z}^d).$$

Example 3.3. Let $d = 2$, $\mathscr{M} = \mathscr{P}\mathscr{N}$ where $\mathscr{P} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\mathscr{N} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Let $\nu = \frac{1}{2\sqrt{2}} \chi_{\mathbb{Z}^d_{\mathscr{N}}}$. Then, $\{\nu_{\bar{m},\bar{k}}\}_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}, \bar{k}\in\mathbb{Z}^d}$ forms a MDTWH Parseval frame for $\ell^2(\mathbb{Z}^d)$. This can be easily verified by observing that

$$\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} \sum_{\bar{k}\in\mathbb{Z}^d} \langle z, \nu_{\bar{m},\bar{k}}\rangle \nu_{\bar{m},\bar{k}}(\bar{n}) = \frac{1}{8} \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} \langle z, E_{\tilde{\mathscr{M}}\bar{m}} T_{\mathscr{N}\bar{k}} \chi_{\mathbb{Z}^d_{\mathscr{N}}}\rangle e^{2\pi i \tilde{\mathscr{M}}\bar{m}(\mathscr{N}\bar{k}+\bar{r})},$$

where $\langle z, E_{\tilde{\mathscr{M}}\bar{m}} T_{\mathscr{N}\bar{k}} \chi_{\mathbb{Z}^d_{\mathscr{N}}}\rangle = \sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{N}\bar{k}}} z(\bar{p}) e^{-2\pi i \tilde{\mathscr{M}}\bar{m}\cdot\bar{p}} = \sum_{\bar{j}\in\mathbb{Z}^d_{\mathscr{N}}} z(\bar{j} + \bar{n} - \bar{r}) e^{-2\pi i \tilde{\mathscr{M}}\bar{m}\cdot(\bar{j}+\mathscr{N}\bar{k})}$, by taking $\bar{j} = \bar{p} - \mathscr{N}\bar{k}$. Thus, we obtain

$$\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} \sum_{\bar{k}\in\mathbb{Z}^d} \langle z, \nu_{\bar{m},\bar{k}}\rangle \nu_{\bar{m},\bar{k}}(\bar{n}) = \frac{1}{8} \sum_{\bar{j}\in\mathbb{Z}^d_{\mathscr{N}}} z(\bar{j} + \bar{n} - \bar{r}) \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} e^{-2\pi i \tilde{\mathscr{M}}\bar{m}\cdot(\bar{j}-\bar{r})} = z(\bar{n}).$$

We define downsampling (decimation) and upsampling (expansion) by positive integers $N_1, N_2, ..., N_d$, which are diagonal entries of diagonal matrix $\mathscr{N}$, defined in Section 2.

Definition 3.4. *Downsampling operator* $D_{\mathscr{N}} : \ell^2(\mathbb{Z}^d) \longrightarrow \ell^2(\mathbb{Z}^d)$ is defined by

$$(D_{\mathscr{N}} z)(\bar{n}) = z(\mathscr{N}\bar{n}) \text{ for all } \bar{n} \in \mathbb{Z}^d.$$

Upsampling Operator $U_{\mathscr{N}} : \ell^2(\mathbb{Z}^d) \longrightarrow \ell^2(\mathbb{Z}^d)$ is defined by

$$(U_{\mathscr{N}} z)(\bar{n}) = \begin{cases} z(\tilde{\mathscr{N}}\bar{n}), \text{ if } n_j \text{ are divisible by } N_j, \text{ for all } j = 1, 2, ..., d; \\ 0, \text{ if any } n_j \text{ is not divisible by } N_j, \text{ for } j = 1, 2, ..., d. \end{cases}$$

In one dimensional space $\ell^2(\mathbb{Z})$, if we downsample and subsequently upsample by factor $L = 2$ and for $\nu \in \ell^2(\mathbb{Z})$, then one can express

$$U_2 D_2 \nu = \frac{1}{2}(\nu + \nu^*),$$

where ν^* is defined as $\nu^*(n) = (-1)^n \nu(n)$, for $n \in \mathbb{Z}$ (see [10]). But this tool becomes tedious when $L > 2$. Moreover, it becomes very complicated in multidimensional spaces. Here, we will give an alternate method, which is very easy to handle. In fact, we will use modulation operator to express the composition of downsampling operator and upsampling operator.

Lemma 3.5. *Let* $z \in \ell^2(\mathbb{Z}^d)$. *Then*

$$\frac{1}{det(\mathscr{N})} \sum_{\bar{k} \in \mathbb{Z}^d_{\mathscr{N}}} E_{-\tilde{\mathscr{N}}\bar{k}} z(\bar{n}) = U_{\mathscr{N}} D_{\mathscr{N}} z(\bar{n}), \bar{n} \in \mathbb{Z}^d.$$

Frame theory is closely associated with sampling theory. Next, we show how downsampling, upsampling and filtering form the frame operator.

Proposition 3.6. Let $\nu \in \ell^2(\mathbb{Z}^d)$ and $\{E_{\tilde{\mathscr{M}}\bar{m}} T_{\bar{k}\mathscr{N}} \nu\}$ be MDTWH Bessel sequence with frame operator S. Then the frame operator S is given by

$$\begin{aligned} Sz &= \sum_{\bar{m} \in \mathbb{Z}^d_{\mathscr{M}}} \sum_{\bar{k} \in \mathbb{Z}^d} \langle z, E_{\tilde{\mathscr{M}}\bar{m}} T_{\mathscr{N}\bar{k}} \nu \rangle E_{\tilde{\mathscr{M}}\bar{m}} T_{\mathscr{N}\bar{k}} \nu \\ &= \sum_{\bar{m} \in \mathbb{Z}^d_{\mathscr{M}}} E_{\tilde{\mathscr{M}}\bar{m}} \nu * \left(E_{\tilde{\mathscr{M}}\bar{m}} \left(U_{\mathscr{N}} D_{\mathscr{N}} \left(E_{-\tilde{\mathscr{M}}\bar{m}} z * \tilde{\nu} \right) \right) \right), \end{aligned}$$

for all $z \in \ell^2(\mathbb{Z}^d)$.

Proof. Note that, for $\bar{k} \in \mathbb{Z}^d$, we have

$$\begin{aligned} & E_{-\tilde{\mathscr{M}}\bar{m}} z * \tilde{\nu}(\mathscr{N}\bar{k}) \\ & \quad = \sum_{\bar{j} \in \mathbb{Z}^d} E_{-\tilde{\mathscr{M}}\bar{m}} z(\bar{j}) \tilde{\nu}(\mathscr{N}\bar{k} - \bar{j}) = \sum_{\bar{j} \in \mathbb{Z}^d} z(\bar{j}) \overline{E_{\tilde{\mathscr{M}}\bar{m}} T_{\mathscr{N}\bar{k}} \nu(\bar{j})} = \left\langle z, E_{\tilde{\mathscr{M}}\bar{m}} T_{\mathscr{N}\bar{k}} \nu \right\rangle. \end{aligned}$$

Hence, we compute

$$
\begin{aligned}
&\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} E_{\tilde{\mathscr{M}}\bar{m}}\nu * \left(E_{\tilde{\mathscr{M}}\bar{m}}\left(U_{\mathscr{N}}D_{\mathscr{N}}\left(E_{-\tilde{\mathscr{M}}\bar{m}}z*\tilde{\nu}\right)\right)\right)(\bar{l})\\
&\quad= \frac{1}{det(\mathscr{N})}\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} E_{\tilde{\mathscr{M}}\bar{m}}\nu * E_{\tilde{\mathscr{M}}\bar{m}}\left(\sum_{\bar{j}\in\mathbb{Z}^d_{\mathscr{N}}} E_{-\tilde{\mathscr{N}}\bar{j}}\left(E_{-\tilde{\mathscr{M}}\bar{m}}z*\tilde{\nu}\right)\right)(\bar{l})\\
&\quad= \frac{1}{det(\mathscr{N})}\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}}\sum_{\bar{n}\in\mathbb{Z}^d} E_{\tilde{\mathscr{M}}\bar{m}}\nu(\bar{l}-\bar{n})E_{\tilde{\mathscr{M}}\bar{m}}\left(\sum_{\bar{j}\in\mathbb{Z}^d_{\mathscr{N}}} E_{-\tilde{\mathscr{N}}\bar{j}}\left(E_{-\tilde{\mathscr{M}}\bar{m}}z*\tilde{\nu}\right)\right)(\bar{n})\\
&\quad= \frac{1}{det(\mathscr{N})}\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}}\sum_{\bar{n}\in\mathbb{Z}^d} e^{2\pi i\tilde{\mathscr{M}}\bar{m}\cdot\bar{l}}\nu(\bar{l}-\bar{n})\sum_{\bar{j}\in\mathbb{Z}^d_{\mathscr{N}}} e^{-2\pi i\tilde{\mathscr{N}}\bar{j}\cdot\bar{n}}\left(E_{-\tilde{\mathscr{M}}\bar{m}}z*\tilde{\nu}\right)(\bar{n})\\
&\quad= \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}}\sum_{\bar{k}\in\mathbb{Z}^d}\left(E_{-\tilde{\mathscr{M}}\bar{m}}z*\tilde{\nu}\right)(\mathscr{N}\bar{k})e^{2\pi i\tilde{\mathscr{M}}\bar{m}\cdot\bar{l}}\nu(\bar{l}-\mathscr{N}\bar{k})\\
&\quad= \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle z, E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\nu\rangle E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\nu(\bar{l}). \qquad \square
\end{aligned}
$$

Let $x \in \ell^2(\mathbb{Z}^d)$. The *convolution operator* is a map $G_x : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$, given by $G_x(z) = z * x$, for $z \in \ell^2(\mathbb{Z}^d)$. Here x is an impulse response, which is often called as *filter*. The convolution with impulse response is called *filtering*. From Proposition 3.6, we observe that

$$\langle z, E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\nu\rangle = D_{\mathscr{N}}E_{-\tilde{\mathscr{M}}\bar{m}}z*\tilde{\nu}(\bar{k}) = D_{\mathscr{N}}G_{\tilde{\nu}}E_{-\tilde{\mathscr{M}}\bar{m}}z(\bar{k}).$$

$D_{\mathscr{N}}G_{\tilde{\nu}}E_{-\tilde{\mathscr{M}}\bar{m}}$ is the sampling operator. Similarly, the interpolation operator is given by $G_{E_{\tilde{\mathscr{M}}\bar{m}}\nu}E_{\tilde{\mathscr{M}}\bar{m}}U_{\mathscr{N}}$.

Next, we give a necessary and sufficient condition for the existence of MDTWH frames using MDT Zak transform. Let $\mathscr{P}$ be a $d \times d$ diagonal matrix whose diagonal entries are $P_1, P_2, ..., P_d \in \mathbb{N}$.

Theorem 3.7. *Let $\nu \in \ell^2(\mathbb{Z}^d)$ and $\mathscr{M} = \mathscr{P}\mathscr{N}$. Then $\{E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\nu\}$ is a multivariate DTWH frame with bounds A and B if and only if*

$$A \leq det(\mathscr{N})\sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}} |\mathscr{Z}^d\nu(\bar{n},\bar{\theta}-\tilde{\mathscr{P}}\bar{p})|^2 \leq B \text{ a.e. } (\bar{n},\bar{\theta}) \in \mathbb{L}^d.$$

Proof. Using Proposition 3.6, we have

$$Sz = \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} E_{\tilde{\mathscr{M}}\bar{m}}\nu * \left(E_{\tilde{\mathscr{M}}\bar{m}}\left(U_{\mathscr{N}}D_{\mathscr{N}}\left(E_{-\tilde{\mathscr{M}}\bar{m}}z*\tilde{\nu}\right)\right)\right).$$

Now,

$$\mathscr{Z}^d\left(\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} E_{\tilde{\mathscr{M}}\bar{m}}\nu * \left(E_{\tilde{\mathscr{M}}\bar{m}}\left(U_{\mathscr{N}}D_{\mathscr{N}}\left(E_{-\tilde{\mathscr{M}}\bar{m}}z*\tilde{\nu}\right)\right)\right)\right)(\bar{n},\bar{\theta})$$

$$= \frac{1}{det(\mathscr{N})}\mathscr{Z}^d\left(\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} E_{\tilde{\mathscr{M}}\bar{m}}\nu * E_{\tilde{\mathscr{M}}\bar{m}}\left(\sum_{\bar{j}\in\mathbb{Z}^d_{\mathscr{N}}} E_{-\tilde{\mathscr{N}}\bar{j}}\left(E_{-\tilde{\mathscr{M}}\bar{m}}z*\tilde{\nu}\right)\right)\right)(\bar{n},\bar{\theta})$$

$$= \frac{1}{det(\mathscr{N})}\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}}\sum_{\bar{s}\in\mathbb{Z}^d_{\mathscr{N}}} \mathscr{Z}^d E_{\tilde{\mathscr{M}}\bar{m}}\nu(\bar{n}-\bar{s},\bar{\theta})\mathscr{Z}^d E_{\tilde{\mathscr{M}}\bar{m}}\left(\sum_{\bar{j}\in\mathbb{Z}^d_{\mathscr{N}}} E_{-\tilde{\mathscr{N}}\bar{j}}\left(E_{-\tilde{\mathscr{M}}\bar{m}}z*\tilde{\nu}\right)\right)(\bar{s},\bar{\theta})$$

$$= \frac{1}{det(\mathscr{N})}\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}}\sum_{\bar{s}\in\mathbb{Z}^d_{\mathscr{N}}} e^{2\pi i\tilde{\mathscr{M}}\bar{m}\cdot(\bar{n}-\bar{s})}\mathscr{Z}^d\nu(\bar{n}-\bar{s},\bar{\theta}-\mathscr{N}\tilde{M}\bar{m})e^{2\pi i\tilde{\mathscr{M}}\bar{m}\cdot\bar{s}}$$

$$\times\mathscr{Z}^d\left(\sum_{\bar{j}\in\mathbb{Z}^d_{\mathscr{N}}} E_{-\tilde{\mathscr{N}}\bar{j}}\left(E_{-\tilde{\mathscr{M}}\bar{m}}z*\tilde{\nu}\right)\right)(\bar{s},\bar{\theta}-\mathscr{N}\tilde{\mathscr{M}}\bar{m})$$

$$= \frac{1}{det(\mathscr{N})}\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}}\sum_{\bar{s}\in\mathbb{Z}^d_{\mathscr{N}}} e^{2\pi i\tilde{\mathscr{M}}\bar{m}\cdot\bar{n}}\mathscr{Z}^d\nu\left(\bar{n}-\bar{s},\bar{\theta}-\mathscr{N}\tilde{\mathscr{M}}\bar{m}\right)$$

$$\times\sum_{\bar{j}\in\mathbb{Z}^d_{\mathscr{N}}} e^{-2\pi i\tilde{\mathscr{N}}\bar{j}\cdot\bar{s}}\mathscr{Z}^d\left(E_{-\tilde{\mathscr{M}}\bar{m}}z*\tilde{\nu}\right)(\bar{s},\bar{\theta}-\mathscr{N}\tilde{\mathscr{M}}\bar{m}+\bar{j}).$$

Also, we have

$$\mathscr{Z}^d\left(E_{-\tilde{\mathscr{M}}\bar{m}}z*\tilde{\nu}\right)(\bar{s},\bar{\theta}-\mathscr{N}\tilde{\mathscr{M}}\bar{m})$$

$$= \sum_{\bar{r}\in\mathbb{Z}^d_{\mathscr{N}}} e^{-2\pi i\tilde{\mathscr{M}}\bar{m}\cdot\bar{r}}\mathscr{Z}^d z(\bar{r},\bar{\theta}-\mathscr{N}\tilde{\mathscr{M}}\bar{m}+\mathscr{N}\tilde{\mathscr{M}}\bar{m})\overline{\mathscr{Z}^d\nu\left(\bar{r}-\bar{s},\bar{\theta}-\mathscr{N}\tilde{\mathscr{M}}\bar{m}\right)}.$$

Therefore, we obtain

$$\mathscr{Z}^d Sz(\bar{n},\bar{\theta})$$

$$= \frac{1}{det(\mathscr{N})}\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}}\sum_{\bar{s}\in\mathbb{Z}^d_{\mathscr{N}}} e^{2\pi i\tilde{\mathscr{M}}\bar{m}\cdot\bar{n}}\mathscr{Z}^d\nu\left(\bar{n}-\bar{s},\bar{\theta}-\mathscr{N}\tilde{\mathscr{M}}\bar{m}\right)\sum_{\bar{j}\in\mathbb{Z}^d_{\mathscr{N}}} e^{-2\pi i\tilde{\mathscr{N}}\bar{j}\cdot\bar{s}}$$

$$\times\sum_{\bar{r}\in\mathbb{Z}^d_{\mathscr{N}}} e^{-2\pi i\tilde{\mathscr{M}}\bar{m}\cdot\bar{r}}\mathscr{Z}^d z(\bar{r},\bar{\theta})\overline{\mathscr{Z}^d\nu\left(\bar{r}-\bar{s},\bar{\theta}-\mathscr{N}\tilde{\mathscr{M}}\bar{m}\right)}$$

$$= \frac{1}{det(\mathscr{N})}\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}}\sum_{\bar{r}\in\mathbb{Z}^d_{\mathscr{N}}}\sum_{\bar{s}\in\mathbb{Z}^d_{\mathscr{N}}}\mathscr{Z}^d\nu\left(\bar{n}-\bar{s},\bar{\theta}-\mathscr{N}\tilde{\mathscr{M}}\bar{m}\right)\mathscr{Z}^d z(\bar{r},\bar{\theta})$$

$$\times\overline{\mathscr{Z}^d\nu\left(\bar{r}-\bar{s},\bar{\theta}-\mathscr{N}\tilde{\mathscr{M}}\bar{m}\right)}e^{-2\pi i\tilde{\mathscr{M}}\bar{m}\cdot(\bar{r}-\bar{n})}\sum_{\bar{j}\in\mathbb{Z}^d_{\mathscr{N}}} e^{-2\pi i\tilde{\mathscr{N}}\bar{j}\cdot\bar{s}}$$

$$= \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}}\sum_{\bar{r}\in\mathbb{Z}^d_{\mathscr{N}}}\mathscr{Z}^d\nu\left(\bar{n},\bar{\theta}-\mathscr{N}\tilde{\mathscr{M}}\bar{m}\right)\mathscr{Z}^d z(\bar{r},\bar{\theta})\overline{\mathscr{Z}^d\nu\left(\bar{r},\bar{\theta}-\mathscr{N}\tilde{\mathscr{M}}\bar{m}\right)}e^{-2\pi i\tilde{\mathscr{M}}\bar{m}\cdot(\bar{r}-\bar{n})}.$$

Let $\mathbb{Z}^d_{\mathscr{P}} = \{(p_1,p_2,\ldots,p_d) : 0\le p_j\le P_j-1 \text{ and } 1\le j\le d\}$. Take $\bar{m}=\mathscr{P}\bar{m}'+\bar{p}$, where $\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}$ and $\bar{m}'\in\mathbb{Z}^d_{\mathscr{N}}$. Let $\tilde{\mathscr{P}}$ be the inverse of $\mathscr{P}$. Then

$$\mathscr{Z}^d Sz(\bar{n},\bar{\theta})$$

$$= \sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}}\sum_{\bar{m}'\in\mathbb{Z}^d_{\mathscr{N}}}\sum_{\bar{r}\in\mathbb{Z}^d_{\mathscr{N}}}\mathscr{Z}^d\nu\left(\bar{n},\bar{\theta}-\bar{m}'-\tilde{\mathscr{P}}\bar{p}\right)\mathscr{Z}^d z(\bar{r},\bar{\theta})\overline{\mathscr{Z}^d\nu\left(\bar{r},\bar{\theta}-\bar{m}'-\tilde{\mathscr{P}}\bar{p}\right)}$$

$$\times e^{-2\pi i\tilde{\mathscr{P}}\tilde{\mathscr{N}}\bar{p}\cdot(\bar{r}-\bar{n})}e^{-2\pi i\tilde{\mathscr{N}}\bar{m}'\cdot(\bar{r}-\bar{n})}$$

$$= \sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}}[\sum_{\bar{r}\in\mathbb{Z}^d_{\mathscr{N}}}(\mathscr{Z}^d\nu\left(\bar{n},\bar{\theta}-\tilde{\mathscr{P}}\bar{p}\right)\mathscr{Z}^d z(\bar{r},\bar{\theta})\overline{\mathscr{Z}^d\nu\left(\bar{r},\bar{\theta}-\tilde{\mathscr{P}}\bar{p}\right)}$$

$$\times e^{-2\pi i\tilde{P}\tilde{\mathscr{N}}\bar{p}\cdot(\bar{r}-\bar{n})}\sum_{\bar{m}'\in\mathbb{Z}^d_{\mathscr{N}}} e^{-2\pi i\tilde{\mathscr{N}}\bar{m}'\cdot(\bar{r}-\bar{n})})]$$

$$= det(\mathscr{N}) \sum_{\bar{p}\in\mathbb{Z}^d_P} \mathscr{Z}^d\nu\left(\bar{n},\bar{\theta}-\tilde{\mathscr{P}}\bar{p}\right)\mathscr{Z}^d z(\bar{n},\bar{\theta})\overline{\mathscr{Z}^d\nu\left(\bar{n},\bar{\theta}-\tilde{\mathscr{P}}\bar{p}\right)}.$$

Let $\{E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\nu\}$ be a MDTWH frame for $\ell^2(\mathbb{Z}^d)$. Then

$$\langle Sz, z\rangle = \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}}\sum_{\bar{k}\in\mathbb{Z}^d}|\langle z, E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\nu\rangle|^2, \text{ for all } z\in\ell^2(\mathbb{Z}^d).$$

Also,

$$\langle\mathscr{Z}^d Sz, \mathscr{Z}^d z\rangle = \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}}\int_{I^d}\mathscr{Z}^d Sz(\bar{n},\bar{\theta})\overline{\mathscr{Z}^d z(\bar{n},\bar{\theta})}d\bar{\theta}$$

$$= det(\mathscr{N})\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{N}}}\int_{I^d}|\mathscr{Z}^d z(\bar{n},\bar{\theta})|^2\sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}}|\mathscr{Z}^d\nu\left(\bar{n},\bar{\theta}-\tilde{\mathscr{P}}\bar{p}\right)|^2 d\bar{\theta}.$$

Moreover, we have

$$A\|\mathscr{Z}^d z\|^2_{\mathscr{L}^2(\mathbb{L}^d)} \le \langle\mathscr{Z}^d Sz, \mathscr{Z}^d z\rangle \le B\|\mathscr{Z}^d z\|^2_{\mathscr{L}^2(\mathbb{L}^d)}, \text{ for all } z\in\ell^2(\mathbb{Z}^d).$$

Hence, we obtain

$$A \le det(\mathscr{N})\sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}}|\mathscr{Z}^d\nu\left(\bar{n},\bar{\theta}-\tilde{\mathscr{P}}\bar{p}\right)|^2 \le B \text{ a.e. } (\bar{n},\bar{\theta})\in\mathbb{L}^d.$$

Conversely, for $z\in\ell^2(\mathbb{Z}^d)$, we have

$$A\|\mathscr{Z}^d z\|^2 \le det(\mathscr{N})\|\mathscr{Z}^d z\|^2\sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}}|\mathscr{Z}^d\nu\left(\bar{n},\bar{\theta}-\tilde{\mathscr{P}}\bar{p}\right)|^2 \le B\|\mathscr{Z}^d z\|^2, \text{ a.e. } (\bar{n},\bar{\theta})\in\mathbb{L}^d$$

and

$$\begin{aligned} det(\mathscr{N})\|\mathscr{Z}^d z\|^2\sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}}|\mathscr{Z}^d\nu\left(\bar{n},\bar{\theta}-\tilde{\mathscr{P}}\bar{p}\right)|^2 &= \sum_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}}\int_{I^d}\mathscr{Z}^d Sz(\bar{n},\bar{\theta})\overline{\mathscr{Z}^d z(\bar{n},\bar{\theta})}d\bar{\theta} \\ &= \langle\mathscr{Z}^d Sz, \mathscr{Z}^d z\rangle. \end{aligned}$$

Therefore, we get

$$A\|z\|^2 \le \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}}\sum_{\bar{k}\in\mathbb{Z}^d}|\langle z, E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\nu\rangle|^2 \le B\|z\|^2, \text{ for all } z\in\ell^2(\mathbb{Z}^d). \qquad \square$$

Remark 3.8. If $\{E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\nu\}$ be a MDTWH frame with frame operator S. Then S commutes with the modulation and translation operator involved, that is,

$$SE_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}} = E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}S.$$

Similarly, one can observe that S^{-1} commutes with $E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}$. Consequently, we can compute that $\{E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}S^{-1}\nu\}$ is also a MDTWH frame for $\ell^2(\mathbb{Z}^d)$.

Next, we give a characterization of DMDTWHT frames using MDT Zak transform. Notice that the notion of the DMDTWHT frame is different from the biorthogonal basis as DMDTWHT frames may not be linearly independent. In signal processing and other applications, the freedom of redundancy is quite useful as it possesses the capacity to minimize the presence of uncorrelated noise.

Theorem 3.9. *Let ν, $w \in \ell^2(\mathbb{Z}^d)$ and $\mathscr{M} = \mathscr{P}\mathscr{N}$. Then a pair $(\{E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\nu\}, \{E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}w\})$ is a DMDTWHT frame in $\ell^2(\mathbb{Z}^d)$ if and only if*

$$\sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}} \mathscr{Z}^d\nu(\bar{n}, \bar{\theta} - \tilde{\mathscr{P}}\bar{p})\ \overline{\mathscr{Z}^d w(\bar{n}, \bar{\theta} - \tilde{\mathscr{P}}\bar{p})} = (det(\mathscr{N}))^{-1} \text{ a.e. } (\bar{n}, \bar{\theta}) \in \mathbb{L}^d.$$

Proof. $(\{E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\nu\}, \{E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}w\})$ is a DMDTWHT frame in $\ell^2(\mathbb{Z}^d)$. From the proof of Proposition 3.6, we observe that

$$\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} \sum_{\bar{k}\in\mathbb{Z}^d} \langle z, E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}w\rangle E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\nu = \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} E_{\tilde{\mathscr{M}}\bar{m}}\nu * \left(E_{\tilde{\mathscr{M}}\bar{m}}\left(U_{\mathscr{N}}D_{\mathscr{N}}\left(E_{\tilde{\mathscr{M}}\bar{m}}z * \tilde{w}\right)\right)\right),$$

for all $z \in \ell^2(\mathbb{Z}^d)$. Also, from the proof of the Theorem 3.7, we have

$$\begin{aligned}&\mathscr{Z}^d(\textstyle\sum_{\bar{m}\in\mathbb{Z}^d_M} E_{\tilde{\mathscr{M}}\bar{m}}\nu * \left(E_{\tilde{\mathscr{M}}\bar{m}}\left(U_{\mathscr{N}}D_{\mathscr{N}}\left(E_{\tilde{\mathscr{M}}\bar{m}}z * \tilde{w}\right)\right)\right))(\bar{n}, \bar{\theta})\\ &\quad = det(\mathscr{N}) \sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}} \mathscr{Z}^d\nu\left(\bar{n}, \bar{\theta} - \tilde{\mathscr{P}}\bar{p}\right) \mathscr{Z}^d z(\bar{n}, \bar{\theta})\overline{\mathscr{Z}^d w\left(\bar{n}, \bar{\theta} - \tilde{\mathscr{P}}\bar{p}\right)}.\end{aligned}$$

So, we have

$$\mathscr{Z}^d z(\bar{n}, \bar{\theta}) = det(\mathscr{N})\mathscr{Z}^d z(\bar{n}, \bar{\theta}) \sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}} \mathscr{Z}^d\nu\left(\bar{n}, \bar{\theta} - \tilde{\mathscr{P}}\bar{p}\right) \overline{\mathscr{Z}^d w\left(\bar{n}, \bar{\theta} - \tilde{\mathscr{P}}\bar{p}\right)}.$$

Thus, we obtain

$$\sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}} \mathscr{Z}^d\nu\left(\bar{n}, \bar{\theta} - \tilde{\mathscr{P}}\bar{p}\right) \overline{\mathscr{Z}^d w\left(\bar{n}, \bar{\theta} - \tilde{\mathscr{P}}\bar{p}\right)} = (det(\mathscr{N}))^{-1}, \text{ a.e. } (\bar{n}, \bar{\theta}) \in \mathbb{L}^d.$$

Conversely, we have

$$\begin{aligned}&\mathscr{Z}^d\left(\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} \sum_{\bar{k}\in\mathbb{Z}^d} \langle z, E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}w\rangle E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\nu\right)(\bar{n}, \bar{\theta})\\ &\qquad = det(\mathscr{N})\mathscr{Z}^d z(\bar{n}, \bar{\theta}) \sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}} \mathscr{Z}^d\nu\left(\bar{n}, \bar{\theta} - \tilde{\mathscr{P}}\bar{p}\right) \overline{\mathscr{Z}^d w\left(\bar{n}, \bar{\theta} - \tilde{\mathscr{P}}\bar{p}\right)}\\ &\qquad = \mathscr{Z}^d z(\bar{n}, \bar{\theta}), \text{ a.e. } (\bar{n}, \bar{\theta}) \in \mathbb{L}^d.\end{aligned}$$

Hence,

$$z = \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} \sum_{\bar{k}\in\mathbb{Z}^d} \langle z, E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}w\rangle E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\nu, \text{ for all } z \in \ell^2(\mathbb{Z}^d). \qquad \square$$

If we take $\nu = w$ in Theorem 3.9, then we immediately have the condition for the existence of MDTWH Parseval frames.

Corollary 3.10. *Let $\nu \in l^2(\mathbb{Z}^d)$ and $\mathscr{M} = \mathscr{P}\mathscr{N}$. Then $\{E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\nu\}$ is a MDTWH Parseval frame if and only if*

$$\sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}} |\mathscr{Z}^d\nu(\bar{n}, \bar{\theta} - \tilde{\mathscr{P}}\bar{p})|^2 = (det(\mathscr{N}))^{-1} \text{ a.e. } (\bar{n}, \bar{\theta}) \in \mathbb{L}^d.$$

Example 3.11. For $\nu = \frac{1}{2\sqrt{2}}\chi_{\mathbb{Z}^d_{\mathscr{N}}} \in \ell^2(\mathscr{Z}^d)$ and $d = 2$ as in Example 3.3, the condition $\sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}} |\mathscr{Z}^d\nu(\bar{n}, \bar{\theta} - \tilde{\mathscr{P}}\bar{p})|^2 = (det(\mathscr{N}))^{-1}$ a.e $(\bar{n}, \bar{\theta}) \in \mathbb{L}^d$ can be easily verified as $\sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}} |\mathscr{Z}^d\nu(\bar{n}, \bar{\theta} - \tilde{\mathscr{P}}\bar{p})|^2 = |\mathscr{Z}^d\nu(\bar{n}, \bar{\theta})|^2 + |\mathscr{Z}^d\nu(\bar{n}, \bar{\theta} - \tilde{\mathscr{P}} \cdot (0, 1))|^2$
and $\mathscr{Z}^d\nu(\bar{n}, \bar{\theta}) = \frac{1}{2\sqrt{2}}$ for all $(\bar{n}, \bar{\theta}) \in \mathbb{L}^d$.

Remark 3.12. Let $\mathscr{M} = \mathscr{N}$, $\nu \in l^2(\mathbb{Z}^d)$ and $\nu_{\bar{m},\bar{k}} = E_{\tilde{\mathscr{N}}\bar{m}} T_{\mathscr{N}\bar{k}}\nu$. If $\{\nu_{\bar{m}\bar{k}}\}$ is a MDTWH frame and S is its frame operator, then

$$\mathscr{Z}^d Sz(\bar{n}, \bar{\theta}) = \mathscr{Z}^d \left(\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} \sum_{\bar{k}\in\mathbb{Z}^d} \langle z, \nu_{\bar{m},\bar{k}}\rangle \nu_{\bar{m},\bar{k}} \right) (\bar{n}, \bar{\theta}) = det(\mathscr{N})\mathscr{Z}^d z(\bar{n}, \bar{\theta})|\mathscr{Z}^d\nu(\bar{n}, \bar{\theta})|^2.$$

Moreover, we have

$$\mathscr{Z}^d S^{-1}z(\bar{n}, \bar{\theta}) = \mathscr{Z}^d z(\bar{n}, \bar{\theta})(det(\mathscr{N}))^{-1} \left(|\mathscr{Z}^d\nu(\bar{n}, \bar{\theta})|^2\right)^{-1}.$$

To justify the Remark, from the proof of Theorem 3.7, we have

$$\begin{aligned}\mathscr{Z}^d Sz(\bar{n}, \bar{\theta}) &= \sum_{\bar{r}\in\mathbb{Z}^d_{\mathscr{N}}} \mathscr{Z}^d z(\bar{r}, \bar{\theta})\mathscr{Z}^d\nu(\bar{n}, \bar{\theta} - \bar{m})\overline{\mathscr{Z}^d\nu(\bar{n}, \bar{\theta} - \bar{m})} \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} e^{-2\pi i\tilde{\mathscr{N}}\bar{m}\cdot(\bar{r}-\bar{n})} \\ &= det(\mathscr{N})\mathscr{Z}^d z(\bar{n}, \bar{\theta})|\mathscr{Z}^d\nu(\bar{n}, \bar{\theta})|^2\end{aligned}$$

Replacing z by $S^{-1}z$, we get

$$\mathscr{Z}^d S^{-1}z(\bar{n}, \bar{\theta}) = (det(\mathscr{N}))^{-1}\mathscr{Z}^d z(\bar{n}, \bar{\theta}) \left(|\mathscr{Z}^d\nu(\bar{n}, \bar{\theta})|^2\right)^{-1}.$$

Now, we are returning to critical sampling problem. We show that, if we take $\mathscr{M} = \mathscr{N}$, then MDTWH frame is a MDTWH Riesz basis.

Theorem 3.13. *If $\mathscr{M} = \mathscr{N}$, $\nu \in \ell^2(\mathbb{Z}^d)$, then $\{\nu_{\bar{m},\bar{k}}\}$ is a Riesz basis with bounds A and B if and only if*

$$A \le det(\mathscr{N})|\mathscr{Z}^d\nu(\bar{n}, \bar{\theta})|^2 \le B \text{ a.e. } (\bar{n}, \bar{\theta}) \in \mathbb{L}^d.$$

Proof. If $\{\nu_{\bar{m},\bar{k}}\}$ is a Riesz basis with bounds A and B, then it is a frame for H with same bounds. Let S denote the frame operator of $\{\nu_{\bar{m},\bar{k}}\}$. By Remark 3.12, we have

$$\mathscr{Z}^d Sz(\bar{n}, \bar{\theta}) = det(\mathscr{N})\mathscr{Z}^d z(\bar{n}, \bar{\theta})|\mathscr{Z}^d\nu(\bar{n}, \bar{\theta})|^2.$$

Also, we have

$$A\|\mathscr{Z}^d z\|^2 \le \langle\mathscr{Z}^d Sz, \mathscr{Z}^d z\rangle \le B\|\mathscr{Z}^d z\|^2$$

and

$$\langle\mathscr{Z}^d Sz, \mathscr{Z}^d z\rangle = det(\mathscr{N}) \sum_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} \int_{I^d} |\mathscr{Z}^d z(\bar{n}, \bar{\theta})|^2|\mathscr{Z}^d\nu(\bar{n}, \bar{\theta})|^2 d\bar{\theta}.$$

Thus, we obtain

$$A\|\mathscr{Z}^d z\|^2 \le \det(\mathscr{N}) \sum_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} \int_{I^d} |\mathscr{Z}^d z(\bar{n}, \bar{\theta})|^2|\mathscr{Z}^d\nu(\bar{n}, \bar{\theta})|^2 d\bar{\theta} \le B\|\mathscr{Z}^d z\|^2.$$

So, we conclude that

$$A \le det(\mathscr{N})|\mathscr{Z}^d\nu(\bar{n}, \bar{\theta})|^2 \le B \text{ a.e. } (\bar{n}, \bar{\theta}) \in \mathbb{L}^d.$$

Conversely, let $\{\nu_{\bar{m},\bar{k}}\}$ be a frame for $\ell^2(\mathbb{Z}^d)$ with bounds A and B. Then we compute

$$\begin{aligned}
& \langle \nu_{\bar{m},\bar{k}}, S^{-1}\nu_{\bar{m}',\bar{k}'}\rangle \\
= \; & \langle \mathscr{Z}^d \nu_{\bar{m},\bar{k}}, \mathscr{Z}^d S^{-1}\nu_{\bar{m}'\bar{k}'}\rangle \\
= \; & \sum_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} \int_{I^d} \mathscr{Z}^d \nu_{\bar{m},\bar{k}}(\bar{n},\bar{\theta}) \overline{\mathscr{Z}^d S^{-1}\nu_{\bar{m}'\bar{k}'}(\bar{n},\bar{\theta})} d\bar{\theta} \\
= \; & \sum_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} \int_{I^d} \mathscr{E}_{\tilde{\mathscr{N}}\bar{m},\bar{k}} \cdot \mathscr{Z}^d \nu(\bar{n},\bar{\theta}) \; \overline{\mathscr{E}_{\tilde{\mathscr{N}}\bar{m}',\bar{k}'} \cdot \mathscr{Z}^d \nu(\bar{n},\bar{\theta})(det(\mathscr{N}))^{-1}(|\mathscr{Z}^d\nu(\bar{n},\bar{\theta})|^2)^{-1}} \; d\bar{\theta} \\
= \; & \frac{1}{det(\mathscr{N})} \sum_{\bar{n}\in\mathbb{Z}^d_{\mathscr{N}}} \int_{I^d} e^{-2\pi i \tilde{\mathscr{N}}(\bar{m}'-\bar{m})\cdot\bar{n}} \; e^{-2\pi i(\bar{k}-\bar{k}')\cdot\bar{\theta}} \; d\bar{\theta} \\
= \; & \delta_{\bar{m}\bar{m}'} \; \delta_{\bar{k}\bar{k}'}. \qquad \square
\end{aligned}$$

Finally, we state the following corollary:

Corollary 3.14. *If $\mathscr{M} = \mathscr{N}$, then $\{\nu_{\bar{m},\bar{k}}\}$ is an orthonormal basis if and only if*

$$|\mathscr{Z}^d \nu(\bar{n},\bar{\theta})|^2 = (det(\mathscr{N}))^{-1} \text{ a.e. } (\bar{n},\bar{\theta}) \in \mathbb{L}^d.$$

Although frames for $\mathrm{L}^2([0,1))$ are not very common in literature, there are many advantages to consider them. For instance, the space $\mathrm{L}^2([0,1))$ is a good model of analog signals with finite duration because wavelet frames can be discretized easily. For $0 < b \leq 1$, the sequence $\{e^{-2\pi ibk\theta}\}_{k\in\mathbb{Z}}$ is a frame for $\mathrm{L}^2([0,1))$ (see [5, 13]). Given $\phi \in \mathrm{L}^2([0,1))$, the system of weighted exponentials, generated by ϕ [13], is

$$\mathscr{W}(\phi) = \{e^{-2\pi ik\theta}\phi(\theta)\}_{k\in\mathbb{Z}} = \{E_{-k}\phi(\theta)\}_{k\in\mathbb{Z}}.$$

Lemma 3.15. [13] *Given $\phi \in L^2([0,1))$, the following statements hold.*

(a) *$\mathscr{W}(\phi)$ is complete in $L^2([0,1))$ if and only if $\phi(\theta) \neq 0$ a.e. $\theta \in [0,1)$.*

(b) *$\mathscr{W}(\phi)$ is Riesz basis in $L^2([0,1))$ if and only if there exist constants $A, B > 0$ such that $A \leq |\phi(\theta)|^2 \leq B$ a.e. $\theta \in [0,1)$.*

(c) *$\mathscr{W}(\phi)$ is an orthonormal basis for $L^2([0,1))$ if and only if $|\phi(\theta)| = 1$ a.e. $\theta \in [0,1)$.*

Now, an interesting question arises: will the system $\{E_{-kN}\phi\}$ be complete orthonormal set in $\mathrm{L}^2([0,1))$, for $\phi \in \mathrm{L}^2([0,1))$ and $N \geq 2$? By ([10], page no. 318), it is observed that, if $N \geq 2$, then the system $\{E_{-kL}\phi\}$ cannot be even complete. This very fact pave us the way to define multivariate Weyl Heisenberg system (MWH system) in periodic space $\mathrm{L}^2([0,1)^d)$ and discuss the reconstruction properties of MWH systems.

Definition 3.16. Let $\phi, \psi \in \mathrm{L}^2([0,1)^d)$. Let $\phi_{\bar{m},\bar{k}} = T_{\tilde{\mathscr{M}}\bar{m}} E_{-\mathscr{N}\bar{k}}\phi$ and $\psi_{\bar{m},\bar{k}} = T_{\tilde{\mathscr{M}}\bar{m}} E_{-\mathscr{N}\bar{k}}\psi$, for $\bar{m} \in \mathbb{Z}^d_{\mathscr{M}}$ and $k \in \mathbb{Z}^d$.

(a) Let $\{\phi_{\bar{m},\bar{k}}\}$ and $\{\psi_{\bar{m},\bar{k}}\}$ be Bessel sequences in $\mathrm{L}^2([0,1)^d)$. A pair $(\{\phi_{\bar{m},\bar{k}}\}, \{\psi_{\bar{m},\bar{k}}\})$ constitutes a *dual multivariate Weyl Heisenberg tight frame* (DMWHT frame) in $\mathrm{L}^2([0,1)^d)$, if

$$f = \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} \sum_{\bar{k}\in\mathbb{Z}^d} \langle f, \psi_{\bar{m},\bar{k}}\rangle \phi_{\bar{m},\bar{k}}, \text{ for all } f \in \mathrm{L}^2([0,1)^d).$$

(b) $\{\phi_{\bar{m},\bar{k}}\}$ is a *multivariate Weyl Heisenberg frame* (MWH frame) in $\mathrm{L}^2([0,1)^d)$, if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} \sum_{\bar{k}\in\mathbb{Z}^d} |\langle f, \phi_{\bar{m},\bar{k}}\rangle|^2 \leq B\|f\|^2 \text{ for all } f \in \mathrm{L}^2([0,1)^d).$$

(c) $\{\phi_{\bar{m},\bar{k}}\}$ is a *multivariate Weyl Heisenberg Parseval frame* (MWH Parseval frame) in $\mathrm{L}^2([0,1)^d)$, if

$$\sum_{\bar{m}\in\mathbb{Z}^d_{\mathscr{M}}} \sum_{\bar{k}\in\mathbb{Z}^d} |\langle f, \phi_{\bar{m},\bar{k}}\rangle|^2 = \|f\|^2 \text{ for all } f \in \mathrm{L}^2([0,1)^d).$$

Remark 3.17. For $\nu \in \ell^2(\mathbb{Z})$, $\mathscr{F}(E_{\mathscr{M}\bar{m}}T_{\mathscr{N}\bar{k}}\nu)(\theta) = T_{\tilde{\mathscr{M}}\bar{m}}E_{-\mathscr{N}\bar{k}}\mathscr{F}\nu(\theta)$.

We give the applications of MDTWH frames in the existence of MWH frames in the periodic space $\mathrm{L}^2([0,1)^d)$. Let $\mathscr{P}$ be a diagonal matrix of order $d \times d$, whose entries are $P_1, P_2, ..., P_d \in \mathbb{N}$.

Theorem 3.18. *Let $\phi, \psi \in L^2([0,1)^d)$. The following statements hold.*

(a) *Let $\mathscr{M} = \mathscr{P}\mathscr{N}$. Then $\{\phi_{\bar{m},\bar{k}}\}$ is a MWH frame in $L^2([0,1)^d)$ with bounds A and B if and only if*

$$A \leq det(\mathscr{N}) \sum_{\bar{p}\in\mathscr{P}} \left|\phi\left(\tilde{\mathscr{N}}(\bar{\theta} - \tilde{\mathscr{P}}\bar{p})\right)\right|^2 \leq B \text{ a.e. } \bar{\theta} \in [0,1)^d.$$

(b) *Let $\mathscr{M} = \mathscr{P}\mathscr{N}$. Then a pair $(\{\phi_{\bar{m},\bar{k}}\}, \{\psi_{\bar{m},\bar{k}}\})$ constitutes a DMWHT frame in $L^2([0,1)^d)$ if and only if*

$$\sum_{\bar{p}\in\mathscr{P}} \phi\left(\tilde{\mathscr{N}}(\bar{\theta} - \tilde{\mathscr{P}}\bar{p})\right) \overline{\psi\left(\tilde{\mathscr{N}}(\bar{\theta} - \tilde{\mathscr{P}}\bar{p})\right)} = (det(\mathscr{N}))^{-1} \text{ a.e. } \bar{\theta} \in [0,1)^d.$$

(c) *Let $\mathscr{M} = \mathscr{P}\mathscr{N}$. Then $\{\phi_{\bar{m},\bar{k}}\}$ is a MWH Parseval frame in $L^2([0,1)^d)$ if and only if*

$$\sum_{\bar{p}\in\mathscr{P}} \left|\phi\left(\tilde{\mathscr{N}}(\bar{\theta} - \tilde{\mathscr{P}}\bar{p})\right)\right|^2 = (det(\mathscr{N}))^{-1} \text{ a.e. } \bar{\theta} \in [0,1)^d.$$

(d) *Let $\mathscr{M} = \mathscr{N}$. Then $\{\phi_{\bar{m},\bar{k}}\}$ is a MWH Riesz basis in $L^2([0,1)^d)$ with bounds A and B if and only if*

$$A \leq det(\mathscr{N}) \left|\phi\left(\tilde{\mathscr{N}}\bar{\theta}\right)\right|^2 \leq B \text{ a.e. } \bar{\theta} \in [0,1)^d.$$

(e) *Let $\mathscr{M} = \mathscr{N}$. Then $\{\phi_{\bar{m},\bar{k}}\}$ is an orthonormal basis in $L^2([0,1)^d)$ if and only if* $\left|\phi(\tilde{\mathscr{N}}\bar{\theta})\right|^2 = (det(\mathscr{N}))^{-1}$ a.e. $\bar{\theta} \in [0,1)^d$.

Proof. First, we give a relation between MDT Zak transform and MDT Fourier transform. Let $\nu \in \ell^2(\mathbb{Z}^d)$, then we have

$$\begin{aligned} E_{\bar{n}}\mathscr{F}^d\nu(\tilde{\mathscr{N}}\bar{\theta}) &= e^{2\pi i\tilde{\mathscr{N}}\bar{\theta}\cdot\bar{n}} \sum_{\bar{j}\in\mathbb{Z}^d} \nu(\bar{j})e^{-2\pi i\tilde{\mathscr{N}}\bar{\theta}\cdot\bar{j}} = \sum_{\bar{j}\in\mathbb{Z}^d} \nu(\bar{j})e^{-2\pi i(\tilde{\mathscr{N}}\bar{j} - \tilde{\mathscr{N}}\bar{n})\cdot\bar{\theta}} \\ &= \sum_{\bar{k}\in\mathbb{Z}^d} \nu(\bar{n} + \mathscr{N}\bar{k})e^{-2\pi i\bar{k}\cdot\bar{\theta}} = \mathscr{Z}^d\nu(\bar{n}, \bar{\theta}). \end{aligned}$$

By using Remark 3.17 and Theorem 3.7, $\{\phi_{\bar{m},\bar{k}}\}$ is a MWH frame in $\mathrm{L}^2([0,1)^d)$ if and only if $\{E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\mathscr{F}^{d*}\phi\}$ is a MDTWH frame in $\ell^2(\mathbb{Z}^d)$ if and only if

$$A \leq det(\mathscr{N}) \sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}} |\mathscr{Z}^d\mathscr{F}^{d*}\phi(\bar{n},\bar{\theta}-\tilde{\mathscr{P}}\bar{p})|^2 \leq B \text{ a.e. } (\bar{n},\bar{\theta}) \in \mathbb{L}^d$$

if and only if

$$A \leq det(\mathscr{N}) \sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}} \left|\phi\left(\tilde{\mathscr{N}}(\bar{\theta}-\tilde{\mathscr{P}}\bar{p})\right)\right|^2 \leq B \text{ a.e. } \bar{\theta} \in [0,1)^d.$$

Also, using Remark 3.17 and Theorem 3.9, a pair $(\{\phi_{\bar{m},\bar{k}}\},\{\psi_{\bar{m},\bar{k}}\})$ constitutes a DMWHT frame in $\mathrm{L}^2([0,1)^d)$ if and only if a pair $(\{E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\mathscr{F}^{d*}\phi\},\{E_{\tilde{\mathscr{M}}\bar{m}}T_{\mathscr{N}\bar{k}}\mathscr{F}^{d*}\psi\})$ constitutes a DMWHT frame in $\ell^2(\mathbb{Z}^d)$ if and only if

$$\sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}} \mathscr{Z}^d\mathscr{F}^{d*}\phi(\bar{n},\bar{\theta}-\tilde{\mathscr{P}}\bar{p})\overline{\mathscr{Z}^d\mathscr{F}^{d*}\psi(\bar{n},\bar{\theta}-\tilde{\mathscr{P}}\bar{p})} = (det(\mathscr{N}))^{-1} \text{ a.e. } (\bar{n},\bar{\theta}) \in \mathbb{L}^d$$

if and only if

$$\sum_{\bar{p}\in\mathbb{Z}^d_{\mathscr{P}}} \phi\left(\tilde{\mathscr{N}}(\bar{\theta}-\tilde{\mathscr{P}}\bar{p}\right)\overline{\psi\left(\tilde{\mathscr{N}}(\bar{\theta}-\tilde{\mathscr{P}}\bar{p}\right)} = (det(\mathscr{N}))^{-1} \text{ a.e. } \bar{\theta} \in [0,1)^d. \qquad \square$$

References

[1] R. Balian, Un principe d'incertitude fort en théorie du signal ou en mécanique quantique, C. R. Acad. Sci. Paris Sér. II Méc. Phys. Chim. Sci. Univers Sci. Terre 292(1981), no. 20, 1357-1362.

[2] J. Benedetto, H. Heil and D. Walnut, Differentiation and the Balian-Low Theorem, J. Fourier Anal. Appl., 1(1995), no. 4, 355-402.

[3] J. Benedetto, H. Heil and D. Walnut, Uncertainty Principles for time-frequency operators, Oper. Theory Adv. Appl., 58(1992), 1-25.

[4] H. Bölcskei and F. Hlawatsch, Discrete Zak transforms, polyphase transforms, and applications, IEEE Trans. Signal Process., 45(1997), no. 4, 851-866.

[5] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Bostan-Basel-Berlin, 2003.

[6] Z. Cvetkovic, M. Vetterli, Tight Weyl-Heisenberg Frames in $\ell^2(\mathbb{Z})$, IEEE Trans. Signal Process., 46(1998), no. 5, 1256-1259.

[7] I. Daubechies, The wavelet transform, time-frequency localization and signal analysis, IEEE Trans. Inform. Theory, 36(1990), no. 5, 961-1005.

[8] D. E. Dudgeon and R. M. Mersereau, Multidimensional Digital Signal Processing, Prentice Hall P T R, 1990.

[9] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., 72(1952), 341-366.

[10] M. Frazier, An introduction to wavelets through linear algebra, Springer-Verlag, New York, 1999.

[11] D. Gabor, Theory of Communication, J. Inst. Electr. Engin. (London) 93(III), 1946, 429-457.

[12] K. Gröchenig, Foundations of Time-Frequency Analysis, Boston, MA: Birkhäuser, 2001.

[13] C. Heil, A basis theory primer, Expanded edition, Applied and Numerical Harmonic Analysis, Birkhäuser Boston Inc., Boston, MA, 2011.

[14] C. Heil, A discrete Zak transform, Tech. Rep. MTR-89W000128, The MITRE Corp., Bedford, MA, 1989.
[15] C. Heil and D. Walnut, Continuous and discrete wavelet transforms, SIAM Rev., 31(1989), no. 4, 628-666.
[16] A. J. E. M. Janssen, The Zak Transform: A Signal Transform for Sampled Time-Continuous Signals, Philips J. Res., 43(1988), 23-69.
[17] F. Low, Complete sets of wave-packets, in A Passion for Physics: Essays in Honor of Geoffrey Chew, Singapore: World Scientific, 1985, pp. 17-22.
[18] V. K. Madisetti and D. B. Williams, Digital Signal Processing, CRC Press LLC, 1999.
[19] M. Vetterli, J. Kovačević, and V. K. Goyal, Foundations of Signal Processing, Cambridge University Press, 2014.
[20] P. P. Vaidyanathan, Multirate Systems and Filter Banks, Prentice Hall P T R, Englewood Cliffs, New Jersey, 1993.
[21] J. W. Woods, Multidimensional Signal, Image and Video Processing and Coding, Elsevier Inc., 2006.
[22] M. Zibulski and Y. Y. Zeevi, Oversampling in the Gabor Scheme, in Proc. IEEE ICASSP, 3(1992), 281-284.

(K. T. Poumai) Department of Mathematics, Motilal Nehru College, University of Delhi, Delhi-110021, India
E-mail address: kholetim@yahoo.co.in

(S. K. Kaushik) Department of Mathematics, Kirori mal College, University of Delhi, Delhi-110007, India
E-mail address: shikk2003@yahoo.co.in

(Poonam Mantry) Department of Mathematics, Daulat Ram College, University of Delhi, Delhi-110007, India
E-mail address: poonam.mantry@gmail.com