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ON CONVOLUTION OF BOAS TRANSFORM OF WAVELETS

NIKHIL KHANNA† AND LEENA KATHURIA

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Abstract. In this paper, we show that the Boas transform of Convolution (cross-correlation) of two wavelets satisfies the required admissibility and regularity conditions. These resulting new wavelets are then employed to analyze Boas transform of convolved (cross-correlated) signals by means of Boas transform wavelet convolution and cross-correlation theorems. Analogously to Bedrosian theorem, Boas transform product theorem is given.

1. Introduction

Boas [2] introduced an integral transform associated to the Hilbert transform, which emerged due to the study of the class of functions having Fourier transform, which vanishes on a finite interval. This transform was known by Boas transform, which finds an application in the theory of filters in electrical engineering. A filter is a system, having some frequency selective mechanism. Recall from [30], that the system transfer function of a high pass filter is given by $H(w) = \begin{cases} Ae^{it_o w}, & \text{if } |w| \geq 1; \\ 0, & \text{otherwise} \end{cases}$. Thus, any finite energy signal f passing through a high pass filter gives an output g such that $\hat{g}(w) = H(w)\ \hat{f}(w)$. Thus, $\hat{g}$ vanishes on $(-1, 1)$. Using Boas' theorems, one can characterize the output of the high pass filter in two ways: (i) a signal g is the output of a high pass filter if and only if $\mathcal{B}(\mathcal{B}g) = -g$, (ii) if g is an output of high pass filter, then $\mathcal{B}g = \mathcal{H}g$. Boas transform was further studied by Goldberg [8], Heywood [10] and Zaidi [29]. For various details related to Boas transform, one may refer to [30] and see [19] for details on Hilbert transform. The wavelet transform finds a significant

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†Corresponding author.

role in the analysis of non-stationary signals as it serves a purpose of classical linear time-frequency representation with better time and frequency localization properties [24]. Further, it plays an important role in removal of the non-stationary properties of the signals involved. Thus, the conventional estimation algorithm (such as Wiener filters) for stationary signals can be applied directly in each scale of the wavelet domain [6]. The received signal during signal processing system is modeled as a convolution of the input signal and the impulse response of the system with additive noise. The problem of signal restoration (deconvolution) [27, 31] has attracted the attention of signal processing groups, but due to lack of exiguity of convolution theorems for the continuous or discrete wavelet transform impedes us to locate optimal deconvolution filters in the wavelet domain. At present, filtering operation for signals is widely used in different areas of signal and image processing and using convolution theorem, this operation is carried out using Fourier transform which is best suited for the stationary signals. But practically, in many cases, signal under analysis is non-stationary and has a different frequency content at different localizations. Thus, the convolution theorem does not give impressive results in case of time-varying filters, which give rise to joint time-frequency representations [23]. In this regard, Pérez-Rendon and Robles [22] gave the convolution theorem for the continuous wavelet transform. Later, Jarrah and Khanna [11], in 2018, gave Hilbert transform convolution (cross-correlation) theorems to study Hilbert transform of convolved (cross-correlated) signals. Recently, Khanna et al. [18] introduced Boas transform of wavelets and found that since Boas transform shares an intimate relation with Fourier transform vanishing on a particular interval, it is one of the reasons for its possible applications in engineering and signal processing. The wavelet theory has a vast range of applications. For details, one may see [4, 5, 7, 9, 12, 13, 14, 15, 16, 17, 20, 21, 25, 26, 28].

In this paper, Boas transform of convolved (cross-correlated) wavelets has been studied and is used to analyze Boas transform of convolved (cross-correlated) signals by means of convolution (cross-correlation) theorems. Boas transform product theorem has been given and employed to show that Boas transform of product of wavelets and their convolution satisfy the required admissibility condition.

2. Boas transform of Wavelets

In terms of principal value integral, the Boas transform of a function $f \in L^2(\mathbb{R})$, denoted by $\mathcal{B}f(x)$, is defined as

$$\begin{aligned}
\mathcal{B}f(x) &= \frac{1}{\pi} p.v. \int_0^\infty \frac{f(x+z) - f(x-z)}{z^2} \sin(z)\; dz \\
&= \frac{1}{\pi} P \int_0^\infty \frac{f(x+z)}{z^2} \sin(z)\; dz
\end{aligned}$$

for any x for which the integral exists.

If $f \in L^2(\mathbb{R})$, then

$$f(x) = \frac{1}{\pi}\int_0^{\infty} (\mathcal{B}f(x-z) - \mathcal{B}f(x+z))\ h(z)\ dz$$
$$= -\frac{1}{\pi}\int_0^{\infty} \mathcal{B}f(x+z)\ h(z)\ dz,$$

where

$$h(z) = \frac{\cos(z)}{z} + \int_0^1 \frac{\sin(zt)}{t}\ dt$$

is known as the inversion formula for the Boas transform.
Note that the relationship between the Boas transform and the Hilbert transform of a function is given by

$$(\mathcal{B}f)(x) = (\mathcal{H}f)(x) - \{\mathcal{H}f * g\}(x), \tag{2.1}$$

where

$$g(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{1-\cos(x)}{\pi x^2}\right),$$

and the relationship between Fourier transform and Hilbert transform is specified by

$$\widehat{\mathcal{H}f}(\gamma) = \begin{cases} -\iota \hat{f}(\gamma), & \text{if } \gamma > 0, \\ \iota \hat{f}(\gamma), & \text{if } \gamma < 0, \\ 0, & \text{if } \gamma = 0. \end{cases} \tag{2.2}$$

Taking Fourier transform on both the sides of (2.1), we have

$$\widehat{\mathcal{B}f}(\gamma) = \widehat{\mathcal{H}f}(x) - \mathcal{F}\{\mathcal{H}f * g\}(\gamma). \tag{2.3}$$

If $\mathcal{H}f(x) \in L^1(\mathbb{R})$, then using (2.1), (2.2), and (2.3), we obtain

$$\widehat{\mathcal{B}f}(\gamma) = -i\ \mathrm{sgn}(\gamma)\ \hat{f}(\gamma)(1-\hat{g}(\gamma)),$$

where

$$\hat{g}(\gamma) = \begin{cases} 0, & \text{if } |\gamma| \geq 1, \\ 1-|\gamma|, & \text{if } |\gamma| < 1. \end{cases}$$

Recall, that a function ψ with finite energy, $\mathcal{E}\{\psi\}$, i.e., $\psi \in L^2(\mathbb{R})$, is said to be a wavelet, if it satisfies the admissibility condition, given by $C_\psi = \int_{\mathbb{R}} \frac{|\hat{\psi}(\gamma)|^2}{|\gamma|} d\gamma < +\infty$, where $\hat{\psi}$ denotes the Fourier transform of a wavelet, ψ. Further, there is another way of constructing wavelets that involves the concept of multiresolution analysis or MRA. This gives the framework for constructing wavelet function $\psi \in L^2(\mathbb{R})$ such that the collection $\{\psi_{j,k}(x)\}_{j,k\in\mathbb{Z}} = \{2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}} = \{D_{2^j} T_k \psi(x)\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis in $\mathbb{R}$. Such collection $\{\psi_{j,k}(x)\}_{j,k\in\mathbb{Z}}$ is called a wavelet orthonormal basis on $\mathbb{R}$.

In [18], a sufficient condition on ψ has been imposed under which Boas transform of a wavelet is again a wavelet. The result is given below:

Theorem 2.1. *Let $\psi \in L^1(\mathbb{R})$ be a wavelet such that $\hat{\psi} \in L^1(\mathbb{R})$ with $\hat{\psi}(0) = 0$. Then $\mathcal{B}\psi$ is again a wavelet.*

It is proved in [18], giving a counter example of Haar wavelet

$$\psi(x) = \begin{cases} 1, & \text{if } \ 0 \le x < \frac{1}{2} \\ -1, & \text{if } \ \frac{1}{2} \le x < 1 \\ 0, & \text{if otherwise} \end{cases},$$

that the condition of $\psi, \hat{\psi} \in L^1(\mathbb{R})$ such that $\hat{\psi}(0) = 0$ is not necessary for $\mathcal{B}\psi$ to be a wavelet.

3. Boas transform of Convolved and Cross Correlated Wavelets

The wavelet theory acts with the general properties of the wavelets and the wavelet transform. We prefer to choose wavelet function according to the application. A wavelet, which is localized in terms of both spatial width and frequency bandwidth, is employed for space-frequency analysis, whereas, while dealing with smooth signals, a smooth wavelet is more appropriate. In case of examining a signal with certain discontinuities, we require wavelets with good spatial localization to scrupulously track swift changes in the signal.

In this section, we analyze Boas transform of convolved (cross-correlated) signals and the best way to study or analyze them is using the Boas transform of convolved (cross-correlated) wavelets, as they satisfy the required admissibility and regularity conditions. Recall from [3, 22] that the convolution of two complex-valued functions $f(x)$ and $g(x)$ of a real variable x, denoted as $f * g$, is defined by $(f * g)(x) = \int_{\mathbb{R}} f(t)\ g(x-t)\ dt$, provided that the integral exists.

Remark 3.1. The Boas transform of a convolution of two functions f and g can be written in terms of a convolution of one of the functions with the Boas transform of the other function, that is, $\mathcal{B}\{f * h\}(x) = (\mathcal{B}f * h)(x) = (f * \mathcal{B}h)(x)$. Indeed, we have

$$\begin{aligned} \mathcal{B}\{f * h\}(x) &= \mathcal{H}\{f * h\}(x) - (\mathcal{H}\{f * h\} * g)(x) = (\mathcal{H}f * h)(x) - ((\mathcal{H}f * h) * g)(x) \\ &= (\mathcal{H}f * h)(x) - (\mathcal{H}f * (h * g))(x) = (\mathcal{H}f * h)(x) - (\mathcal{H}f * (g * h))(x) \\ &= (\mathcal{H}f * h)(x) - ((\mathcal{H}f * g) * h)(x) = ((\mathcal{H}f - (\mathcal{H}f * g)) * h)(x) \\ &= (\mathcal{B}f * h)(x). \end{aligned}$$

Similarly, one can show that $\mathcal{B}\{f * h\}(x) = (f * \mathcal{B}h)(x)$.

In the following result, we give a sufficient condition under which Boas transform of convolved wavelets form a wavelet.

Proposition 3.2. Let ψ_1, ψ_2 be wavelets such that for $i = 1, 2,\ \psi_i, \hat{\psi_i} \in L^1(\mathbb{R})$ with $\hat{\psi_2}(0) = 0$. Then $\mathcal{B}\{\psi_1 * \psi_2\}$ is a wavelet.

Proof. In view of Remark 3.1, we can write

$$\|\mathcal{B}\{\psi_1 * \psi_2\}\|_2 = \|\psi_1 * \mathcal{B}\psi_2\|_2 \le \|\psi_1\|_1\ \|\mathcal{B}\psi_2\|_2 < +\infty.$$

Further, note that, since $\mathcal{H}\{\psi_1 * \psi_2\} \in L^1(\mathbb{R})$, we have

$$\begin{aligned} C_{\mathcal{B}\{\psi_1*\psi_2\}} &= \int_{\mathbb{R}} \frac{|\mathcal{B}\{\psi_1 * \psi_2\}^\wedge(\gamma)|^2}{|\gamma|}\, d\gamma \\ &= \int_{\mathbb{R}} \frac{|\mathcal{H}\{\psi_1 * \psi_2\}^\wedge(\gamma)(1-\hat{g}(\gamma))|^2}{|\gamma|}\, d\gamma \\ &= \int_{\mathbb{R}} \frac{|\widehat{\psi_1 * \psi_2}(\gamma)(1-\hat{g}(\gamma))|^2}{|\gamma|}\, d\gamma \\ &\le \int_{\mathbb{R}} \frac{|\hat{\psi}_1(\gamma)\ \hat{\psi}_2(\gamma)|^2}{|\gamma|}\, d\gamma \\ &\le K^2 \int_{\mathbb{R}} \frac{|\hat{\psi}_2(\gamma)|^2}{|\gamma|}\, d\gamma < +\infty, \text{ where } K \text{ is constant.} \end{aligned}$$

Thus, $\mathcal{B}\{\psi_1 * \psi_2\}$ is an admissible wavelet. □

An important property of the wavelet transform is the number of vanishing moments that represents the regularity of the wavelet function and ability of wavelet transform to capture the localized information. A wavelet $\psi(x)$ is said to have d vanishing moments if $\int_{\mathbb{R}} x^j\ \psi(x)\ dx = 0$, for $0 \le j \le d-1$. Note that the wavelet series of a smooth function will converge very rapidly to the function as long as the wavelet has a lot of vanishing moments. This means that relatively few wavelet coefficients will be required in order to get a good approximation. Therefore, during image compression, it requires only to keep a few coefficients, where the image is smooth and, in contrast to this, we need more coefficients at the edges. In other words, a wavelet, having d vanishing moments, forces all wavelet coefficients $\langle f, \psi_{j,k}\rangle$ of any polynomial of degree d or less to vanish. In fact, if a smooth function is interrupted by occasional singularity (or discontinuity), then the wavelet coefficients on the smooth parts will be very small or zero if at that point the function behaves like a polynomial of certain order. For details, see [12, 13, 26].

In the following result, we prove that the number of vanishing moments of the Boas transform of convolved wavelets is the sum of the number of vanishing moments of wavelets involved.

Theorem 3.3. *Let ψ_1, ψ_2 be wavelets such that*

(i) $\psi_i, \hat{\psi}_i \in L^1(\mathbb{R})$ *for* $i = 1, 2$,
(ii) $\hat{\psi}_2(0) = 0$ *and* $\psi_2^{(1)} \in L^1(\mathbb{R})$, *and*
(iii) $\int_{\mathbb{R}} x^q G(x)\ dx = 0$ *for* $0 \le q \le n$, *where* $G(x) = \int_{-1-x}^{1-x} \widehat{\psi^{(1)}}(\gamma)\ d\gamma$.

If ψ_1 and ψ_2 have m and n vanishing moments, respectively, then $\mathcal{B}\{\psi_1 * \psi_2\}$ has $(m+n)$ vanishing moments, provided that $x^n \psi_2(x) \in L^2(\mathbb{R})$.

Proof. We have

$$\begin{aligned}
\int_{\mathbb{R}} x^q\ \mathcal{B}\{\psi_1 * \psi_2\}(x)\ dx &= \int_{\mathbb{R}} x^q\ (\psi_1 * \mathcal{B}\psi_2)(x)\ dx \\
&= \int_{\mathbb{R}} x^q \int_{\mathbb{R}} \psi_1(y)\ \mathcal{B}\psi_2(x-y)\ dy\ dx \\
&= \sum_{p=0}^{q} {}^qC_p\ \mathrm{Mom}_p(\psi_1)\ \mathrm{Mom}_{q-p}(\mathcal{B}\psi_2) \\
&= \sum_{p=0}^{q} {}^qC_p\ \mathrm{Mom}_p(\psi_1) \Big(\mathrm{Mom}_{q-p}(\mathcal{H}\psi_2) - \mathrm{Mom}_{q-p}(\mathcal{H}\psi_2 * g)\Big) \\
&= \sum_{p=0}^{q} {}^qC_p\ \mathrm{Mom}_p(\psi_1) \Big(\mathrm{Mom}_{q-p}(\mathcal{H}\psi_2) \\
&\qquad - \int_{\mathbb{R}} x^{q-p} \int_{\mathbb{R}} \widehat{\mathcal{H}T_x\psi_2}(\gamma)\ \hat{g}(\gamma)\ d\gamma\ dx\Big) \\
&= \sum_{p=0}^{q} {}^qC_p\ \mathrm{Mom}_p(\psi_1) \Big(\mathrm{Mom}_{q-p}(\mathcal{H}\psi_2) \\
&\qquad + i \int_{\mathbb{R}} x^{q-p} \int_{-1}^{1} \gamma\widehat{T_x\psi_2}(\gamma)\ d\gamma\ dx\Big).
\end{aligned}$$

Since $\psi_2, \psi_2^{(1)}, \hat{\psi}_2 \in L^1(\mathbb{R})$, we have

$$\begin{aligned}
\int_{\mathbb{R}} x^q\ \mathcal{B}\{\psi_1 * \psi_2\}(x)\ dx &= \sum_{p=0}^{q} {}^qC_p\ \mathrm{Mom}_p(\psi_1) \Big(\mathrm{Mom}_{q-p}(\mathcal{H}\psi_2) \\
&\qquad + \frac{1}{2\pi} \int_{\mathbb{R}} x^q \int_{-1-x}^{1-x} \widehat{\psi^{(1)}}(\gamma)\ d\gamma\ dx\Big) \\
&= \sum_{p=0}^{q} {}^qC_p\ \mathrm{Mom}_p(\psi_1) \Big(\mathrm{Mom}_{q-p}(\mathcal{H}\psi_2) + \frac{1}{2\pi}\int_{\mathbb{R}} x^q G(x)\ dx\Big) \\
&= \sum_{p=0}^{q} {}^qC_p\ \mathrm{Mom}_p(\psi_1)\ \mathrm{Mom}_{q-p}(\mathcal{H}\psi_2).
\end{aligned}$$

Again, since ψ_2 has n vanishing moments and $x^n\psi_2(x) \in L^2(\mathbb{R})$, using moment formula for the Hilbert transform, we have

$$\int_{\mathbb{R}} x^p\ \mathcal{B}\psi_2(x)\ dx = 0 \text{ for } 0 \le p \le n.$$

Let $q \le m+n$. If $q-p \le n$, then $\mathrm{Mom}_{q-p}(\mathcal{H}\psi_2) = 0$, otherwise $p \le m-1$, which gives $\mathrm{Mom}_p(\psi_1) = 0$.
Thus, the number of vanishing moments of $\mathcal{B}\{\psi_1 * \psi_2\}$ is $m_1 + m_2$. $\square$

Let us now recall, that the continuous wavelet transform (CWT) of a function $f \in L^2(\mathbb{R})$ is defined as

$$W_\psi f(\alpha,\beta) = \langle f, \psi_{\alpha,\beta}\rangle = \int_{\mathbb{R}} f(t)\overline{\psi_{\alpha,\beta}(t)}dt,$$

where $\overline{\psi_{\alpha,\beta}(t)}$ is the complex conjugate of the scaled and translated wavelet function $\psi_{\alpha,\beta}(t)$.

The following result states that the continuous wavelet transform of a signal, which is Boas transform of convolved signal, i.e., $u * v$ with respect to Boas transform of convolved wavelet, i.e., $\psi_1 * \psi_2$ can be decomposed as the convolution of the continuous wavelet transform of a signal, which is Boas transform of signal u with respect to wavelet ψ_1 and the continuous wavelet transform of the signal v with respect to Boas transform of wavelet ψ_2. Precisely, we have

Theorem 3.4. *Let ψ_1, ψ_2 be wavelets such that for $i = 1, 2$, $\psi_i, \hat{\psi}_i \in L^1(\mathbb{R})$ with $\hat{\psi}_2(0) = 0$ and let $\mathcal{W}_{\psi_1}\mathcal{B}u$ and $\mathcal{W}_{\mathcal{B}\psi_2}v$ be the continuous wavelet transforms of $\mathcal{B}u$ with respect to wavelet ψ_1 and that of v with respect to wavelet $\mathcal{B}\psi_2$, respectively, where $u \in L^2(\mathbb{R})$ and $v, \hat{v} \in L^1(\mathbb{R})$. If $z = u * v$ and $\psi = \psi_1 * \psi_2$, then*

$$\mathcal{W}_{\mathcal{B}\psi}\mathcal{B}z(\alpha,\beta) = \frac{1}{|\alpha|^{\frac{1}{2}}}(\mathcal{W}_{\psi_1}\mathcal{B}u * \mathcal{W}_{\mathcal{B}\psi_2}v)(\alpha,\beta).$$

Proof. The continuous wavelet transform of $\mathcal{B}z$ with respect to $\mathcal{B}\psi$ is given by

$$\mathcal{W}_{\mathcal{B}\psi}\mathcal{B}z(\alpha,\beta) = \frac{1}{|\alpha|^{\frac{1}{2}}}\int_{\mathbb{R}} \mathcal{B}z(x)\ \overline{\mathcal{B}\psi\left(\frac{x-\beta}{\alpha}\right)}\ dx.$$

This can be rewritten as

$$\begin{aligned}
\mathcal{W}_{\mathcal{B}\psi}\mathcal{B}z(\alpha,\beta) &= \frac{1}{|\alpha|^{\frac{1}{2}}}\int_{\mathbb{R}} (\mathcal{B}u * v)(x)\ \overline{(\psi_1 * \mathcal{B}\psi_2)\left(\frac{x-\beta}{\alpha}\right)}\ dx.\\
&= \frac{1}{|\alpha|^{\frac{1}{2}}}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{B}u(p)\ v(x-p)\ dp \int_{\mathbb{R}} \overline{\psi_1(q)}\ \overline{\mathcal{B}\psi_2\left(\frac{x-\beta}{\alpha}-q\right)}\ dq\ dx\\
&= \frac{1}{|\alpha|^{\frac{3}{2}}}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{B}u(p)\ \overline{\psi_1\left(\frac{p-(\beta-s)}{\alpha}\right)}\ dp\int_{\mathbb{R}} v(r)\ \overline{\mathcal{B}\psi_2\left(\frac{r-s}{\alpha}\right)}\ dr\ ds,\\
&\qquad\qquad\qquad\qquad (x-p=r \text{ and } \beta+\alpha q-p=s)\\
&= \frac{1}{|\alpha|^{\frac{1}{2}}}\int_{\mathbb{R}} \mathcal{W}_{\psi_1}\mathcal{B}u(\alpha,\beta-s)\ \mathcal{W}_{\mathcal{B}\psi_2}v(\alpha,s)\ ds\\
&= \frac{1}{|\alpha|^{\frac{1}{2}}}(\mathcal{W}_{\psi_1}\mathcal{B}u * \mathcal{W}_{\mathcal{B}\psi_2}v)(\alpha,\beta).
\end{aligned}$$

□

The main use of convolution is in specifying the output of a linear time-invariant (LTI) system. The input-output behavior of LTI system can be characterized by means of its impulse response, and the output of LTI system for any input signal $f(x)$ can be expressed as the convolution of the input signal with the system's impulse response. Note that convolution is a measure of effect of one signal on the other, whereas the cross-correlation of two signals is a measure of how similar the two signals are to each other.

Further, recall from [3, 22] that the cross-correlation of two complex-valued functions $f(x)$ and $g(x)$ of a real variable x, denoted as $f \star g$ is defined by $(f \star g)(x) = \int_{\mathbb{R}} \overline{f(t-x)}\, g(t)\, dt = \int_{\mathbb{R}} \overline{f(t)}\, g(x+t)\, dt$, provided that the integral exists, where $\overline{f}$ is the complex conjugate of f.

Remark 3.5. The Boas transform of a cross-correlation of two functions f and g can be written in terms of a cross-correlation of one of the functions with the Boas transform of the other function, that is, $\mathcal{B}\{f \star h\}(x) = (\mathcal{B}f \star h)(x) = (f \star \mathcal{B}h)(x)$. Indeed, we have

$$
\begin{aligned}
&\mathcal{B}\{f \star h\}(x)\\
&= \mathcal{H}\{f \star h\}(x) - (\mathcal{H}\{f \star h\} * g)(x) = (\mathcal{H}f \star h)(x) - (\mathcal{H}f \star h)(x) * g(x)\\
&= (\mathcal{H}f \star h)(x) - g(x) * (\mathcal{H}f \star h)(x) = (\mathcal{H}f \star h)(x) - g(x) * (\overline{\mathcal{H}f(-x)} * h(x))\\
&= (\mathcal{H}f \star h)(x) - (g(x) * \overline{\mathcal{H}f(-x)}) * h(x) = (\mathcal{H}f \star h)(x) - \overline{(g * \mathcal{H}f)(-x)} * h(x),\\
&= (\mathcal{H}f \star h)(x) - \overline{(\mathcal{H}f * g)(-x)} * h(x) = (\mathcal{H}f \star h)(x) - ((\mathcal{H}f * g) \star h)(x),\\
&= ((\mathcal{H}f - \mathcal{H}f * g) \star h)(x) = (\mathcal{B}f \star h)(x),
\end{aligned}
$$

where $g(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{1-\cos(x)}{\pi x^2}\right)$. Similarly, one can show that $\mathcal{B}\{f \star h\}(x) = (f \star \mathcal{B}h)(x)$.

Next result gives a sufficient condition under which Boas transform of cross-correlated wavelets forms a wavelet.

Proposition 3.6. Let ψ_1, ψ_2 be wavelets such that for $i = 1, 2$, $\psi_i, \hat{\psi}_i \in L^1(\mathbb{R})$, with $\hat{\psi}_2(0) = 0$. Then $\mathcal{B}\{\psi_1 \star \psi_2\}$ is a wavelet.

Proof. Follows in view of proof for Proposition 3.2. □

In the following result, we prove that the number of vanishing moments of the Boas transform of cross-correlated wavelets is the sum of the number of vanishing moments of the wavelets involved.

Theorem 3.7. *Let ψ_1, ψ_2 be wavelets such that*

(i) $\psi_i, \hat{\psi}_i \in L^1(\mathbb{R})$ *for* $i = 1, 2$,

(ii) $\hat{\psi}_2(0) = 0$ *and* $\psi_2^{(1)} \in L^1(\mathbb{R})$, *and*

(iii) $\int_{\mathbb{R}} x^q G(x)\, dx = 0$ *for* $0 \le q \le n$, *where* $G(x) = \int_{-1-x}^{1-x} \widehat{\psi^{(1)}}(\gamma)\, d\gamma$.

If ψ_1 and ψ_2 have m and n vanishing moments, respectively, then $\mathcal{B}\{\psi_1 \star \psi_2\}$ has $(m+n)$ vanishing moments, provided that $x^n \psi_2(x) \in L^2(\mathbb{R})$.

Proof. The proof can be worked out on the lines of the Theorem 3.3. □

The next result states that the continuous wavelet transform of a signal, which is Boas transform of cross-correlated signal, i.e., $u \star v$ with respect to Boas transform of cross-correlated wavelet, i.e., $\psi_1 \star \psi_2$ can be decomposed as the cross-correlation of the continuous wavelet transform of a signal which is Boas transform of signal u with respect to wavelet ψ_1 and the continuous wavelet transform of the signal v with respect to Boas transform of wavelet ψ_2. Precisely, we have

Theorem 3.8. *Let ψ_1, ψ_2 be wavelets such that for $i = 1, 2$, $\psi_i, \hat{\psi}_i \in L^1(\mathbb{R})$, with $\hat{\psi}_2(0) = 0$, and let $\mathcal{W}_{\psi_1}\mathcal{B}u$ and $\mathcal{W}_{\mathcal{B}\psi_2}v$ be the continuous wavelet transforms of $\mathcal{B}u$ with respect to wavelet ψ_1 and that of v with respect to wavelet $\mathcal{B}\psi_2$, respectively, where $u \in L^2(\mathbb{R})$ and $v, \hat{v} \in L^1(\mathbb{R})$. If $z = u \star v$ and $\psi = \psi_1 \star \psi_2$, then*

$$\mathcal{W}_{\mathcal{B}\psi}\mathcal{B}z(\alpha, \beta) = \frac{1}{|\alpha|^{\frac{1}{2}}}(\mathcal{W}_{\psi_1}\mathcal{B}u \star \mathcal{W}_{\mathcal{B}\psi_2}v)(\alpha, \beta).$$

Proof. The continuous wavelet transform of $\mathcal{B}z$ with respect to $\mathcal{B}\psi$ is given by

$$\mathcal{W}_{\mathcal{B}\psi}\mathcal{B}z(\alpha, \beta) = \frac{1}{|\alpha|^{\frac{1}{2}}} \int_{\mathbb{R}} \mathcal{B}z(x)\ \overline{\mathcal{B}\psi\left(\frac{x-\beta}{\alpha}\right)}\ dx.$$

This can be rewritten as

$$\begin{aligned}
\mathcal{W}_{\mathcal{B}\psi}\mathcal{B}z(\alpha, \beta) &= \frac{1}{|\alpha|^{\frac{1}{2}}} \int_{\mathbb{R}} (\mathcal{B}u \star v)(x)\ \overline{(\psi_1 \star \mathcal{B}\psi_2)\left(\frac{x-\beta}{\alpha}\right)}\ dx. \\
&= \frac{1}{|\alpha|^{\frac{1}{2}}} \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\mathcal{B}u(p)}\ v(x+p)\ dp \int_{\mathbb{R}} \psi_1(q)\ \overline{\mathcal{B}\psi_2\left(\frac{x-\beta}{\alpha} + q\right)}\ dq\ dx \\
&= -\frac{1}{|\alpha|^{\frac{3}{2}}} \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\mathcal{B}u(p)}\ \psi_1\left(\frac{p-(s-\beta)}{\alpha}\right) dp \int_{\mathbb{R}} v(r)\ \overline{\mathcal{B}\psi_2\left(\frac{r-s}{\alpha}\right)} drds, \\
&\qquad\qquad (x+p = r \text{ and } \beta + p - \alpha q = s) \\
&= \frac{1}{|\alpha|^{\frac{1}{2}}} \int_{\mathbb{R}} \overline{\mathcal{W}_{\psi_1}\mathcal{B}u(\alpha, s-\beta)}\ \mathcal{W}_{\mathcal{B}\psi_2}v(\alpha, s)\ ds \\
&= \frac{1}{|\alpha|^{\frac{1}{2}}}(\mathcal{W}_{\psi_1}\mathcal{B}u \star \mathcal{W}_{\mathcal{B}\psi_2}v)(\alpha, \beta).
\end{aligned}$$

□

In 1963, Bedrosian [1] gave the Hilbert transform product theorem, or Bedrosian theorem, which states that the Hilbert transform of the product of a low-pass and a high-pass signal with non-overlapping spectra is given by the product of the low-pass signal and the Hilbert transform of the high-pass signal. Usually, the Bedrosian theorem is used to ensure the Bedrosian identity $\mathcal{H}\{A(t)\ \cos(\theta(t)\} = A(t)\ \mathcal{H}\{\cos(\theta(t)\}$, where $A(t)$ and $\theta(t)$ stand for instantaneous amplitude and phase, respectively. It plays a significant role in demodulation of signals in the non-stationary signal processing. The following result is the generalization of the Bedrosian theorem in terms of Boas transform.

Theorem 3.9 (**Boas Transform Product Theorem**). *Let f_1, f_2 be functions such that*

(i) $f_i, \hat{f}_i \in L^1(\mathbb{R})$ *for* $i = 1, 2$,

(ii) $\hat{f}_1(\gamma)$ *vanishes for* $|\gamma| > 1$ *and* $\hat{f}_2(\gamma)$ *vanishes for* $|\gamma| \leq 1$.

Then $\mathcal{B}\{f_1(x)\ f_2(x)\} = f_1(x)\ \mathcal{B}f_2(x)$.

Proof. Note that

$$\begin{aligned}\mathcal{B}\{f_1(x)\ f_2(x)\} &= \mathcal{H}\{f_1(x)\ f_2(x)\} - \mathcal{H}\{f_1(x)\ f_2(x)\} * g(x)\\ &= \mathcal{H}\{f_1(x)\ f_2(x)\} - \int_{\mathbb{R}} \mathcal{H}\{f_1(t)\ f_2(t)\}\ g(x-t)\ dt. \end{aligned} \tag{3.4}$$

Since $\hat{f}_1(\gamma)$ vanishes for $|\gamma| > 1$ and $\hat{f}_2(\gamma)$ vanishes for $|\gamma| < 1$, using Bedrosian theorem, we have

$$\begin{aligned}\|\mathcal{H}\{f_1(x)\ f_2(x)\}\|_2 &= \|f_1(x)\ \mathcal{H}f_2(x)\|_2\\ &\leq K^2\ \|\mathcal{H}f_2(x)\|_2,\ \text{(where } K \text{ is constant)}\\ &< +\infty.\end{aligned}$$

We have

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{H}\{f_1(t)\ f_2(t)\}\ g(x-t)\ dt &= \int_{\mathbb{R}} \mathcal{H}\{f_1(t)\ f_2(t)\}^\wedge(\gamma)\ \hat{g}(\gamma - x)\ d\gamma\\ &= -i\ \operatorname{sgn}(\gamma) \int_{\mathbb{R}} (f_1(t)\ f_2(t))^\wedge(\gamma)\ \hat{g}(\gamma - x)\ d\gamma\\ &= -i\ \operatorname{sgn}(\gamma) \int_{\mathbb{R}} \hat{f}_1(\gamma) * \hat{f}_2(\gamma)\ \widehat{T_x g}(\gamma)\ d\gamma,\\ &= -i\ \operatorname{sgn}(\gamma) \int_{\mathbb{R}} \hat{f}_1(\gamma) * \hat{f}_2(\gamma)\ E_{-x}\hat{g}(\gamma)\ d\gamma,\\ &= -i\ \operatorname{sgn}(\gamma) \int_{\mathbb{R}} \hat{f}_1(\gamma) * \hat{f}_2(\gamma)\ e^{-2\pi i x\gamma}\ \hat{g}(\gamma)\ d\gamma\\ &= -i\ \operatorname{sgn}(\gamma)\bigg(\int_{|\gamma|>1} \hat{f}_1(\gamma) * \hat{f}_2(\gamma)\ e^{-2\pi i x\gamma}\ \hat{g}(\gamma)\ d\gamma\\ &\qquad + \int_{|\gamma|\leq 1} \hat{f}_1(\gamma) * \hat{f}_2(\gamma)\ e^{-2\pi i x\gamma}\ \hat{g}(\gamma)\ d\gamma\bigg)\\ &= 0.\end{aligned}$$

Since $\hat{f}_2(\gamma)$ vanishes for $|\gamma| < 1$, it follows that

$$\mathcal{B}\{f_1(x)\ f_2(x)\} = f_1(x)\ \mathcal{H}f_2(x) = f_1(x)\ \mathcal{B}f_2(x).$$

□

In the following result, a sufficient condition has been given to prove that Boas transform of product of two wavelets is again a wavelet. Further, it is also proved that convolution of the resulting wavelets happens to be a wavelet.

Theorem 3.10. *Let ψ_l for $l = 1, 2, 3, 4$ be wavelets such that*

(i) $\psi_l, \hat{\psi}_l \in L^1(\mathbb{R})$ *for $l = 1, 2, 3, 4$, and*
(ii) $\hat{\psi}_i(\gamma),\ i = 1, 3,$ *vanishes for $|\gamma| > 1$ and $\hat{\psi}_j(\gamma),\ j = 2, 4,$ vanishes for $|\gamma| \leq 1$.*

Then $\mathcal{B}\{\psi_i(x)\psi_j(x)\},\ i = 1, 3,\ j = 2, 4,$ forms a wavelet. Further, $\mathcal{B}\{\psi_1(x)\psi_2(x)\} * \mathcal{B}\{\psi_3(x)\psi_4(x)\}$ also forms a wavelet.

Proof. In view of Boas transform product theorem, we have

$$\begin{aligned}\|\mathcal{B}\{\psi_i(x)\ \psi_j(x)\}\|_2 &= \|\psi_i(x)\ \mathcal{B}\psi_j(x)\|_2\\ &\leq C^2\ \|\mathcal{B}\psi_j(x)\|_2,\ \text{where } C \text{ is constant}\\ &< +\infty. \end{aligned}\tag{3.5}$$

Also, we have

$$\begin{aligned}\int_{\mathbb{R}} \frac{|\mathcal{B}\{\psi_i(x)\ \psi_j(x)\}|^2}{|\gamma|}\ d\gamma &= \int_{\mathbb{R}} \frac{|(\psi_i(x)\ \psi_j(x))^\wedge(\gamma)(1-\hat{g}(\gamma))|^2}{|\gamma|}\ d\gamma\\ &= \int_{\mathbb{R}} \frac{|(\psi_i(x)\ \psi_j(x))^\wedge(\gamma)(1-\hat{g}(\gamma))|^2}{|\gamma|}\ d\gamma\\ &= \int_{\mathbb{R}} \frac{|(\hat{\psi}_i(\gamma) * \hat{\psi}_j(\gamma))(1-\hat{g}(\gamma))|^2}{|\gamma|}\ d\gamma\\ &= \int_{|\gamma|>1} \frac{|\hat{\psi}_i(\gamma) * \hat{\psi}_j(\gamma)|^2}{|\gamma|}\ d\gamma + \int_{|\gamma|\leq 1} |\hat{\psi}_i(\gamma) * \hat{\psi}_j(\gamma)|^2|\gamma|\ d\gamma\\ &< +\infty.\end{aligned}$$

Thus, $\mathcal{B}\{\psi_i(x)\psi_j(x)\},\ i = 1,3; j = 2,4$ forms a wavelet. Note that

$$\begin{aligned}\|\mathcal{B}\{\psi_3(x)\ \psi_4(x)\}\|_1 &= \|\psi_3(x)\ \mathcal{B}\psi_4(x)\|_1\\ &\leq M\|\mathcal{B}\psi_4(x)\|_1,\ M \text{ is a constant}\\ &\leq M(\|\mathcal{H}\psi_4(x)\|_1 + \|(\mathcal{H}\psi_4 * g)(x)\|_1).\end{aligned}$$

Since $\psi_4 \in L^1(\mathbb{R})$ and $\hat{\psi}_4(0) = 0$, we have $\|\mathcal{B}\{\psi_3(x)\ \psi_4(x)\}\|_1 < +\infty$. This gives

$$\|\mathcal{B}\{\psi_1(x)\ \psi_2(x)\} * \mathcal{B}\{\psi_3(x)\ \psi_4(x)\}\|_2 \leq \|\mathcal{B}\{\psi_1(x)\ \psi_2(x)\}\|_1\ \|\mathcal{B}\{\psi_3(x)\ \psi_4(x)\}\|_2 < +\infty.$$

Further,

$$\begin{aligned}&\int_{\mathbb{R}} \frac{|(\mathcal{B}\{\psi_1(x)\ \psi_2(x)\} * \mathcal{B}\{\psi_1(x)\ \psi_2(x)\})^\wedge(\gamma)|^2}{|\gamma|}\ d\gamma\\ &= \int_{\mathbb{R}} \frac{|(\mathcal{B}\{\psi_1(x)\ \psi_2(x)\})^\wedge(\gamma)\ (\mathcal{B}\{\psi_1(x)\ \psi_2(x)\})^\wedge(\gamma)|^2}{|\gamma|}\ d\gamma\\ &\leq K^2 \int_{\mathbb{R}} \frac{|(\mathcal{B}\{\psi_1(x)\ \psi_2(x)\})^\wedge(\gamma)|^2}{|\gamma|}\ d\gamma\\ &< +\infty. \qquad \square\end{aligned}$$

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(Nikhil Khanna) Department of Mathematics, Motilal Nehru College, University of Delhi, Delhi 110021, India
E-mail address: nikkhannak232@gmail.com

(Leena Kathuria) Department of Mathematics, Amity Institute of Applied Sciences, Amity University, Sector 125, Noida, India
E-mail address: leenakathuria91@gmail.com