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NEAR EXACT OPERATOR BANACH FRAMES IN BANACH SPACES

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Abstract. Near exact operator Banach frame (near exact OBF) is defined and studied. Examples have been given for the existence of near exact operator Banach frames. Also, a sufficient condition for an operator Banach frame to be a near exact OBF is given. Further, it has been proved that if E and F are Banach spaces having near exact operator Banach frames, then the product space $E \times F$ also has a near exact operator Banach frame. Further, we consider block perturbation of operator Banach frames and prove that a block perturbation of an OBF is also an OBF. Finally, we give an application of near exact OBF related to eigenvalue problem.

1. Introduction

Frames are the redundant systems introduced by Duffin and Schaeffer [9], to study some deep problems in non-harmonic Fourier series. For a nice introduction to frames, one may refer to [7]. Frames were extended to Banach spaces by Feichtinger and Gröchenig [11], who, in fact, introduced the notion of atomic decompositions for Banach spaces. Later, Gröchenig [12] introduced a more general concept called Banach frame for Banach spaces. He gave the following definition of a Banach frame.

Let E be a Banach space and E_d be an associated Banach space of scalar valued sequences indexed by $\mathbb{N}$. Let $\{f_n\} \subset E^*$ and $S : E_d \to E$ be given. The pair $(\{f_n\}, S)$ is called a *Banach frame* for E with respect to E_d, if

(1) $\{f_n(x)\} \in E_d, \quad x \in E;$

(2) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E;$$

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(3) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \quad x \in E.$$

Banach frames were further studied in [15, 16, 17, 18, 19].
The concept of fusion Banach frame for Banach spaces was introduced and studied by Kaushik and Kumar [14] as a generalization of Banach frame. They gave the following definition of a fusion Banach frame.

Definition 1.1. Let E be a Banach space, $\{E_n\}$ be a sequence of closed subspaces of E and let $\{T_n\}$ be non trivial linear projections of E onto E_n. Let $\mathcal{A}$ be an associated Banach space and $S : \mathcal{A} \to E$ be an operator. Then $(\{E_n,\ T_n\},\ S)$ is called a fusion Banach frame for E with respect to $\mathcal{A}$ if

(1) $\{T_n f\} \in \mathcal{A},\ f \in E$;
(2) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_E \leq \|\{T_n f\}\|_{\mathcal{A}} \leq B\|f\|_E, \quad f \in E;$$

(3) S is a bounded linear operator such that

$$S(\{T_n f\}) = f, \quad f \in E.$$

The notions of (p, Y)-Bessel operator sequences, operator frames and (p, Y)-Riesz bases for Banach spaces were introduced and studied by Cao et al. [2] as a generalization of usual concepts in Hilbert spaces and of the g-frames introduced by Sun [20].

Later, Chun-Yan-Li [8] introduced and studied operator Bessel sequences, operator frames, Banach operator frames and observed that frames, g-frames for Hilbert spaces, E_d-frames and (p, Y)-operator frames for Banach spaces can be regarded as special cases of operator frames.

The concept of operator Banach frame for a Banach space is introduced and studied in [4] as an amalgamation of the notions of operator frame and Banach operator frame. It is observed that Banach frames, p-frames, E_d-frames, fusion Banach frames for Banach spaces and frames, g-frames for Hilbert spaces can be regarded as special cases of operator Banach frames.

In this paper, we study near exact operator Banach frames (near exact OBF) and provide examples for the existence of near exact operator Banach frames. Also, a sufficient condition for an operator Banach frame to be a near exact OBF is obtained. Further, it has been proved that if E and F are Banach spaces having near exact operator Banach frames, then the product space $E \times F$ also has a near exact operator Banach frame. Further, we consider block perturbation of operator Banach frames and prove that a block perturbation of an operator Banach frame is also an operator Banach frame. Finally, we give an application of near exact OBF related to eigenvalue problem.

2. Preliminaries

Throughout this paper, E and $E_n,\ n \in \mathbb{N}$ will denote Banach spaces over the scaler field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and E^* denotes the dual space of the Banach space E, $T_n \in B(E, E_n)$, for each $n \in \mathbb{N}$, denotes a bounded linear operator from E into E_n and $\operatorname{ran}(P)$ denotes the range of bounded linear operator P.

We begin this section with the following definition of an operator Banach frame, given in [4].

Definition 2.1. Let E be a Banach space, $\{E_n\}$ be a sequence of Banach spaces and $T_n \in B(E, E_n), n \in \mathbb{N}$. Let $\mathcal{A}$ be an associated Banach space and $S : \mathcal{A} \to E$ be an operator. Then $(\{T_n\},\ S)$ is called an *Operator Banach frame* (OBF) for E with respect to $\mathcal{A}$, if

(1) $\{T_n f\} \in \mathcal{A}, f \in E$;
(2) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_E \leq \|\{T_n f\}\|_{\mathcal{A}} \leq B\|f\|_E, \quad f \in E; \tag{2.1}$$

(3) S is a bounded linear operator such that

$$S(\{T_n f\}) = f, \quad f \in E.$$

The positive constants A and B, respectively, are called lower and upper frame bounds for the OBF $(\{T_n\},\ S)$. The inequality (2.1) is called the frame inequality for the OBF. The operator $S : \mathcal{A} \to\ E$ is called the reconstruction operator. If condition (1) in Definition 2.1 and the upper inequality in (2.1) are satisfied, then we call $\{T_n\}$ an operator Bessel sequence for E, with respect to $\mathcal{A}$. Let us denote by Bess(E) the set of all operator Bessel sequences for E, with respect to $\mathcal{A}$. For a sequence $\mathcal{T} = \{T_n\} \in \text{Bess}(E)$, define $\mathcal{R}_{\mathcal{T}} : E \to \mathcal{A}$ by $\mathcal{R}_{\mathcal{T}} f = \{T_n f\}$ for all $f \in E$. Then $\mathcal{R}_{\mathcal{T}} \in B(E, \mathcal{A})$. We call $\mathcal{R}_{\mathcal{T}}$ the analysis operator for the operator Bessel sequence $\{T_n\}$.
If conditions $(1), (3)$ and only upper inequality in (2.1) is satisfied, then $(\{T_n\}, S)$ is called Banach operator frames (which was introduced by Chun-Yan-Li [8]).

Observations.

(1) One may observe that an operator Banach frame is a Banach operator frame. The converse need not be true.
(2) In case $E_n = \mathbb{K}$ for each $n \in \mathbb{N}$, the notion of operator Banach frame coincides with the notion of Banach frame.
(3) If E_n is closed subspace of E and $\{T_n\}$ is a sequence of non trivial projection of E onto E_n in Definition 2.1, then $(\{E_n,\ T_n\},\ S)$ is a fusion Banach frame for E with respect to $\mathcal{A}$.

Definition 2.2. An OBF $(\{T_n\},\ S)$ for E with respect to $\mathcal{A}$ with frame bounds A and B is called

(1) *tight*, if it is possible to choose $A = B$, satisfying (2.1);
(2) *normalized tight*, if it is possible to choose $A = B = 1$, satisfying (2.1);
(3) *exact*, if for each n_0 there exists no reconstruction operator S_0 such that $(\{T_n\}_{n \neq n_0},\ S_0)$ is an OBF for E.

Definition 2.3. Let E be a Banach space, $\{E_n\}$ be a sequence of Banach spaces and $T_n \in B(E,\ E_n),\ \ n \in \mathbb{N}$. Then $\{T_n\}$ is called *total* on E, if

$$\{f \in E : T_n f = 0, \forall n \in \mathbb{N}\} = \{0\}.$$

The following lemma, proved in [4], will be used in subsequent results.

Lemma 2.4. *Let E be a Banach space, $\{E_n\}$ be a sequence of Banach spaces and $T_n \in B(E,\ E_n),\ n \in I\!N$. If $\{T_n\}$ is total over E, then $\mathcal{A} = \{\{T_n f\} : f \in E\}$ is a Banach space with norm, given by $\|\{T_n f\}\|_{\mathcal{A}} = \|f\|_E,\ f \in E$.*

3. Near exact operator Banach frames

Definition 3.1. An operator Banach frame $(\{T_n\}, S)$ (where $T_n \in B(E, E_n)$, $S : \mathcal{A} \to E$) is said to be near exact if it can be transformed into an exact OBF by deleting a finite number of its elements.

For the existence of near exact OBF, we give following examples:

Example 3.2. Let $E = \ell^1$ and $E_n = \ell^1$ for all $n \in \mathbb{N}$. Define $T_n \in B(E, E_n)$ as

$$\begin{cases} T_1 f = T_2 f = \ldots = T_k f = \{\xi_1, 0, 0, \ldots\} \\ T_n f = \{0, 0, \ldots, \underbrace{\xi_{n-k}}_{(n-k)^{th}-\ position}, \ldots\} \text{ for } n = k+1, k+2, \ldots, \text{ where } f = \{\xi_n\}. \end{cases}$$

Then $T_n f = 0$ for all $n \in I\!N$ implies $\xi_n = 0$ for all $n \in I\!N$. Therefore, by Lemma 2.4, there exist an associated Banach space $\mathcal{A} = \{\{T_n f\} : f \in E\}$ with norm $\|\{T_n f\}\|_{\mathcal{A}} = \|f\|_E$ and a bounded linear operator $S : \mathcal{A} \to E$, defined as $S(\{T_n f\}) = f,\ f \in E$ such that $(\{T_n\}, S)$ is an OBF. Again, by Lemma 2.4, there exists an associated Banach space $\mathcal{A}_0 = \{\{T_n(f)\}_{n \neq 1,2,\ldots k} : f \in E\}$ and a bounded linear operator $U : \mathcal{A}_0 \to E$ such that $(\{T_n(f)\}_{n \neq 1,2,\ldots k}, U)$ is an OBF.

Example 3.3. Let $E = \ell^p$ $(1 \leq p < \infty)$ and $E_n = \ell^p$ for all $n \in \mathbb{N}$. Define $T_n : E \to E_n$ by

$$T_{2n-1} f = T_{2n} f = \{0, 0, \ldots \xi_n \ldots\},\ n \in I\!N. \tag{3.2}$$

Then $T_n f = 0$ gives $f = 0$. Thus, by Lemma 2.4, there exists an associated Banach space $\mathcal{A}$ and a bounded linear operator $S : \mathcal{A} \to E$, defined by $S(\{T_n f\}) = f,\ f \in E$, such that $(\{T_n\}, S)$ is an OBF. Also, by Lemma 2.4, there exists an associated Banach space $\mathcal{A}_0$ and a bounded linear operator S_0 such that $(\{T_{2n-1}\}, S_0)$ is an exact OBF. Hence, $(\{T_n\}, S)$ is not near exact OBF.

In the following result, we give a sufficient condition for constructing a near exact OBF from a given near exact OBF.

Theorem 3.4. *Let E and F be two Banach spaces. Let $(\{T_n\}, S)$ be a near exact OBF for E with respect to E_d. Let $\{R_n\} \subset B(F, E_n)$. If there exists a linear homeomorphism U from E onto F such that $R_n \circ U = T_n,\ n \in I\!N$, then F has a near exact OBF.*

Proof. Since U is onto, for each $g \in F$, there exists an $f \in E$ such that $Uf = g$. Assume that $R_n g = 0$ for all $n \in I\!N$. Then $R_n(Uf) = 0$ for all $n \in I\!N$. This gives $g = 0$. Hence, by Lemma 2.4, there is an associated Banach space $\mathcal{A}_0$ and a bounded linear operator V such that $(\{R_n\}, V)$ is an OBF for F with respect to $\mathcal{A}_0$. Also, $(\{R_n\}, V)$ is a near exact operator Banach frame. If it is not near exact, then there exists a reconstruction operator V_0 such that $(\{R_n\}_{n\neq\sigma(1),\sigma(2),\ldots}, V_0)$ is an exact OBF with bounds A_0 , B_0. Thus, for each $g \in F$, we have

$$A_0\|g\| \leq \|\{R_n g\}_{n\neq\sigma(1),\sigma(2),\ldots}\| \leq B_0\|g\|.$$

This gives

$$A_0\|U(f)\| \leq \|\{R_n U f\}_{n\neq\sigma(1),\sigma(2),\ldots}\| = \|\{T_n f\}_{n\in\sigma(1),\sigma(2),\ldots}\|, \quad f \in E.$$

Now, if $T_n f = 0$ for all $n \neq \sigma(1), \sigma(2), \ldots$, then we have $f = 0$. Thus, there exists a reconstruction operator S_0 such that $(\{T_n\}_{n\neq\sigma(1),\sigma(2),\ldots}, S_0)$ is an OBF which is a contradiction. Hence, $(\{R_n\}, V)$ is a near exact OBF. □

Next, we give a necessary and sufficient condition for a sequence of operators to be a near exact OBF.

Theorem 3.5. *Let $(\{T_n\}, S)$ be a near exact OBF for E with respect to $\mathcal{A}$ with frame bounds A_T and B_T. Let $\{R_n\}$ be a sequence in $B(E, E_n)$ such that $\{R_n f\} \in \mathcal{A}$, $f \in E$ and $Q : \mathcal{A} \to \mathcal{A}$ be a bounded linear invertible operator such that $Q(\{T_n f\}) = \{R_n f\}$, $f \in E$. Then there exists a reconstruction operator P such that $(\{R_i\}, P)$ is a near exact OBF for E with respect to $\mathcal{A}$ if and only if there exists a positive constant γ such that*

$$\|\{R_n f\}\|_{\mathcal{A}} \geq \gamma\|W\{R_n f\}\|_{\mathcal{A}},$$

where $W : \mathcal{A} \to \mathcal{A}$ is a bounded linear operator such that $W(\{R_n f\}) = \{T_n f\}$, $f \in E$.

Proof. Let A_T and B_T be the frame bounds for OBF $(\{T_n\}, S)$. Then

$$\gamma A_T\|f\|_E \leq \gamma\|\{T_n f\}\|_{\mathcal{A}} \leq \|\{R_n f\}\|_{\mathcal{A}} \leq \|Q\|\|\{T_n f\}\|_{\mathcal{A}} \leq B_T\|Q\|\|f\|_E,\ f \in E.$$

Therefore, we obtain

$$\gamma A_T\|f\|_E \leq \|\{R_n f\}\|_{\mathcal{A}} \leq B_T\|Q\|\|f\|_E,\ f \in E.$$

Write $P = SW$. Then $P : \mathcal{A} \to E$ is a bounded linear operator such that

$$P(\{R_n f\}) = f, \quad f \in E.$$

Thus, $(\{R_n\}, P)$ is an OBF for E with respect to $\mathcal{A}$. Now, we shall prove that $(\{R_n\}, P)$ is a near exact OBF. If it is not, then there exists a reconstruction operator P_0 such that $(\{R_n\}_{n\neq\sigma(1),\sigma(2),\ldots}, P_0)$ is an OBF with frame bounds A_0 and B_0. Then, for each $f \in E$, we have

$$A_0\|f\|_E \leq \|\{R_n f\}_{n\neq\sigma(1),\sigma(2),\ldots}\| \leq \|Q\|\|\{T_n f\}_{n\neq\sigma(1),\sigma(2),\ldots}\|.$$

This gives

$$A_0\|Q\|^{-1} \leq \|\{T_n f\}_{n\neq\sigma(1),\sigma(2),\ldots}\|,\ f \in E.$$

Thus, by Lemma 2.4, there exists a reconstruction operator S_0 such that $(\{T_n\}_{n\neq\sigma(1),\sigma(2),\ldots}, S_0)$ is an OBF. This is a contradiction. Hence, $(\{R_n\}, P)$ is a near exact OBF.

Conversely, let $(\{R_n\}, P)$ be a near exact OBF for E with respect to $\mathcal{A}$ and with frame bounds A_R and B_R. Let $U : E \to \mathcal{A}$ be a mapping, defined by $Uf = \{T_n f\}$ and $W = UP$. Then, $W : \mathcal{A} \to \mathcal{A}$ is a bounded linear operator such that

$$W(\{R_n f\}) = \{T_n f\}, \quad f \in E.$$

Thus, there exists a $\gamma > 0$ which depends on bounds of OBF, $(\{T_n\}, S)$ and $(\{R_n\}, P)$ such that

$$\|\{R_n f\}\|_{\mathcal{A}} \geq \gamma \|W(\{R_n f\})\|_{\mathcal{A}}. \qquad \square$$

Next, we show that if two Banach spaces have near exact OBF, then their product space also has a near exact OBF.

Theorem 3.6. *Let $(\{T_n\}, S)$ and $(\{R_n\}, Q)$ be near exact operator Banach frames for Banach spaces E and F with respect to $\mathcal{A}$ and $\mathcal{B}$ respectively. Then there exists a sequence $\{P_n\} \in B(E \times F, Z_n = E_n \times E_n)$, an associated Banach space $\mathcal{C}$, and a reconstruction operator $U : \mathcal{C} \to E \times F$ such that $(\{P_n\}, U)$ is a near exact OBF for $E \times F$ with respect to $\mathcal{C}$.*

Proof. Let $\{P_n\}$ be a sequence in $B(E \times F, Z_n)$ such that

$$P_n(f, g) = (T_n f, R_n g) \quad \text{for all} \quad n \in \mathbb{N}.$$

Now, suppose that

$$P_n(f, g) = 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Then $T_n f = 0 = R_n g = 0$. This gives $(f, g) = 0$. Therefore, there exists an associated Banach space $\mathcal{C} = \{\{P_n\}(f, g) : (f, g) \in E \times F\}$ and a reconstruction operator U such that $(\{P_n\}, U)$ is a near exact OBF for $E \times F$ with respect to $\mathcal{C}$. Now we shall prove that $(\{P_n\}, U)$ is a near exact OBF for $E \times F$. Suppose that $(\{P_n\}, U)$ is not a near exact OBF. Then there exists a reconstruction operator U_0 such that $(\{P_n\}_{n \neq \sigma(1), \sigma(2), \ldots}, U_0)$ is an OBF for $E \times F$ with bounds A_P and B_P. Then, for each $(f, g) \in E \times F$, we have

$$A_P \|(f, g)\| \leq \|\{P_n(f, g)\}_{n \neq \sigma(1), \sigma(2), \ldots}\| = \|\{T_n f, R_n g\}_{n \neq \sigma(1), \sigma(2), \ldots}\| \leq B_P \|(f, g)\|.$$

Now, if $(T_n f, R_n g) = 0$ for all $n \neq \sigma(1), \sigma(2), \ldots$, then $f = 0 = g$. Hence, by Lemma 2.4, there exists reconstruction operators S_0 and Q_0 such that $(\{T_n\}_{n \neq \sigma(1), \sigma(2), \ldots}, S_0)$ and $(\{R_n\}_{n \neq \sigma(1), \sigma(2), \ldots}, S_0)$ are OBF, for E and F. This contradicts the fact that $(\{T_n\}, S)$ and $(\{R_n\}, Q)$ are near exact operator Banach frames. Hence, $(\{P_n\}, U)$ is a near exact OBF for $E \times F$ with respect to $\mathcal{C}$. $\square$

4. Block Perturbation of Operator Banach frames

Let $\{T_n\}$ be a sequence in $B(E, E_n)$ and let $\{m_n\}, \{p_n\}$ be increasing sequence of positive integers such that $m_0 = 0$ and $m_{n-1} + 1 \leq p_n \leq m_n$, for all $n \in \mathbb{N}$.
Define $\{R_n\} \subset B(E, E_n)$, $n \in \mathbb{N}$, by

$$R_k = \begin{cases} T_k, & \text{if } k = p_n, \\ T_{p_n} + U_n, & \text{if } k \neq p_n, \end{cases}$$

where

$$U_n = \sum_{i=m_{n-1}+1}^{p_n-1} \alpha_i T_i + \sum_{i=p_n+1}^{m_n} \alpha_i T_i.$$

Then the sequence $\{R_n\}$ is called block perturbation of $\{T_n\}$.

Theorem 4.1. *Let $(\{T_n\}, S)$ be an OBF for E with respect to $\mathcal{A}$. Let $\{R_n\}$ be a block perturbation of $\{T_n\}$. Then there exists an associated Banach space $\mathcal{B}$ and reconstruction operator S_0 such that $(\{R_n\}, S_0)$ is an OBF for E.*

Proof. Suppose that there exists a non-zero $f \in E$ such that $R_n f = 0$ for all $n \in \mathbb{N}$. Then, from the definition of Block perturbation, it follows that $T_k(f) = 0, \ \ k \neq p_n$ and

$$\begin{aligned} T_{p_n} &= -U_n(f), \ k \neq p_n, \ n \in \mathbb{N} \\ &= -\left\{ \sum_{i=m_{n-1}+1}^{p_n-1} \alpha_i T_i(f) + \sum_{i=p_n+1}^{m_n} \alpha_i T_i(f) \right\} \\ &= 0. \end{aligned}$$

Therefore, $T_k(f) = 0$ for all $k \in \mathbb{N}$. This gives $f = 0$. This is a contradiction. Hence, by Lemma 2.4, there exists an associated Banach space $\mathcal{B}$ and a reconstruction operator S_0 such that $(\{R_n\}, S_0)$ is an OBF. □

5. Application

Finally, we give an application of near exact operator Banach frame related to eigenvalue problem.

Theorem 5.1. *Let $(\{T_n\}, S), (T_n : E \to \ E, \ \ S : \mathcal{A} \to \ E)$ be a near exact operator Banach frame for E with respect to $\mathcal{A}$. Let $\{f_k\}_{k=1}^m \subset E^*$ be a linearly independent set and for each $k(1 \leq k \leq m)$ and let there be an x_k such that*

$$T_n(x_k) = \lim_{l \to \infty} \sum_{\substack{i=1 \\ i \neq n}}^{m_l} \alpha_i^{(l)} T_i(x_k) \quad \text{for all} \quad n \in \mathbb{N},$$

where $\{m_l\}$ is an increasing sequence of positive integers. If, for any arbitrary but fixed $f \in E^*$, $\left(\left\{(f \circ T_n) + \frac{1}{p} \sum_{k=1}^m (\lim_{l \to \infty} \sum_{\substack{i=1 \\ i \neq n}} \alpha_i^{(l)} (f \circ T_i)(x_k))\right\}, U\right)$, where p is any real number, is an operator Banach frame for E with respect to $\mathcal{A}$, then $-p$ is not an eigenvalue of the matrix

$$\begin{pmatrix} f_1(x_1) & f_1(x_2) & \dots & f_1(x_m) \\ f_2(x_1) & f_2(x_2) & \dots & f_2(x_m) \\ \vdots & \dots & \dots & \dots \\ f_m(x_1) & f_m(x_2) & \dots & f_m(x_m) \end{pmatrix}.$$

Proof. The case when $m = 1$ and $m = 2$ are easy to work out. We shall now prove the result for the case when $m = 3$. Suppose that $-p$ is an eigenvalue of the matrix

$$\begin{pmatrix} f_1(x_1) & f_1(x_2) & f_1(x_m) \\ f_2(x_1) & f_2(x_2) & f_2(x_3) \\ f_m(x_1) & f_m(x_2) & f_3(x_3) \end{pmatrix}.$$

Then we obtain

$$\begin{vmatrix} f_1(x_1)+p & f_1(x_2) & f_1(x_m) \\ f_2(x_1) & f_2(x_2)+p & f_2(x_3) \\ f_m(x_1) & f_m(x_2) & f_3(x_3)+p \end{vmatrix} = 0.$$

So, there exist scalars α, β, γ not all zero such that

$$\begin{aligned} \alpha f_1(x_1) + \beta f_1(x_2) + \gamma f_1(x_3) &= -p\alpha, \\ \alpha f_2(x_1) + \beta f_2(x_2) + \gamma f_2(x_3) &= -p\beta, \\ \alpha f_3(x_1) + \beta f_3(x_2) + \gamma f_3(x_3) &= -p\gamma. \end{aligned}$$

Consider a non zero element $x_0 = -p\alpha x_1 - p\beta x_2 - p\gamma x_3$ in E. Then, for any $n \in \mathbb{N}$, we have

$$T_n(x_0) = -p\alpha\left(\lim_{l\to\infty}\sum_{\substack{i=1\\i\neq n}}^{m_l}\alpha_i^{(l)}T_i(x_1)\right) - p\beta\left(\lim_{l\to\infty}\sum_{\substack{i=1\\i\neq n}}^{m_l}\alpha_i^{(l)}T_i(x_2)\right) - p\gamma\left(\lim_{l\to\infty}\sum_{\substack{i=1\\i\neq n}}^{m_l}\alpha_i^{(l)}T_i(x_3)\right).$$

Also, $f_k(x_0) = -p\ \alpha f_k(x_1) - p\ \beta f_k(x_2) - p\ \gamma f_k(x_3)$ for $k = 1, 2, 3$. Therefore, $f_1(x_0) = p^2\alpha, f_2(x_0) = p^2\beta$, and $f_3(x_0) = p^2\gamma$. Thus, we get

$$T_n(x_0) + \frac{1}{p}\sum_{k=1}^{3}\left(\left(\lim_{l\to\infty}\sum_{\substack{i=1\\i\neq n}}^{m_l}\alpha_i^{(l)}f(T_i(x_k))\right)f_k(x_0)\right) = 0.$$

Further, since $\left(\left\{(f\circ T_n) + \frac{1}{p}\sum_{k=1}^{m}(\lim_{l\to\infty}\sum_{\substack{i=1\\i\neq n}}^{m_l}\alpha_i^{(l)}(f\circ T_i)(x_k))\right\}, U\right)$ is an OBF for E, with respect to $\mathcal{A}$, it follows that $x_0 = 0$. This is a contradiction. The other cases can be worked out in a similar fashion. □

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