Poincare Journal of Analysis & Applications
Vol. 2019 (2), 77-83
©Poincare Publishers

ON A NEW KIND OF EINSTEIN WARPED PRODUCT (POLJ)-MANIFOLD

A. PIGAZZINI, C. ÖZEL†, P. LINKER AND S. JAFARI

Date of Receiving	:	02.05.2019
Date of Revision	:	10.08.2019
Date of Acceptance	:	03.09.2019

Abstract. We provide a possible way of constructing a new kind of Einstein warped-product manifold which we will call POLJ-manifold; if we can build the fiber-manifold as a derived-differential-manifold (i.e. the fiber-manifold can admit negative dimension), then we have an Einstein warped product manifold that may have zero dimension ($dimB + dimF = 0$) so the result may be an "invisible"-manifold but made up of two manifolds with 2 and -2 dimension respectively, a kind of "point-manifold" (zero dimension) with "hidden" dimensions. The POLJ-manifold could introduce a new kind of Kaluza-Klein theory with extra negative dimensional spaces, or Kaluza-Klein theory with new kind of time-manifold. In fact POLJ-manifold could describe a new nature of time.

1. Introduction and Preliminaries

The concept of negative dimensional space is already used in linguistic statistics [2]. Also in supersymmetric theories in Quantum Field Theory, negative dimensional spaces were used [3].

Let $E \cong M \times F$ be a fiber bundle with base space M and its fiber F. We will discuss now a case, where the fiber has negative dimension. Note that the total dimension of the fiber bundle is given by the relation $\dim E = \dim M + \dim F$. We will consider the case, where the base manifold has greater positive dimension than the negative dimension of the fiber, i.e. $\dim M > -\dim F$. In this case, the dimension of the total fiber bundle is still positive. Since the base manifold is obtained by projection of the fiber bundle along the fiber by projection operator π_F, we have $\pi_F E = M$ i.e. the projection of the lower-dimensional fiber-bundle along the fiber yields the higher dimensional base manifold space. Therefore, the projection operator π_F along the negative-dimensional fiber, is a suspension operator that raises the dimension of topological spaces.

2010 *Mathematics Subject Classification.* 53C25.
Key words and phrases. Einstein warped product, derived-differentials, negative dimensional manifolds, invisible-manifolds, point-manifolds, derived-manifold.
Communicated by: S. P. Moshokoa
†Corresponding author.

Let $\dim F = -d, d > 0$. Then, the operator π_F will be a d-fold suspension (a single suspension extends a topological space by dimension one). By a slight redefinition of the definition for a fiber bundle to the new mathematical structure, the inverse fiber bundle , we can turn fiber bundles with negative-dimensional fibers into inverse fiber bundles with positive-dimensional fibers. More precisely, we define the inverse fiber bundle as follows: An inverse fiber bundle E^* is locally isomorphic to the space $M^* \times^* F^*$ with some base manifold M^* and a fiber F^*.

However, we have a suspension along the fiber Σ_{F^*} such that $\Sigma_{F^*} E^* = M^*$ instead of a projection along a fiber in an inverse fiber bundle. - Because of the use of suspension instead of projections, the inverse cartesian product $\times^*$ is not the ordinary cartesian product. It can be defined functorially as follows: in the category of fiber bundle topology *Fib*, the ordinary cartesian product is a morphism between some topological spaces (the objects in this category). By defining an inversion functor $\mathscr{F}$ that maps the category *Fib* to the category of inverse fiber bundles *InvFib* by mapping projection operators (are also morphisms) to suspensions $\mathscr{F}(\pi_F) = \Sigma_{F^*}$. Objects remain the same, but on the fiber, the sign of the dimension is changed (negative fibers in *Fib* change to positive ones in *InvFib*). Thus, it holds $\mathscr{F}(\times) = \times^*$. We define this product to be compatible with suspensions instead of projections. The functor also preserves commutative diagrams that are frequently used in the definition of fiber bundles. It is also easy to generalize the above discussion to the case of negative fibers with arbitrary negative dimension. We can treat a fiber bundle with negative-dimensional fiber as an inverse fiber bundle, where the fibers will be positive-dimensional. Cartesian products, however, must be changed to inverse cartesian products. The base manifold in inverse fiber bundles has the highest dimension, since it is obtained by a suspension. Imposing local isomorphisms between an inverse fiber bundle to the inverse cartesian product of base manifold with fiber, we can see that the inverse cartesian product reduces dimensions by some amount. It holds: $\dim E^* = \dim(E^* \times^* F^*) = \dim M^* - \dim F^*$.

The operator $\times^*$ acts similar as the topological quotient, but in such a way that it is compatible with respect to above defined functor. The inverse cartesian product we can also view as a quotient operator obtained by using the $\mathscr{F}$-functor. We can alternatively write: $\times^* = /_{\mathscr{F}}$.

2. Differential Geometry of POLJ-manifolds

Einstein warped product manifolds are mathematical objects that are studied intensively in current research [1, 4]. We will treat an Einstein warped product manifold in this section of the paper.

Definition 2.1. We say that dimension of X is -1 if suspension of X, SX is diffeomorphic to a point. By induction and suspension operation we can define all negative dimensions. Since if X and Y are difeomorphic then their suspensions SX and SY are diffeomorphic. Also a metric on X is compatible with SX. It is also well-known that differentials and suspensions are compatible on manifolds. This means that the differential on the suspension space SX is induced from the differential on X.

Definition 2.2. A warped product manifold $(M, \bar{g}) = (B, g) \times_f (F, \ddot{g})$ (where (B, g) is the base-manifold, $(F, \ddot{g})$ is the fiber-manifold), with $\bar{g} = g + f^2\ddot{g}$, is Einstein if only if:

$$\bar{Ric} = \lambda\bar{g} \Longleftrightarrow \begin{cases} Ric - \frac{m}{f}\nabla^2 f = \lambda g \\ \ddot{Ric} = \mu\ddot{g} \\ f\Delta f + (m-1)|\nabla f|^2 + \lambda f^2 = \mu \end{cases} \tag{2.1}$$

where λ and μ are constants, m is the dimension of F, $\nabla^2 f$, Δf and ∇f are, respectively, the Hessian, the Laplacian and the gradient of f for g, with $f : (B) \to (0, \infty)$ which is a smooth positive function.

Contracting first equation of (1) we get:

$$R_B f^2 - mf\Delta f = nf^2\lambda \tag{2.2}$$

where n and R_B is the dimension and the scalar curvature of B respectively, and from third equation, considering $m \neq 0$ and $m \neq 1$, we have:

$$mf\Delta f + m(m-1)|\nabla f|^2 + m\lambda f^2 = m\mu \tag{2.3}$$

Now from (2) and (3) we obtain:

$$|\nabla f|^2 + [\frac{\lambda(m-n) + R_B}{m(m-1)}]f^2 = \frac{\mu}{(m-1)} \tag{2.4}$$

The following definition of *f-curvature-Base* was considered and introduced by A. Pigazzini [1] as a condition to study the metric of the base-manifold in the Einstein warped product manifolds.

Definition 2.3. Let $(M, \bar{g}) = (B, g) \times_f (F, \ddot{g})$ be an Einstein warped product manifold with $\bar{g} = g + f^2\ddot{g}$. We define the scalar curvature of the base-manifold (B, g) as *f-curvature-Base* (R_{f_B}), if it is a multiple of the warping function f (i.e. $R_{f_B} = cf$ for an arbitrary real constant c).

Case: Einstein warped product manifold Ricci-flat with flat-fiber ($\lambda = \mu = 0$)

In case where $n = 2$, $\mu = 0$ and $R_B = R_{f_B}$, (2) and (3) become:

$$\Delta f - hf^2 = 0 \quad \text{(with } h = c/m.) \tag{2.5}$$

$$f\Delta f + (m-1)|\nabla f|^2 = 0 \tag{2.6}$$

Then (4) becomes:

$$(m-1)|\nabla f|^2 + hf^3 = 0. \tag{2.7}$$

If $h = 0$, then it is easy to see that f must be a constant, so we assume $h \neq 0$. Setting $p = (m-1)$ and $u = -hf$, the above equations (5) and (6) become:

$$\Delta u + u^2 = 0, \tag{2.8}$$

$$u\Delta u + p|\nabla u|^2 = 0 \tag{2.9}$$

Then (7) becomes:

$$p|\nabla u|^2 - u^3 = 0 \tag{2.10}$$

Let g be the metric on B and assume that u is a nonzero (and not necessarily positive) solution to the above system on a simply-connected open subset $B' \subset B$.

The equation (10) implies that $\omega_1 = p^{(1/2)}u^{-3/2}du$ is a 1-form with g-norm 1 on B' and hence g can be written in the form $g = \omega_1^2 + \omega_2^2$ on B' for some ω_2, which is also a unit 1-form. Fix an orientation by requiring that $\omega_1 \wedge \omega_2$ be the g-area form on B', then $\star du = p^{-(1/2)} u^{3/2} \omega_2$ and since $d(\star du) = \Delta u\, \omega_1 \wedge \omega_2$, it follows that:

$$\begin{aligned} \frac{3}{2} u^{1/2} p^{-(1/2)} du \wedge \omega_2 + u^{3/2} p^{-(1/2)} d\omega_2 &= d(p^{-(1/2)} u^{3/2} \omega_2) \\ &= -u^2 \omega_1 \wedge \omega_2 \\ &= -u^{1/2} p^{(1/2)} du \wedge \omega_2 \end{aligned}$$

or $d\omega_2 = -\frac{3}{2} u^{-1} du \wedge \omega_2 - p u^{-1} du \wedge \omega_2$, i.e. $d\omega_2 = (\frac{-3-2p}{2}) u^{-1} du \wedge \omega_2$, which can be written in the form: $d(u^{(\frac{3+2p}{2})} \omega_2) = 0$.

Since B' is simply connected, it follows that there exists a function v on B' such that $\omega_2 = u^{(\frac{-3-2p}{2})} dv$, consequently the metric g has the form:

$$g = \omega_1^2 + \omega_2^2 = p u^{-3} du^2 + u^{-3-2p} dv^2. \tag{2.11}$$

This metric can be placed in polar form by setting $u = 4pr^{-2}$ and $v = (4^{(3+2p)^{1/2}} p^{(3+2p)^{1/2}}) \vartheta$, in which case, we have:

$$g = dr^2 + r^{(6+4p)} d\vartheta^2. \tag{2.12}$$

This is a singular and incomplete metric at $r = 0$, though it is complete at $r = \infty$ and its Gauss curvature is $K = (-6 - 10p - 4p^2) r^{-2}$.

From the initial hypothesis, where we had set $R_B = R_{f_B}$, the Gaussian curvature must be $K = \frac{R_{f_B}}{2} = \frac{cf}{2}$, and remembering $h = \frac{c}{m}$ and $u = -hf = \frac{4p}{r^2}$, then we have:

$$K = (\frac{-6 - 10p - 4p^2}{4p}) u = \frac{cf}{2} = \frac{-m}{2} u. \tag{2.13}$$

Remembering also that $p = (m-1)$, we have: $-6 - 10(m-1) - 4(m-1)^2 = -2m(m-1)$, then $-4m - 2m^2 = 0$ i.e. $m = -2$, and this means that the fiber-manifold has negative dimension ($dimF = -2$).

Then if $dimB = 2$ (the base-manifold is a 2-dimensional manifold) with R_{f_B} (f-*curvature-Base*) and $dimF = -2$, we found that the metric of the base-manifold is the nontrivial metric $g = 3|u^{-3}| du^2 + |u^3| dv^2$, and this means that u will only be negative (see equation (10)) and remembering that $u = -hf$ with $h = \frac{c}{m}$, i.e., $h = -\frac{c}{2}$, we must choose a negative constant c).

Now we set $u = -12r^{-2}$, then $|u| = 12r^{-2}$, and $v = 12^{-\frac{3}{2}} \vartheta$, so it results a metric that in polar form is $g = dr^2 + r^{-6} d\vartheta^2$, which will have negative Gaussian curvature.

Remark 2.4. We point out that the expression of the 1-form ω_2 although it may seem to be complex-values for $p = -3$ but it is actually not. In fact it is sufficient to set u (which we have said to be negative-value) as $u = -s$ for some (real) positive-value function s, and by recalculating we obtain $\omega_1 = 3^{\frac{1}{2}} s^{-\frac{3}{2}} ds$ and $\omega_2 = s^{\frac{3}{2}} dv$.

Definition 2.5. We called POLJ-manifold an Einstein warped product manifold $(M, \bar{g}) = (B, g) \times_f (F, \ddot{g})$, where the scalar curvature of the Riemannian (or pseudo-Riemannian) base-manifold (B, g) is the f-*curvature-Base* (R_{f_B}), and the fiber-manifold $(F, \ddot{g})$ is a derived differential manifold with negative integer dimensions, and these can be uniquely identified by (n, m, g_B, g_F).

Definition 2.6. The points on a derived differential manifold N are morphisms $pt \to N$, which translates to a morphism of $C^\infty(N) \to C^\infty(pt) = \mathbb{R}$ (a morphism in the category of $SC^\infty - rings$ [11] and $\mathbb{R}$ is the set of real numbers equipped with a structure of a $C^\infty - rings$ [12]), which is equivalent with $\pi_0(C^\infty(N)) \to \mathbb{R}$, i.e., a point in the underlying ordinary smooth space of N, where π_0 denotes the set of connected components.

Remark 2.7. With the notation $SC^\infty - rings$, we refer to the category of simplicial $C^\infty - rings$ (for the definition see the reference [11] Sec. 3), which morphisms are used to define the notion of "points" on derived differential manifolds ($C^\infty - rings$ for ordinary manifolds). By the notation $C^\infty - rings$ we refer to smooth algebra, an algebra Ω over the reals $\mathbb{R}$, for which not only the product operation $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ lifts to the algebra product $\Omega \times \Omega \to \Omega$, but for which every smooth map $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ lifts to a map $\Omega(\varphi) : \Omega^n \to \Omega^m$ in a compatible way.

As seen in *Definition 5* (see also *Remarks 2*) points on a derived differential manifold can be considered in usual sense (morphisms), and we will consider them that way. For this reason we will also use the usual Riemannian (or pseudo-Riemannian) metric to describe our fiber-manifold metric.

This metric doesn't take into account the negative dimensions, but we use it to "approximate" the flat-metric on our fiber-manifold.

POLJ metric: In our case we are going to write the metric for a $(2, -2)$-POLJ-manifold, but there could be other POLJ-manifolds suitable for different dimensions of base and fiber. We leave this research open.
Referring to a manifold with two negative dimensions, we denote the pseudo-Riemannian metric of the fiber-manifold with the following notation: $\ddot{g} = (-d\xi^2 - \xi^2 d\varphi^2)_{-2}$.

$$(2, -2) - POLJ = B^2 \times_f S^{-2},$$

where B^2 is our base-manifold, f is our warping function, S^{-2} is double desuspension of a point (negative dimensions) and represents our fiber-manifold.

$$\bar{g}POLJ = g + f^2 \ddot{g}.$$

Recall that from our calculations the warping function is $f = -24r^{-2}c^{-1}$, (then $f^2 = 576r^{-4}c^{-2}$), and the POLJ metric is:

$$ds^2 = dr^2 + r^{-6} d\vartheta^2 - (576c^{-2}r^{-4})(d\xi^2 + \xi^2 d\varphi^2)_{-2}.$$

Conclusions and Remarks

This is a possible way of construction of a new kind of Einstein warped-product manifold. If we are able to construct the fiber-manifold as a derived differential manifold (i.e. the fiber-manifold can admit negative dimension), then we have an Einstein warped product manifold that may have zero dimension ($dimB + dimF = 0$). One can interpret the outcome as an "invisible"- manifold but made up of two manifolds with 2 and -2 dimensions respectively, a kind of "point-manifold" (zero dimension) with "hidden" dimensions. This new kind of manifold, for example, could describe a new nature of time, in fact the negative dimension of the fiber-manifold (being unexplored for an object with positive dimension) could represent the "past" time (in fact this could represent the impossibility of travelling back in the time), the positive dimension of the base-manifold could represent the time "future" (2-dimensional time "future"), while the "present" time which is instantaneous, (because it immediately falls back into the past), could be represented by a point (dimension zero dim(M)=0). It is well-known that

Einstein considered the five dimensional Kaluza-Klein theory in his investigation for a unified field theory, i.e., a unification of gravity and electromagnetism, during the years 1938-1943. But by finding a non-singular particle solution, he discarded this theory as an impossible model for a unified field theory. Since then some other theories have been formulated which are based on Kalazu-Klein's theory [[5],[6], [7]]. In this note we gave some ideas concerning a supersymmetric Kaluza-Klein theory in which POLJ-manifold can be considered as hidden extra 2-time-like dimensions in a theory with more than 10 dimensions. To get rid of massless ghosts we have a 4-dimensional spacetime, a spontaneous compactification of ground state is necessary by assuming that the cosmological constant is zero. Indeed, in Einstein theory the vierbein field of the 4-dimensional spacetime has 10 degrees of freedom. But 5-dimensional Kaluza-Klein theory the metric field has 5 more degrees of freedom. Suppose that ground state is $\Sigma_5 \times S_1$ with a radius of Planck length. Moreover, let us assume that the metric field is independent of the fifth coordinate. The provided action creates a theory of Σ_4 which together with the Brans-Dicke scalar field has 15 degrees of freedom. It is well-known that 11-dimensional supergravity is an important theory among other supergravity theories since eleven is the maximal space-time dimension which does not allow particles with helicity greater than two (see: [8]). Eleven dimensional supergravity has been discarded as it does not provide a realistic model in 4 dimensions. But it is considered as description of low-energy dynamics of M-theory (see: [7]). It is true that the eleven-dimensional super gravity constructed by Cremmer et al. [9], did not consider the chiral phenomenology since it is eleven-dimensional and eleven is odd. But as Wang Mian [10] proposed, one can construct a consistent supergravity theory with an extra hidden time-like dimension. We think that it is possible to construct a consistent theory of supergravity with two hidden extra time-like dimensions, and these two hidden extra time-like dimensions are identified with our POLJ-manifold which we denote by M_{POLJ}. Our M_{POLJ} is a hidden extra time-like manifold, or in other words hidden extra 2-time-like dimensions. In fact the "role" of the negative 2-dimensions of the fiber-manifold is to hide the 2-dimensions of the base-manifold making our M_{POLJ} as a point-manifold (recall that $dimM_{POLJ} = dimB + dimF = 2 - 2 = 0$). For this reason, we call the extra 2-time-like dimensions as "hidden".

Now it is possible to have the topology of vaccum as $\Sigma_{12} = \Sigma_4 \times S_2 \times S_2 \times S_1 \times H^3 \backslash \Gamma$, where Σ_4 is the 4-spacetime with the condition that the cosmological constant is zero. Moreover $H^3 \backslash \Gamma$ is a hyperbolic manifold and Γ is a discrete isometry which is not acting freely. This implies that there is no Killing vectors in this quotient space. This suggests that we have no gauge field which is associated with the hidden extra 2-time-like dimension, and our gravitational model can be written as: $\Sigma_{12} \times M_{POLJ}$.

We believe that the theory can successfully describe a super-Yang-Mills field theory coupled with gravitation after spontaneous compactification.

References

[1] A. Pigazzini, On the (2+2)-Einstein warped product manifolds with f-curvature-base, Ital. J. Pure Appl. Math., 43 (2020).

[2] V. P. Maslov, Negative dimension in general and asymptotic topology, eprint arXiv:math/0612543v1 [math.GM], Dec 2006.

[3] P. Cvitanović, Negative dimensions and E7 symmetry, Nucl. Phys. B, 188(1981), no. 2, 373–396,
[4] Chenxu He et al., On the classification of warped product Einstein metrics, Mathematics Faculty Scholarship, 130 (2010).
[5] M. J. Duff, Kaluza-Klein theory in perspective, in the Proceedings of the Nobel Symposium Oskar Klein Centenary, Stockholm, September 1994, (Ed Lindstrom, World Scientific 1995), hep-th/9410046.
[6] M. J. Duff, An introduction to Kaluza-Klein theories, Ed. by H. C. Lee, World Scientific, 1983.
[7] M. J. Duff, The world in eleven dimensions: supergravity, supermembranes and M-theory, Ed. M. J. Duff, IOP Publishing, 1999.
[8] W. Nahm, Supersymmetries and their representations, Nucl. Phys. B, 135(1978), 149.
[9] E. Cremmer, B. Julia and J. Scherk, Supergravity theory in 11 dimensions, Phys. Lett. B, 76(1978), 409.
[10] W. Mian, Proc. of the Fourth Marcel Grossmann Meeting on General Relativity, Ed. R. Ruffini (1986), p. 1459.
[11] D. Borisov and J. Noel, Simplicial approach to derived differential manifolds, eprint arXiv:1112.0033v1 [math.DG], Nov 2011.
[12] J. Nestruev, Smooth Manifolds and Observables, Springer, 2003.

(Alexander Pigazzini) IT-Impresa srl, 20900 Monza, Italy.
E-mail address: `pigazzinialexander18@gmail.com`

(Cenap Ozel) King Abdulaziz University, Department of Mathematics, 21589 Jeddah KSA.
E-mail address: `cenap.ozel@gmail.com`

(Patrick Linker) Department of Materials Testing, University of Stuttgart, 70569 Stuttgart, Germany.
E-mail address: `Mrpatricklinker@gmail.com`

(Saeid Jafari) College of Vestsjaelland South, Herrestraede 11, 4200 Slagelse, Denmark.
E-mail address: `jafaripersia@gmail.com`