Poincare Journal of Analysis & Applications
Vol. 2019 (2), 85-95

A NOTE ON Λ-BANACH FRAMES AND O-FRAMES

M. P. GOSWAMI[†] AND H. K. PATHAK

Date of Receiving : 06.02.2019
Date of Revision : 29.08.2019
Date of Acceptance : 09.12.2019

Abstract. In this paper, we define and study near-exact Λ-Banach frame for operator spaces with some illustrative examples. A sufficient condition for a Λ-Banach frame to be near-exact has been given. In the sequel, we deal with a Krien-Milman-Rutman type stability result for Λ-Banach frame. Moreover, we demonstrate a method to construct Λ-Banach frames associated with O-frames. Also, some results regarding characterizations of O-frames have been given.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [5] In 1952. Recall that, a sequence $\{x_n\}_{n=1}^{\infty}$ of elements in a Hilbert space H is said to be a frame for H if there exist constants with $0 < A_1 \leq A_2 < \infty$ such that

$$A_1\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq A_2\|x\|^2, \qquad x \in H. \tag{1.1}$$

Nowadays, frames play an important role in many applications in mathematics and engineering. In particular, frames are widely used in wavelet theory, signal processing, wireless sensor network, image processing and many more. In view of theoretical developments, many concepts and notions from Banach spaces and operator theory are used to study frames.

In 1991, Hilbert frame was extended to Banach spaces by Grochenig [7] who introduced the notion of Banach frame in Banach spaces. Since then a number of generalizations and variations of frames in Banach spaces have been introduced and studied by various authors, namely, retro Banach frames [8], fusion Banach frames [10], operator frames [13], operator Banach frames [16] etc. Besides, reconstruction property in the context of frames in Banach spaces was studied in [1] and [11]. In 2015, O. Reinov

---

2010 *Mathematics Subject Classification.* 42C15, 42A38.
Key words and phrases. Banach frame, Λ-Banach frame, O-frame.
The authors are indebted to the anonymous referees for their helpful comments and suggestions toward the improvement of the paper.
Communicated by: L. K. Vashisht
[†]Corresponding author.

[15] introduced the notion of O-frames for an operator as a generalization of Schauder frames. Very recently, Goswami and Pathak [6] studied frames for the space of bounded linear operators and introduced the notion of Λ-Banach frames.

In the present paper, we further study the Λ-Banach frame for operator spaces and define near-exact Λ-Banach frame for operator spaces. The property of near exactness in the context of Banach frame and retro Banach frame was previously studied by Jain et al.[9] and Kumar et al. [12], respectively. We obtain a sufficient condition for the near exactness of the Λ-Banach frame and provide a Krien-Milman-Rutman type stability result for Λ-Banach frame. We construct Λ-Banach frames as the image of a bounded linear operator on the underlying space. Further, we correlate Λ-Banach frame to the O-frame. In fact, we demonstrate a method to construct a Λ-Banach frame associated with O-frame. Also, we deal with some characterizations of O-frames for operators in Banach spaces which are motivated by [1] and [11].

2. Basic definitions and needed results

Throughout this paper, X and Y will denote Banach spaces over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and X^* and Y^* denote the dual spaces of X and Y, respectively. $B(X,Y)$ be the family of bounded linear operators from X into Y and for $\Lambda \in B(X,Y)$ we shall refer $R(\Lambda)$ as the range of Λ. The closed linear span of $\{x_n\}$ in the norm topology of X is denoted by $\overline{span}\{x_n\}$. We define $\mathscr{B}_d = \{\{\Lambda(x_n)\} : \Lambda \in B(X,Y)\}$, a Banach space of vector valued sequences associated with Y.

In order to make the paper self-contained, we list the following definitions.

Definition 2.1. [7] Let X be a Banach space and X_d be an associated Banach space of scalar valued sequences, indexed by $\mathbb{N}$. Let $\{f_n\} \subset X^*$ and $S : X_d \to X$ be given. The pair $(\{f_n\}, S)$ is called a *Banach frame* for X with respect to X_d if

(i) $\{f_n(x)\} \in X_d$, for each $x \in X$,

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that
$$A\|x\|_X \leq \|\{f_n(x)\}\|_{X_d} \leq B\|x\|_X, \qquad x \in X, \tag{2.1}$$

(iii) S is a bounded linear operator such that
$$S(\{f_n(x)\}) = x, \qquad x \in X.$$

The positive constants A and B, respectively, are called *lower* and *upper frame bounds* of the Banach frame $(\{f_n\}, S)$. The operator $S : X_d \to X$ is called the *reconstruction operator* (or, the *pre frame operator*). The inequality (2.1) is called the *frame inequality.*

Definition 2.2. [8] Let X be a Banach space and X^* be its conjugate space. Let $(X^*)_d$ be a Banach space of scalar valued sequences indexed by $\mathbb{N}$ associated with X^*. Let $\{x_n\}$ be a sequence in X and $T : (X^*)_d \to X^*$ be given. A pair $(\{x_n\}, T)$ is called a *retro Banach frame* for X^* with respect to $(X^*)_d$ if

(i) $\{f(x_n)\} \in (X^*)_d$, for each $f \in X^*$

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that
$$A\|f\|_{X^*} \leq \|\{f(x_n)\}\|_{(X^*)_d} \leq B\|f\|_{X^*}, \qquad \text{for all } f \in X^* \tag{2.2}$$

(iii) T is a bounded linear operator such that
$$T(\{f(x_n)\}) = f, \qquad f \in X^*.$$

The positive constants A and B are called, *lower* and *upper frame bounds* of the retro Banach frame $(\{x_n\}, T)$, respectively. The operator $T : (X^*)_d \to X^*$ called the *reconstruction operator* for the retro Banach frame $(\{x_n\}, T)$. The inequality (2.2) is called the *retro frame inequality.*

Definition 2.3 ([6])**.** Let $\{x_n\}$ be a sequence in X, $\Lambda \in B(X, Y)$ and $S : \mathscr{B}_d \to B(X, Y)$ be an operator, where $\mathscr{B}_d$ be a Banach space of vector valued sequences associated with Y. Then $(\{x_n\}, \Lambda, S)$ is called a Λ-*Banach frame* for $B(X, Y)$ with respect to $\mathscr{B}_d$, if

(i) $\{\Lambda(x_n)\} \in \mathscr{B}_d$, $\Lambda \in B(X, Y)$

(ii) there exist constants $0 < A \leq B < \infty$ such that

$$A\|\Lambda\| \leq \|\{\Lambda(x_n)\}\|_{\mathscr{B}_d} \leq B\|\Lambda\|, \qquad \Lambda \in B(X, Y) \tag{2.3}$$

(iii) S is a bounded linear operator such that

$$S(\{\Lambda(x_n)\}) = \Lambda, \qquad \Lambda \in B(X, Y).$$

The positive constants A and B are called lower and upper frame bounds, respectively, for the Λ-Banach frame $(\{x_n\}, \Lambda, S)$. The inequality (2.3) is called the Λ-frame inequality and $S : \mathscr{B}_d \to B(X, Y)$ is called the reconstruction operator.

Recall that, if removal of any one x_k from the Λ-Banach frame renders the collection $\{x_n\}_{n \neq k}$ no longer a Λ-Banach frame for the underlying space, then $(\{x_n\}, \Lambda, S)$ is said to be an *exact* Λ-*Banach frame* for $B(X, Y)$ with respect to $\mathscr{B}_d$. The Λ-Banach frame $(\{x_n\}, \Lambda, S)$ is said to be *tight* if $A = B$, and *normalized tight* if $A = B = 1$. A sequence $\{x_n\}$ in X is said to be *total* over $B(X, Y)$ if

$$\{\Lambda \in B(X, Y) : \Lambda(x_n) = 0, \ \forall n \in \mathbb{N}\} = \{0\}.$$

Definition 2.4. [15] Let X and Y be infinite dimensional separable Banach spaces over the scalar field ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and let $(\{f_n\}, \{y_n\}) \subset X^* \times Y$ and $\Lambda \in B(X, Y)$ be given. Then the pair $(\{f_n\}, \{y_n\})$ is called an O-frame for the operator Λ if

$$\Lambda(x) = \sum_{n=1}^{\infty} f_n(x) y_n, \qquad x \in X, \tag{2.4}$$

where, the series in (2.4) converges in the norm topology of Y.

We finish this section with the following result proved in [6].

Lemma 2.5 ([6])**.** *Let X and Y be Banach spaces and $\Lambda \in B(X, Y)$. If $\{x_n\} \subset X$ is total over $B(X, Y)$, then $\mathscr{B}_d = \{\{\Lambda(x_n)\} : \Lambda \in B(X, Y)\}$ is a Banach space with norm given by $\|\{\Lambda(x_n)\}\|_{\mathscr{B}_d} = \|\Lambda\|$, $\Lambda \in B(X, Y)$.*

3. Near-exact Λ-Banach frame

Let us define a new class of Λ-Banach frames, called near-exact Λ-Banach frame.

Definition 3.1. A Λ-Banach frame $(\{x_n\}, \Lambda, S)$ $(\{x_n\} \in X, \Lambda \in B(X, Y), S : \mathscr{B}_d \to B(X, Y))$ for $B(X, Y)$ is said to be *near-exact* Λ-*Banach frame* for $B(X, Y)$ if, it can be transformed into an exact Λ-Banach frame by omitting a finite number of its elements.

In order to ensure the existence of near-exact Λ-Banach frame, we furnish the following example.

Example 3.2. Let $\{e_n\}_{n\in\mathbb{N}}$ be a sequence of unit vectors in $X = c_0$. Define a sequence $\{x_n\} \subset X$ by

$$\begin{aligned} x_n =& e_1, \ 1 \le n \le k, \\ x_n =& e_{n-k}, \qquad n \ge k+1. \end{aligned}$$

Then, by Lemma 2.5, there exists a reconstruction operator $S : \mathscr{B}_d \to B(X,Y)$ such that $(\{x_n\}, \Lambda, S)$ is a Λ-Banach frame for $B(X,Y)$. Further, since for each $1 \le n \le k$, $x_n \in \overline{span}\{x_i\}_{i\neq n}$, $(\{x_n\}, \Lambda, S)$ is non exact. Again, by Lemma 2.5, there exists a reconstruction operator $U : \mathscr{B}_d \to B(X,Y)$ such that $(\{x_n\}_{n\neq 1,2,3,\ldots,k}, \Lambda, U)$ is an exact Λ-Banach frame for $B(X,Y)$. Hence, $(\{x_n\}, \Lambda, S)$ is a near-exact Λ-Banach frame for $B(X,Y)$.

Example 3.3. Let $X = Y = l^p$ $(1 \le p < \infty)$. Let $\{x_n\}$ be the sequence of unit vectors in X. Define $\{y_n\} \subset X$ by $y_{2n-1} = y_{2n} = x_n$, $n \in \mathbb{N}$. Then by Lemma 2.5, there exists a reconstruction operator $S : \mathscr{B}_d \to B(X,Y)$ such that $(\{y_n\}, \Lambda, S)$ is a Λ-Banach frame for $B(X,Y)$. Further, one may observe that $(\{y_n\}, \Lambda, S)$ is not a near-exact Λ-Banach frame.

We now obtain a necessary and sufficient condition for the exactness of Λ-Banach frame. Lemma 3.4 follows using similar arguments given in the proof of Lemma 3.1 in [8]. For the sake of completeness, here we give the complete proof.

Lemma 3.4. *Let $(\{x_n\}, \Lambda, S)$ (where $\{x_n\} \subset X, \Lambda \in B(X,Y), S : \mathscr{B}_d \to B(X,Y)$) be a Λ-Banach frame for $B(X,Y)$ with respect to $\mathscr{B}_d$. Then $(\{x_n\}, \Lambda, S)$ is exact if and only if $x_n \notin \overline{span}\{x_i\}_{i\neq n}$.*

Proof. Assume that $(\{x_n\}, \Lambda, S)$ is an exact Λ-Banach frame. Thus, we can not find a bounded linear operator $T_0 : \mathscr{B}_d \to B(X,Y)$ such that $(\{x_i\}_{i\neq n}, \Lambda, T_0)$ is a Λ-Banach frame for $B(X,Y)$. Therefore, by Λ-frame inequality $\overline{span}\{x_i\}_{i\neq n} \neq X$. Hence, $x_n \notin \overline{span}\{x_i\}_{i\neq n}$.

For the reverse part, on contrary, assume that $(\{x_n\}, \Lambda, S)$ is not exact, then there exists a reconstruction operator $T_1 : \mathscr{B}_d \to B(X,Y)$ such that $(\{x_i\}_{i\neq n}, \Lambda, T_1)$ is a Λ-Banach frame for $B(X,Y)$. Therefore, by Λ-frame inequality $\overline{span}\{x_i\}_{i\neq n} = X$. This gives, $x_n \in \overline{span}\{x_i\}_{i\neq n}$, a contradiction. □

In the next result, we obtain a sufficient condition for a Λ-Banach frame to be near-exact. This result generalizes Theorem 3.4 of [9].

Theorem 3.5. *Let X^* (the dual of X) be a separable Banach space. A Λ-Banach frame $(\{x_n\}, \Lambda, S)$ (where $\{x_n\} \subset X, \Lambda \in B(X,Y), S : \mathscr{B}_d \to B(X,Y)$) for $B(X,Y)$ is near-exact if for every infinite sequence $\{\alpha(k)\}_{k=1}^{\infty}$ of positive integers*

$$\overline{span}\{x_i\}_{i\neq\alpha(1),\alpha(2),\ldots} \neq X.$$

Proof. Suppose, by omitting a finite number of its elements, that $(\{x_n\}, \Lambda, S)$ is a non-exact Λ-Banach frame. Then, by Lemma 3.4, there exists an index $\alpha(1)$ such that

$$x_{\alpha(1)} \in \overline{span}\{x_i\}_{i\neq\alpha(1)} = X \text{ and } \|x_{\alpha(1)}\| = 1.$$

Since X^* is separable, it follows [see Theorem 3.11 of [14]] that the unit ball B of X is metrizable in the weak topology. Let ϱ be a metric on B. Since $x_{\alpha(1)} \in B \subset X$, there exists an index $n_1 > \alpha(1)$ and scalars $\beta_1^{(1)}, \beta_2^{(1)}, ...\beta_{n_1}^{(1)}$ such that

$$\sum_{i=1}^{n_1} \beta_i^{(1)} x_i \in B \text{ and } \varrho\left(x_{\alpha(1)}, \sum_{i=1, i\neq\alpha(1)}^{n_1} \beta_i^{(1)} x_i\right) < \frac{1}{2}.$$

By assumption $\{x_i\}_{i=n_1+1}^{\infty}$ is not exact. Therefore, by Lemma 3.4, there exists an index $\alpha(2) \geq n_1 + 1$ such that

$$x_{\alpha(2)} \in \overline{span}\{x_i\}_{i=n_1+1, i\neq\alpha(2)}^{\infty} = \overline{span}\{x_i\}_{i=n_1+1}^{\infty} \text{ and } \|x_{\alpha(2)}\| = 1.$$

Then $x_{\alpha(2)} \in B$. Thus there exists an index $n_2 \geq \alpha(2)$ and scalars $\beta_1^{(2)}, \beta_2^{(2)}, ...\beta_{n_2}^{(2)}$ such that

$$\sum_{i=1}^{n_2} \beta_i^{(2)} x_i \in B \text{ and } \varrho\left(x_{\alpha(2)}, \sum_{i=1, i\neq\alpha(1),\alpha(2)}^{n_2} \beta_i^{(2)} x_i\right) < \frac{1}{4}.$$

Since $\overline{span}\{x_i\}_{i\neq\alpha(1),\alpha(2)} = \overline{span}\{x_i\}_{i=1}^{\infty}$, we can choose the index n_2 so that

$$\varrho\left(x_{\alpha(1)}, \sum_{i=1, i\neq\alpha(1),\alpha(2)}^{n_2} \beta_i^{(2)} x_i\right) < \frac{1}{4}.$$

Proceeding in this manner, we obtain infinite sets $\{\alpha(n)\}_{n=1}^{\infty}$ and $\{n_i\}_{i=1}^{\infty}$, with $n_{k-1} + 1 \leq \alpha(k) \leq n_k$ and $n_0 = 0$ and scalars $\beta_i^{(k)}$, for $i, k \in \mathbb{N}$ such that

$$\sum_{i=1}^{n_k} \beta_i^{(k)} x_i \in B \text{ and } \varrho\left(x_{\alpha(j)}, \sum_{i=1, i\neq\alpha(1),\alpha(2),...,\alpha(k)}^{n_k} \beta_i^{(k)} x_i\right) < \frac{1}{2^k}, \ (j = 1, 2, ...k; k \in \mathbb{N}).$$

Thus $x_{\alpha(j)} \in \overline{span}\{x_i\}_{i\neq\alpha(1),\alpha(2),...}$ for all $j \in \mathbb{N}$. Hence $X = \overline{span}\{x_i\}_{i=1}^{\infty} = \overline{span}\{x_i\}_{i\neq\alpha(1),\alpha(2),...}$ □

Remark 3.6. Very recently, Verma et al. [17] studied the concept of finite excess of retro Banach frames. Theorem 3.5 is equivalent to Theorem 2.1 of [17] which was proved for finite excess of retro Banach frames.

We now establish the Krein-Milman-Rutman type stability result for Λ-Banach frames. This result is inspired by [12].

Theorem 3.7. *Let X and Y be separable Banach spaces. Let $(\{x_n\}, \Lambda, S)$ be a non near-exact Λ-Banach frame for $B(X, Y)$. Then, for any sequence of positive numbers $\{\varrho_n\}$, there exists Λ-Banach frame $(\{y_n\}, \Lambda, T)$ for $B(X, Y)$ such that*

$$\|x_n - y_n\| \leq \varrho_n, \qquad \textit{for all } n \in \mathbb{N}.$$

Proof. Since $(\{x_n\}, \Lambda, S)$ is a non near-exact Λ-Banach frame for $B(X, Y)$, therefore for an infinite sequence of indices $\{\alpha_k\}$ we have

$$\overline{span}\{x_i\}_{i\neq\alpha_1,\alpha_2,...} = \overline{span}\{x_n\}.$$

Due to separability of X, we can find a complete sequence $\{z_n\}$ in X with $\|z_n\| = 1$, for all $n \in \mathbb{N}$. Define

$$y_{\alpha_n} = x_{\alpha_n} + \varrho_{\alpha_n} z_n, \qquad n \in \mathbb{N},$$
$$y_{\beta_n} = x_{\beta_n}, \qquad n \in \mathbb{N},$$

where $\{\beta_n\} = \mathbb{N} \setminus \{\alpha_n\}$. Then $\{y_n\} \subset X$ is such that $\|x_n - y_n\| \leq \varrho_n$, for all $n \in \mathbb{N}$.

Now, let $\Lambda \in B(X,Y)$ be such that $\Lambda(y_n) = 0$, for all $n \in \mathbb{N}$. Then, $\Lambda(x_n) = 0$, for all $n \in \mathbb{N}$. Therefore,

$$\varrho_{\alpha_n}\Lambda(z_n) = -\Lambda(x_{\alpha_n}) = 0.$$

This gives $\Lambda(z_n) = 0$, since $\varrho_{\alpha_n} \neq 0$, for all $n \in \mathbb{N}$. Since, $\overline{span}\{z_n\} = X$, $\Lambda = 0$. Thus $\overline{span}\{y_n\} = X$. Therefore, by Lemma 2.5, there exists a reconstruction operator $U : \mathscr{B}_d \to B(X,Y)$ such that $(\{y_n\}, \Lambda, U)$ is a Λ-Banach frame for $B(X,Y)$. $\square$

Next, we construct a Λ-Banach frame using a bounded linear operator on the underlying space with respect to the same associated Banach space of vector valued sequences. This result extend Theorem 3.4 of [4].

Theorem 3.8. *Let $(\{x_n\}, \Lambda, S)$ be a Λ-Banach frame for $B(X,Y)$ with respect to $\mathscr{B}_d$ and let $U \in B(X,X)$ be serjective. Let $\Psi : \mathscr{B}_d \to \mathscr{B}_d$ be a bounded linear operator defined by $\Psi(\{\Lambda(x_n)\}) = \{\Lambda(U(x_n))\}$, $\Lambda \in B(X,Y)$. Then, there exists a reconstruction operator S_U such that $(\{U(x_n)\}, \Lambda, S_U)$ is a Λ-Banach frame for $B(X,Y)$ if and only if*

$$\|\Psi(\{\Lambda(x_n)\})\| \geq \gamma\|\Phi(\{\Lambda(U(x_n))\})\|, \qquad \Lambda \in B(X,Y) \tag{3.1}$$

where $\gamma > 0$ is a constant and $\Phi : \mathscr{B}_d \to \mathscr{B}_d$ is a bounded linear operator given by,

$$\Phi(\{\Lambda(U(x_n))\}) = \{\Lambda(x_n)\}, \qquad \Lambda \in B(X,Y).$$

Proof. Assume that $(\{U(x_n)\}, \Lambda, S_U)$ is a Λ-Banach frame for $B(X,Y)$ with respect to $\mathscr{B}_d$ having A_U and B_U as the corresponding Λ-frame bounds. Then we have the Λ-frame inequality

$$A_U\|\Lambda\| \leq \|\{\Lambda(U(x_n))\}\|_{\mathscr{B}_d} \leq B_U\|\Lambda\|, \qquad \Lambda \in B(X,Y).$$

Let $T : B(X,Y) \to \mathscr{B}_d$ be the analysis operator associated with the Λ-Banach frame $(\{x_n\}, \Lambda, S)$, which is given by

$$T(\Lambda) = \{\Lambda(x_n)\}, \qquad \Lambda \in B(X,Y).$$

Choose $\Phi = TS_U$ and $\gamma = \frac{A_U}{\|T\|} > 0$. Then

$$\begin{aligned}
\|\Psi(\{\Lambda(x_n)\})\| =& \|\{\Lambda(U(x_n))\}\|_{\mathscr{B}_d} \\
\geq& A_U\|\Lambda\| \\
=& \gamma\|T\|\|\Lambda\| \\
\geq& \gamma\|T(\Lambda)\| \\
=& \gamma\|\{\Lambda(x_n)\}\|_{\mathscr{B}_d} \\
=& \gamma\|\Phi(\{\Lambda(U(x_n))\})\|, \qquad \Lambda \in B(X,Y).
\end{aligned}$$

Now, for the reverse part, assume that the inequality (3.1) holds. We compute,

$$\begin{aligned}
\|\{\Lambda(U(x_n))\}\|_{\mathscr{B}_d} = & \|\Psi(\{\Lambda(x_n)\})\| \\
\leq& \|\Psi\|\|\{\Lambda(x_n)\}\|_{\mathscr{B}_d} \\
=& \|\Psi\|\|T(\Lambda)\| \\
\leq& \|\Psi\|\|T\|\|\Lambda\|, \qquad \Lambda \in B(X,Y).
\end{aligned}$$

Now, using Λ-frame inequality for the Λ-Banach frame $(\{x_n\}, \Lambda, S)$ which has the lower bound A (say) and the inequality (3.1), we compute,

$$\begin{aligned}\gamma A\|\Lambda\| \leq &\gamma\|\{\Lambda(x_n)\}\|_{\mathscr{B}_d}\\ =&\gamma\|\Phi(\{\Lambda(U(x_n))\})\|\\ \leq&\|\Psi(\{\Lambda(x_n)\})\|\\ =&\|\{\Lambda(U(x_n))\}\|_{\mathscr{B}_d}.\end{aligned}$$

Hence we get the required Λ-frame inequality as

$$\gamma A\|\Lambda\| \leq \|\{\Lambda(U(x_n))\}\|_{\mathscr{B}_d} \leq \|\Psi\|\|T\|\|\Lambda\|.$$

Put $S_U = S\Phi$. Then $S_U : \mathscr{B}_d \to B(X, Y)$ is a bounded linear operator such that $S_U(\{\Lambda(U(x_n))\}) = \Lambda$, $\Lambda \in B(X, Y)$. Hence, $(\{U(x_n)\}, \Lambda, S_U)$ is a Λ-Banach frame for $B(X, Y)$ with respect to $\mathscr{B}_d$ having Λ-frame bounds γA and $\|\Psi\|\|T\|$.

□

4. O-Frame

In the first result we construct a Λ-Banach frame associated with O-frame for operator spaces.

Theorem 4.1. *Let $\{x_n\} \subset X$, $\{y_n\} \subset Y$, $\{f_n\} \subset X^*$. If $(\{f_n\}, \{y_n\})$ is an O-frame for the operator $\Lambda \in B(X, Y)$. Then there exists a bounded linear operator S such that $(\{x_n\}, \Lambda, S)$ is a Λ-Banach frame for $B(X, Y)$.*

Proof. For each $n \in \mathbb{N}$, define $\Lambda_n : X \to Y$ by

$$\Lambda_n(x) = \sum_{k=1}^{n} f_k(x)y_k, \qquad x \in X.$$

Then,

$$\lim_{n\to\infty} \Lambda_n(x) = \sum_{k=1}^{\infty} f_k(x)y_k = \Lambda(x).$$

Thus, $\sup_{1\leq n<\infty} \|\Lambda_n(x)\| < \infty$, for all $x \in X$. Therefore, by the Banach-Steinhaus Theorem we have, $\sup_{1\leq n<\infty} \|\Lambda_n\| < \infty$. Let us fix $\Lambda \in B(X, Y)$ and compute

$$\begin{aligned}\|\Lambda\| &= \sup_{x\in X, \|x\|\leq 1} \|\Lambda(x)\|\\ &= \sup_{x\in X, \|x\|\leq 1} \|\sum_{k=1}^{\infty} f_k(x)y_k\|\\ &= \sup_{x\in X, \|x\|\leq 1} \|\lim_{n\to\infty}\sum_{k=1}^{n} f_k(x)y_k\|\\ &\leq \sup_{1\leq n<\infty} \|\Lambda_n\|\\ &= \|\{\Lambda(x_n)\}\|_{\mathscr{B}_d}.\end{aligned} \tag{4.1}$$

Moreover, we define the sequence of finite rank operators $P_n : X \to X$ by

$$P_n(x) = \sum_{k=1}^{n} f_k(x)x_k.$$

Then, for all $x \in X$, we have

$$\begin{aligned} \|\sum_{k=1}^{n} f_k(x)\Lambda(x_k)\| \leq& \|\Lambda\| \|\sum_{k=1}^{n} f_k(x)x_k\| \\ =& \|\Lambda\| \|P_n(x)\| \\ \leq& \|\Lambda\| B\|x\|, \end{aligned}$$

where $B = \sup \|P_n\|$. This gives,

$$\sup_n \|\sum_{k=1}^{n} f_k(x)\Lambda(x_k)\| \leq B\|\Lambda\|\|x\|.$$

Hence, for all $\Lambda \in B(X, Y)$ we have,

$$\|\{\Lambda(x_k)\}\|_{\mathscr{B}_d} = \sup_n \sup_{x \in X, \|x\| \leq 1} \|\sum_{k=1}^{n} f_k(x)\Lambda(x_k)\| \leq B\|\Lambda\|. \tag{4.2}$$

Hence, by (4.1) and (4.2) we get the required Λ-frame inequality for $A = 1$,

$$A\|\Lambda\| \leq \|\{\Lambda(x)\}\|_{\mathscr{B}_d} \leq B\|\Lambda\|, \qquad \Lambda \in B(X, Y).$$

□

We now give the following result regarding characterizations of O-frames in Banach spaces.

Theorem 4.2. *Let $(\{f_n\}, \{x_n\}) \subset X^* \times X$ be an O-frame for $\Lambda \in B(X)$ and let $\{g_n\} \in X^*$ be given. If*

$$\sum_{n=1}^{\infty} \|g_n - f_n\|\|x_n\| < \infty, \tag{4.3}$$

then $\Theta : R(\Lambda) \to X$ given by

$$\Theta(\Lambda x) = \sum_{n=1}^{\infty} (g_n - f_n)(x)x_n, \qquad x \in X,$$

defines a compact linear operator on $R(\Lambda)$.

Proof. For $n \geq m$ we have

$$\begin{aligned} \left\| \sum_{k=m}^{n} (g_k - f_k)(x)x_k \right\| \leq& \sum_{k=m}^{n} |(g_k - f_k)(x)|\|x_k\| \\ \leq& \left(\sum_{k=m}^{n} \|g_k - f_k\|\|x_k\| \right) \|x\|. \end{aligned}$$

Therefore, $\Theta : R(\Lambda) \to Y$ is a well defined bounded linear operator on $R(\Lambda)$. For each $n \in \mathbb{N}$, define $\Theta_n : R(\Lambda) \to X$ by

$$\Theta_n(\Lambda x) = \sum_{k=1}^{n} (g_k - f_k)(x)x_k, \qquad n \in \mathbb{N}.$$

Then we have,

$$\|\Theta - \Theta_n\| \leq \sum_{k=n+1}^{\infty} \|g_k - f_k\| \|x_k\| \to 0, \text{ as } n \to \infty.$$

Since each Θ_n is a finite dimensional and continuous operator, Θ is compact. □

In 2008, Casazza and Christensen [1] introduced the reconstruction property in Banach spaces and discussed some theoretical consequences related to the reconstruction property. In this direction, they proved the following

Proposition 4.3 ([1])**.** If $\{f_i, f_i^*\}$ (where $\{f_i\} \subset X$, $\{f_i^*\} \subset X^*$) has the reconstruction property for X, then $\overline{span}\{f_i^*\}$ norms X.

Motivated by Casazza and Christensen [1], we give the following result as a theoretical consequence related to the O-frame. In this regard, we first recall the following

Definition 4.4. Let $\Lambda : X \to Y$ be a bounded linear operator mapping from X onto Y. We say that a subspace $M \subset Y^*$ norms Y if there is a constant $A > 0$ so that for all $x \in X$ we have

$$A\|\Lambda(x)\| \leq \sup_{\|g\|=1, g \in M} |g(\Lambda(x))|.$$

Theorem 4.5. *Let $\Lambda : X \to X$ be a bounded linear operator on X. If Λ has an O-frame $(\{f_n\}, \{\xi_n\})$ (where $\{f_n\} \subset X^*$, $\{\xi_n\} \subset X$) then $\overline{span}\{f_n\}_{n=1}^{\infty} = M \subset X^*$ norms X.*

Proof. For all $x \in X$, we have

$$\Lambda(x) = \sum_{i=1}^{\infty} f_i(x)\xi_i.$$

It follows that the finite rank operators

$$\sigma_n(\Lambda x) = \sum_{i=1}^{n} f_i(x)\xi_i, \qquad n \in \mathbb{N},$$

are pointwise bounded on $R(\Lambda)$. By the uniform boundedness principle, there exists a constant K so that for all n and all $x \in X$,

$$\|\sum_{i=1}^{n} f_i(x)\xi_i\| \leq K\|\Lambda x\|.$$

Now for every $g \in X^*$, we have

$$\begin{aligned}\|\sum_{i=1}^{n} g(\xi_i)f_i\| &= \sup_{\|x\|=1} |(\sum_{i=1}^{n} g(\xi_i)f_i)x| \\ &= \sup_{\|x\|=1} |\sum_{i=1}^{n} g(\xi_i)f_i(x)| \\ &= \sup_{\|x\|=1} |g(\sum_{i=1}^{n} f_i(x)\xi_i)| \\ &\leq \|g\| \sup_{\|x\|=1} \|\sum_{i=1}^{n} f_i(x)\xi_i\| \\ &\leq K\|g\|\|\Lambda x\|.\end{aligned}$$

Now, choose $g \in X^*$ so that $\|g\| = 1$ and $\|\Lambda x\| \leq (1+\varepsilon)|g(\Lambda x)|$. Then, for sufficiently large n, we have

$$\begin{aligned}\|\Lambda x\| &\leq (1+\varepsilon)|g(\Lambda x)| \\ &\leq (1+\varepsilon)|\sum_{i=1}^{\infty} f_i(x)g(\xi_i)| \\ &\leq (1+\varepsilon)^2|\sum_{i=1}^{n} f_i(x)g(\xi_i)| \\ &\leq (1+\varepsilon)^2|(\sum_{i=1}^{n} g(\xi_i)f_i)(x)|.\end{aligned}$$

Since $\{\sum_{i=1}^{n} g(\xi_i)f_i\} \subset M$ and since the norms of these vectors are uniformly bounded it follows that the space M norms $R(\Lambda)$. $\square$

In the support of Theorem 4.5, we furnish the following example.

Example 4.6. Let $X = c_0$. Define sequences $\{f_n\} \subset X^*$ and $\{\xi_n\} \subset X$ by $f_n(x) = x_n$, $x = \{x_j\} \in X$, and $\xi_n = e_{n+1}$, $n \in \mathbb{N}$. Define $\Lambda : X \to X$ by

$$\Lambda(x) = \{0, x_1, x_2, ...\}, \qquad x = \{x_j\} \in X.$$

Then $\Lambda \in B(X)$ and for every $x \in X$ we have

$$\Lambda(x) = \sum_{n=1}^{\infty} f_n(x)\xi_n.$$

Hence, $(\{f_n\}, \{\xi_n\})$ is an O-frame for X.

Moreover, let $M = \overline{[f_n]} \subset X^*$. Then, we obtain

$$\|\Lambda(x)\| = \sup_{\|g\|=1, g\in M} |g(\Lambda(x))|.$$

Consequently, for $0 < A \leq 1$, we see that M norms X.

Remark 4.7. Converse of Theorem 4.5 is not generally true. Indeed, in Example 4.6, if we define $\{\xi_n\} \subset X$, by $\xi_n = e_n$, $n \in \mathbb{N}$, then we can see that $(\{f_n\}, \{\xi_n\})$ is not an O-frame for X. However, $M = \overline{[f_n]} \subset X^*$ norms X.

References

[1] P. G. Casazza and O. Christensen, The reconstruction property in Banach spaces and a perturbation theorem, Canad. Math. Bull., 51 (2008), 348-358.

[2] O. Christensen, An introduction to frames and Riesz bases, Birkhäuser, Boston, 2003.

[3] O. Christensen, Frames and bases: an introductory course, Birkhäuser, Boston, 2008.

[4] R. Chugh, M. Singh and L. K. Vashisht, Shadow of operators, TWMS J. App. Eng. Math., 5(2015), no. 1, 132-144.

[5] R. J. Duffin and A. C. Scheaffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., 72(1952), 341-366.

[6] M. P. Goswami and H. K. Pathak, Some results on Λ-Banach frames for operator spaces, Jordan J. Math. Stat., 11(2018), no. 2, 169-194.

[7] K. Grochenig, Describing functions: atomic decompositions versus frames, Monatsh. Math., 112(1991), 1-41.

[8] P. K. Jain, S. K. Kaushik and L. K. Vashisht, Banach frames for conjugate Banach spaces, Z. Ana. Anwendungen, 23(2004), no. 4, 713-720.

[9] P. K. Jain, S. K. Kaushik and Nisha Gupta, On near-exact Banach frames in Banach spaces, Bull. Austral. Math. Soc., 78(2008), no. 2, 335-342.

[10] P. K. Jain, S. K. Kaushik and Varinder Kumar, Frames of subspaces for Banach spaces, Int. J. Wavelets, Multiresolut. Inf. Process., 8(2010), no. 2, 243-252.

[11] S. K. Kaushik, L. K. Vashisht and G. Khattar, Reconstruction Property and Frames in Banach Spaces, Palest. J. Math., 3(2014), no. 1, 11-26.

[12] R. Kumar and S. K. Sharma, A note on retro Banach frames, Int. J. Pure Appl. Math., 48(2008), no. 1, 111-115.

[13] C.-Y. Li, Operator frames for Banach spaces, Complex Anal. Oper. Theory, 6(2012), 1-21.

[14] T. J. Morrison, Functional analysis: an introduction to Banach space theory, John Wiley and Sons, New York, 2000.

[15] O. Reinov, O-frames for operators in Banach spaces, Poincare J. Anal. Appl., 2015(2015), no. 2, 65-75.

[16] C. Shekhar, Operator Banach frames in Banach spaces, J. Math. Anal., 6(2015), no. 3, 17-26.

[17] G. Verma, L. K. Vashisht and M. Singh, On excess of retro Banach frames, Izv. Nats. Akad. Nauk Armenii Mat., 54(2019), no. 3, 87–92; reprinted in J. Contemp. Math. Anal., 54(2019), no. 3, 143-146.

School of Studies in Mathematics, Pt. Ravishankar Shukla University, Raipur (C.G.) 492010 India.

E-mail address, M. P. Goswami: `mayurpuri89@gmail.com`

E-mail address, H. K. Pathak: `hkpathak05@gmail.com`