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# ON $|K^\lambda|$ SUMMABILITY OF ORTHOGONAL SERIES

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***Abstract***. In this paper we have proved two theorems pertaining to $|K^\lambda|$ summability of orthogonal series.

## 1. Introduction

Let $\{\psi_n(x)\}$ be an orthonormal system defined in the interval $(a, b)$. We assume that $f(x)$ belongs to $L^2(a, b)$ and

$$f(x) \sim \sum_{n=0}^{\infty} c_n \psi_n(x), \tag{1.1}$$

where $c_n = \int_a^b f(x)\psi_n(x)dx$, $(n = 0, 1, 2, \dots)$. By The Riesz–Fischer theorem, for the existence of the function $f$ such that $c_n = \int_a^b f(x)\psi_n(x)dx$ for every $n$, a necessary and sufficient condition is the convergence of the series $\sum a_n^2$.

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with its partial sums $s_n$ and let $\sigma_n^\alpha$ denotes the $n$th Cesàro mean of order $\alpha$ (see [4]) such that

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty. \tag{1.2}$$

A series $\sum_{n=0}^{\infty} a_n$ is said to be absolutely summable by Cesàro method of $\alpha > -1$ and index $k \geq 1$, symbolically

$$\sum_{n=0}^{\infty} a_n \in |C, \alpha|, \quad (\alpha > -1, k \geq 1),$$

if (1.2) holds.

Let $p_n$ be a sequence of positive numbers such that

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$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{and} \quad \frac{p_n}{P_n} \to 0 \quad \text{as} \quad n \to \infty.$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \tag{1.3}$$

defines the sequence $\{t_n\}$ of the Nörlund means of the sequence $\{s_n\}$, generated by the sequence of the coefficients $\{p_n\}$, where the sequence $\{s_n\}$ is a sequence of partial sums of $\sum_{n=0}^{\infty} a_n$.

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|N, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{1.4}$$

Among others, Y. Okuyama [8] concerning the $|N, p_n|_k$, $1 \leq k \leq 2$, summability of orthogonal series (1.1) proved the following theorem.

**Theorem 1.1.** *Let $1 \leq k \leq 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. If $\{p_n\}$ is a positive non-increasing sequence and the series $\sum_{n=1}^{\infty} |c_n|^2 \Omega^{\frac{2}{k}-1}(n) w^{(k)}(n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty} c_n \psi_n(x)$ is $|N, p_n|_k$ summable almost everywhere, where*

$$w^{(k)}(j) = j^{-1} \sum_{n=j}^{\infty} \frac{n^{\frac{2}{k}} p_n^{\frac{2}{k}} p_{n-j}^2}{P_n^{2+\frac{2}{k}}} \left( \frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right)^2.$$

In fact, Theorem 1.1 generalize a theorem presented in [7].

For $n = 0, 1, 2, \ldots$, define the numbers $\begin{bmatrix} n \\ k \end{bmatrix}$ for $0 \leq k \leq n$ by

$$\prod_{v=0}^{n-1} (x+v) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \quad x > 0, \tag{1.5}$$

where

$$\prod_{v=0}^{n-1} (x+v) = x(x+1)\ldots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}.$$

It is obvious that $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ whenever $k < 1$ and $k > n$. We are going to use the convention that $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$. The numbers $\begin{bmatrix} n \\ k \end{bmatrix}$ are knwon as Stirling's numbers of first kind. It is knwon that these numbers satisfy the following relation

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \tag{1.6}$$

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with its sequence $\{s_n\}$ of partial sums whose general term is $s := s_n = \sum_{k=0}^{n} a_k$. Let $\lambda > 0$ be a real number, the $K^{\lambda}$-mean $T_n$ of the sequence

$\{s_n\}$ is defined by ([5])

$$T_n(s) = \frac{\Gamma(\lambda)}{\Gamma(\lambda+n)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k s_k.$$

If $\lim_{n\to\infty} T_n = s$, then we say that sequence $\{s_n\}$ (or the series $\sum_{n=0}^{\infty} a_n$) is summable $K^\lambda$ to $s$.

The $K^\lambda$-summability is a regular method only for $\lambda > 0$ and this is why $\lambda$ is assumed to be positive. The particular case $\lambda = 1$ implies $K^1$-summability which has been reintroduced by A. V. Lototsky in [6].

The series $\sum_{n=0}^{\infty} a_n$ (or the sequence $\{s_n\}$) is said to be absolutely $K^\lambda$-summable or $|K^\lambda|$-summable (see [3], [9]) if $\{T_n\}$ is of bounded variation sequence i.e.,

$$\sum_{n=0}^{\infty} |T_n - T_{n-1}| < \infty.$$

It is of great importance to note here, it has been proved in [3] on page 37, that $|K^\lambda|$-method is absolutely conservative; that is $|C, 0| \subset |K^\lambda|$; $\lambda > 0$, where $|C, 0|$ denotes the absolute convergence.

The main aim of the present paper is to prove an analogous theorem with Theorem 1.1 finding conditions under which the orthogonal series (1.1) will be absolutely conservative $K^\lambda$-summable.

The following lemma due to B. Levi is often used in the theory of functions. It will help us to prove main results.

**Lemma 1.2** ([2], p. 11)**.** *If $h_n(t) \in L(U)$ are non-negative functions and*

$$\sum_{n=1}^{\infty} \int_U h_n(t)dt < \infty, \tag{1.7}$$

*then the series $\sum_{n=1}^{\infty} h_n(t)$ converges (absolutely) almost everywhere on $U$ to a function $h(t) \in L(U)$.*

Throughout this paper $K$ denotes a positive constant that it may depends only on $k$, and be different in different relations.

## 2. Main results

We prove the following two theorems.

**Theorem 2.1.** *If the series*

$$\sum_{n=1}^{\infty} \left( \sum_{j=0}^{n} |\widehat{C}_{n,j}(\lambda)|^2 |c_j|^2 \right)^{\frac{1}{2}}$$

*converges, where $\lambda > 0$, $j = 0, 1, 2, \ldots n$,*

$$\widehat{C}_{n,j}(\lambda) := \overline{C}_{n,j}(\lambda) - \overline{C}_{n-1,j}(\lambda), \quad \textit{and} \quad \overline{C}_{n,j}(\lambda) := \frac{\Gamma(\lambda)}{\Gamma(\lambda+n)} \sum_{k=j}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k,$$

*then the orthogonal series*

$$\sum_{n=0}^{\infty} c_n \psi_n(x) \tag{2.1}$$

*is absolutely conservative $K^\lambda$-summable almost everywhere.*

*Proof.* For the $T_n(s)(x)$ mean of the partial sums of the orthogonal series $\sum_{n=0}^{\infty} c_n \psi_n(x)$ we have

$$\begin{aligned} T_n(s)(x) &= \frac{\Gamma(\lambda)}{\Gamma(\lambda+n)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k s_k(x) \\ &= \frac{\Gamma(\lambda)}{\Gamma(\lambda+n)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \sum_{j=0}^{k} c_j \psi_j(x) \\ &= \frac{\Gamma(\lambda)}{\Gamma(\lambda+n)} \sum_{j=0}^{n} c_j \psi_j(x) \sum_{k=j}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k = \sum_{j=0}^{n} \overline{C}_{n,j}(\lambda) c_j \psi_j(x) \end{aligned}$$

where

$$\overline{C}_{n,j}(\lambda) := \frac{\Gamma(\lambda)}{\Gamma(\lambda+n)} \sum_{k=j}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k.$$

Whence,

$$\begin{aligned} T_n(s)(x) - T_{n-1}(s)(x) &= \sum_{j=0}^{n} \overline{C}_{nj}(\lambda) c_j \psi_j(x) - \sum_{j=0}^{n-1} \overline{C}_{n-1,j}(\lambda) c_j \psi_j(x) \\ &= \overline{C}_{n,n}(\lambda) c_n \psi_n(x) + \sum_{j=0}^{n-1} \left( \overline{C}_{n,j}(\lambda) - \overline{C}_{n-1,j}(\lambda) \right) c_j \psi_j(x) \\ &= \widehat{C}_{n,n}(\lambda) c_n \psi_n(x) + \sum_{j=0}^{n-1} \widehat{C}_{n,j}(\lambda) c_j \psi_j(x) \\ &= \sum_{j=0}^{n} \widehat{C}_{n,j}(\lambda) c_j \psi_j(x). \end{aligned}$$

Using Cauchy-Schwartz inequality and the property of orthogonality, we have

$$\begin{aligned} \int_a^b |T_n(s)(x) - T_{n-1}(s)(x)|\, dx &\le \sqrt{b-a} \left( \int_a^b |T_n(s)(x) - T_{n-1}(s)(x)|^2\, dx \right)^{\frac{1}{2}} \\ &= \sqrt{b-a} \left( \int_a^b \left| \sum_{j=0}^{n} \widehat{C}_{n,j}(\lambda) c_j \psi_j(x) \right|^2 dx \right)^{\frac{1}{2}} \\ &= \sqrt{b-a} \left( \sum_{j=0}^{n} |\widehat{C}_{n,j}(\lambda)|^2 |c_j|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Subsequently, the series

$$\sum_{n=1}^{\infty}\int_a^b |T_n(s)(x)-T_{n-1}(s)(x)|\,dx \le \sqrt{b-a}\sum_{n=1}^{\infty}\left(\sum_{j=0}^{n}|\widehat{C}_{n,j}(\lambda)|^2|c_j|^2\right)^{\frac{1}{2}} \tag{2.2}$$

converges by the assumption.

Now, according to the Lemma 1.2 the proof of the theorem is completed. □

Let us consider the special case $\lambda = 1$, in Theorem 2.1. Namely, using the identity (see [1], page 106, or just put $x = 1$ in (1.5))

$$\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix} = n!,$$

we obtain

$$\begin{aligned}
\widehat{C}_{n,j} &= \frac{1}{n!}\sum_{k=j}^{n}\begin{bmatrix} n\\ k\end{bmatrix} - \frac{1}{(n-1)!}\sum_{k=j}^{n-1}\begin{bmatrix} n-1\\ k\end{bmatrix}\\
&= \frac{1}{n!}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix} - \sum_{k=0}^{j-1}\begin{bmatrix} n\\ k\end{bmatrix}\right) - \frac{1}{(n-1)!}\left(\sum_{k=0}^{n-1}\begin{bmatrix} n-1\\ k\end{bmatrix} - \sum_{k=0}^{j-1}\begin{bmatrix} n-1\\ k\end{bmatrix}\right)\\
&\overset{(1.6)}{=} \frac{1}{n!}\left(n! - \sum_{k=0}^{j-1}\begin{bmatrix} n\\ k\end{bmatrix}\right) - \frac{1}{(n-1)!}\left((n-1)! - \sum_{k=0}^{j-1}\begin{bmatrix} n-1\\ k\end{bmatrix}\right)\\
&= \frac{1}{n!}\sum_{k=0}^{j-1}\left(n\begin{bmatrix} n-1\\ k\end{bmatrix} - \begin{bmatrix} n\\ k\end{bmatrix}\right)\\
&= \frac{1}{n!}\sum_{k=0}^{j-1}\left((n-1)\begin{bmatrix} n-1\\ k\end{bmatrix} - \begin{bmatrix} n\\ k\end{bmatrix} + \begin{bmatrix} n-1\\ k\end{bmatrix}\right)\\
&\overset{(1.6)}{=} \frac{1}{n!}\sum_{k=0}^{j-1}\left(\begin{bmatrix} n-1\\ k\end{bmatrix} - \begin{bmatrix} n-1\\ k-1\end{bmatrix}\right)\\
&= \frac{1}{n!}\begin{bmatrix} n-1\\ j-1\end{bmatrix}; \quad \text{by convention} \quad \begin{bmatrix} n-1\\ -1\end{bmatrix} := 0.
\end{aligned}$$

This equality implies an corollary in which imposed condition is simpler and which ensure that the series (2.1) will be absolute conservative $K^1$-summable almost everywhere.

**Corollary 2.2.** *If the series*

$$\sum_{n=1}^{\infty}\frac{1}{n!}\left(\sum_{j=0}^{n}\left|\begin{bmatrix} n-1\\ j-1\end{bmatrix}\right|^2|c_j|^2\right)^{\frac{1}{2}}$$

*converges, then the orthogonal series (2.1) is absolutely conservative $K^1$-summable almost everywhere.*

If we put

$$w(j;\lambda) := \frac{1}{j}\sum_{n=j}^{\infty} n^2 |\widehat{C}_{n,j}(\lambda)|^2 \tag{2.3}$$

then the following theorem holds true.

**Theorem 2.3.** *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$ converges. If the series $\sum_{j=1}^{\infty}|c_j|^2\Omega(j)w(j;\lambda)$ converges, then the orthogonal series (2.1) is absolutely conservative $K^{\lambda}$-summable almost everywhere, where $w(j;\lambda)$ is defined by (2.3).*

*Proof.* Using (2.2) and applying Cauchy-Schwartz inequality, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\int_a^b |T_n(s)(x) - T_{n-1}(s)(x)|\,dx \\
&\leq K\left(\sum_{j=0}^{n}|\widehat{C}_{n,j}(\lambda)|^2|c_j|^2\right)^{\frac{1}{2}} \\
&= K\sum_{n=1}^{\infty}\frac{1}{(n\Omega(n))^{\frac{1}{2}}}\left(n\Omega(n)\sum_{j=0}^{n}|\widehat{C}_{n,j}(\lambda)|^2|c_j|^2\right)^{\frac{1}{2}} \\
&\leq K\left(\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}\right)^{\frac{1}{2}}\left[\sum_{n=1}^{\infty}(n\Omega(n))\sum_{j=0}^{n}|\widehat{C}_{n,j}(\lambda)|^2|c_j|^2\right]^{\frac{k}{2}} \\
&\leq K\left\{\sum_{j=1}^{\infty}|c_j|^2\sum_{n=j}^{\infty}\left(\frac{\Omega(n)}{n}\right)n^2|\widehat{C}_{n,j}(\lambda)|^2\right\}^{\frac{1}{2}} \\
&\leq K\left\{\sum_{j=1}^{\infty}|c_j|^2\left(\frac{\Omega(j)}{j}\right)\sum_{n=j}^{\infty}n^2|\widehat{C}_{n,j}(\lambda)|^2\right\}^{\frac{1}{2}} \\
&= K\left\{\sum_{j=1}^{\infty}|c_j|^2\Omega(j)w(j;\lambda)\right\}^{\frac{1}{2}},
\end{aligned}$$

which is finite by virtue of the hypothesis of the theorem, and this completes the proof of the theorem. $\square$

We have showed above that for $\lambda = 1$ the equality $\widehat{C}_{n,j} = \frac{1}{n!}\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}$ hods true, and thus Theorem 2.3 reduces to the following simpler statement.

**Corollary 2.4.** *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$ converges. If the series $\sum_{j=1}^{\infty}|c_j|^2\Omega(j)w(j) < \infty$, then the orthogonal series (2.1) is absolutely conservative $K^1$-summable almost everywhere, where $w(j)$ is defined by $w(j) := \frac{1}{j}\sum_{n=j}^{\infty}\left(\frac{1}{(n-1)!}\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}\right)^2$.*

Let $E_n^{(2)}(f)$ denotes the best approximation of $f$ in the metric $L^2$ by means of polynomials $\sum_{j=0}^{n-1} c_j\psi_j(x)$, i.e., $E_n^{(2)}(f) = \left(\sum_{j=n}^{\infty} |c_j|^2\right)^{\frac{1}{2}}$. Then, assuming that $\Omega(0) = 0$ and $w(0;\lambda) = 0$, we have

$$\begin{aligned}\sum_{j=1}^{\infty} |c_j|^2\Omega(j)w(j;\lambda) &= \sum_{j=1}^{\infty} |c_j|^2 \sum_{i=1}^{j} \Delta\left(\Omega(i)w(i;\lambda)\right) \\ &= \sum_{i=1}^{\infty} \Delta\left(\Omega(i)w(i;\lambda)\right) \sum_{j=i}^{\infty} |c_j|^2 \\ &= \sum_{i=1}^{\infty} \Delta\left(\Omega(i)w(i;\lambda)\right) (E_i^{(2)}(f))^2,\end{aligned}$$

where

$$\Delta\left(d_s\right) := d_s - d_{s+1},\ (s = 1, 2, \dots).$$

The above equality enable us to reformulate Theorem 2.3 and Corollary 2.4 in their equivalent form.

**Theorem 2.5.** *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. If the series $\sum_{j=1}^{\infty} \Delta\left(\Omega(j)w(j;\lambda)\right) (E_j^{(2)}(f))^2$ converges, then the orthogonal series (2.1) is absolutely conservative $K^\lambda$-summable almost everywhere, where $w(j;\lambda)$ is defined by (2.3).*

**Corollary 2.6.** *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. If the series $\sum_{j=1}^{\infty} \Delta\left(\Omega(j)w(j)\right) (E_j^{(2)}(f))^2$, then the orthogonal series (2.1) is absolutely conservative $K^1$-summable almost everywhere, where $w(j)$ is defined by $w(j) := \frac{1}{j}\sum_{n=j}^{\infty} \left|\frac{1}{(n-1)!}\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}\right|^2$.*

**Remark 2.7.** We have noted at the beginning of this study that $|C, 0| \subset |K^\lambda|$, from which we conclude that conditions imposed in our results ensure the absolute convergence of orthogonal series (2.1).

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