Poincare Journal of Analysis & Applications
Vol. 2019 (2), 105-114

NEW INEQUALITIES FOR *AH*-CONVEX FUNCTIONS USING BETA AND HYPERGEOMETRIC FUNCTIONS

HURIYE KADAKAL[†] AND KERIM BEKAR

Date of Receiving	:	16.05.2019
Date of Revision	:	30.07.2019
Date of Acceptance	:	11.09.2019

Abstract. In this manuscript, we obtain some new general inequalities in the form of Hermite-Hadamard type and Bullen type for functions whose third derivatives in absolute value at certain power are arithmetically-harmonically convex via beta and hypergeometric functions. Some applications to special means of real numbers are also given.

1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \varnothing$. This definition is well known in the literature.

Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. Many papers have been written by researchers concerning inequalities for different classes of convex functions see for instance the recent papers [1, 3, 4, 6, 7, 8, 14, 13, 16] with the other references mentioned therein.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

holds. The inequality (1.1) is known in the literature as Hermite-Hadamard integral inequality for convex functions. Moreover, it is known that some of the classical

2010 *Mathematics Subject Classification.* 26A51, 26D15.
Key words and phrases. Convex function, *AH*-convex function, beta and hypergeometric functions, Hermite-Hadamard and Bullen type inequality.
Communicated by: Nikhil Khanna
[†]Corresponding author.

inequalities for means can be derived from (1.1) for appropriate particular selections of the function f. See [6, 10, 15, 17], for the results of the generalization, improvement and extention of the famous integral inequality (1.1).

Theorem 1.1 ([18])**.** *Suppose that $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$. Then we have the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2} \end{aligned} \tag{1.2}$$

The third inequality in (1.2) is known in the literature as Bullen's inequality.

Definition 1.2 ([5, 20])**.** A function $f : I \subset \mathbb{R} \to (0,\infty)$ is said to be arithmetic-harmonically (AH) convex function if for all $x, y \in I$ and $t \in [0,1]$ the equality

$$f(tx+(1-t)y) \leq \frac{f(x)f(y)}{tf(y)+(1-t)f(x)} \tag{1.3}$$

holds. If the inequality (1.2) is reversed then the function $f(x)$ is said to be arithmetic-harmonically (AH) concave function.

An important case which provides many examples is that one in which the function is assumed to be positive for any $x \in C$. In that situation the inequality (AH) is equivalent to

$$(1-\lambda)\frac{1}{f(x)}+\lambda\frac{1}{f(y)} \leq (\geq) \frac{1}{f((1-\lambda)x+\lambda y)}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

Therefore, the following Criterion can be stated:

Criterion 1.3 ([5])**.** *Let X be a linear space and C a convex subset in X. The function $f : C \to (0,\infty)$ is AH-convex (concave) on C if and only if $\frac{1}{f}$ is concave (convex) on C in the usual sense.*

Example 1.4. If $-p > 1$ or $-p < 0$, i.e. $p \in (-\infty,-1) \cup (0,\infty)$ then the function $f(x) = t^p, t > 0$ is arithmetic-harmonically (AH) concave function. If $p \in (-1,0)$ then the function $f(x) = t^p, t > 0$ is arithmetic-harmonically (AH) convex function.

Readers can find more informations on arithmetic-harmonically convex functions in [2, 5, 9, 20] and references therein.

Definition 1.5. (Beta Function) The Beta function denoted by $\beta(\alpha,\beta)$ is defined by

$$\beta(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, \quad \alpha,\beta > 0.$$

Definition 1.6. (Incomplete Beta Function) The incomplete beta function is defined by

$$\beta_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}dt$$

with $funcRep > 0$, $funcReq > 0$, $0 \leq x \leq 1$.

Definition 1.7. The hypergeometric function defined by

$$ {}_2F_1\left(a,b;c;z\right)=\frac{1}{\beta\left(b,c-b\right)}\int_0^1\frac{t^{b-1}\left(1-t\right)^{c-b-1}}{\left(1-zt\right)^{a}}dt,\ c>b>0,\ |z|<1. $$

In order to establish some inequalities of Hermite-Hadamard type integral inequalities for arithmetic-harmonically convex functions, we will use the following lemma.

Lemma 1.8 ([19])**.** *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a three times differentiable mapping on I° such that $f'''\in L\left[a,b\right]$, where $a,b\in I^\circ$ with $a<b$, then the following identity holds:*

$$ J_n\left(f,a,b\right):=\sum_{k=1}^{n}\frac{(b-a)^3}{12n^4}\left[\int_0^1 t^2\left(3-2t\right)f'''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)dt\right] \tag{1.4} $$

for all $n\in\mathbb{N}$, where

$$ \begin{aligned} &J_n\left(f,a,b\right)\\ &=\sum_{k=1}^{n}\frac{1}{2n}\left[f\left(a+\frac{(k-1)(b-a)}{n}\right)+f\left(a+\frac{k(b-a)}{n}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx. \end{aligned} $$

In this study, using Hölder integral inequality and the identity (1.4) in order to provide inequality for functions whose third derivatives in absolute value at certain power are arithmetic-harmonically-convex functions.

Throughout this paper, we will use the following notations for special means of two nonnegative numbers a,b with $b>a$ [6]:

1. The arithmetic mean
$$ A:=A(a,b)=\frac{a+b}{2},\quad a,b>0, $$
2. The geometric mean
$$ G:=G(a,b)=\sqrt{ab},\quad a,b\geq 0 $$
3. The harmonic mean
$$ H:=H(a,b)=\frac{2ab}{a+b},\quad a,b>0, $$
4. The logarithmic mean
$$ L:=L(a,b)=\begin{cases}\frac{b-a}{\ln b-\ln a}, & a\neq b\\ a, & a=b\end{cases};\quad a,b>0 $$
5. The p-logarithmic mean
$$ L_p:=L_p(a,b)=\begin{cases}\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a\neq b, p\in\mathbb{R}\backslash\left\{-1,0\right\}\\ a, & a=b\end{cases};\quad a,b>0. $$

These means are often used in numerical approximation and in other areas. However, the following simple relationship is known in the literature: $H\leq G\leq L\leq I\leq A$. It is also known that L_p is monotonically increasing over $p\in\mathbb{R}$, denoting $L_0=I$ and $L_{-1}=L$. In addition,

$$ A_{n,k}=A_{n,k}\left(a,b\right)=\frac{1+n-k}{n}a+\frac{k-1}{n}b,\ n\in\mathbb{N},\ k=1,2,...,n. $$

Previously, in the papers written by functions Kadakal and İşcan, $|f'|^q$, $q>1$ and $|f''|^q$, $q>1$ functions were considered and related inequalities were obtained for arithmetically harmonically-convex functions [11, 12]. In this study, we obtained similar inequalities for $|f'''|^q$, $q>1$ function by using the studies of Kadakal and İşcan. Our main purpose in doing this is to generalize for n th derivatives by using the studies made for the first, second and third derivatives mentioned above in our later works.

2. Main results

Throughtout this section, we will use $C_q(u,v)$ integral defined in the following Lemma:

Lemma 2.1. *It is easily seen that the following identities hold for $u,v>0$:*

$$\begin{aligned} C_q(u,v) &= \int_0^1 \frac{t^q}{tu+(1-t)v}dt = \int_0^1 \frac{(1-t)^q}{tv+(1-t)u}dt \\ &= \begin{cases} \frac{1}{u(q+2)} \cdot {}_2F_1\left(1,1;q+2;1-\frac{v}{u}\right) & u>v \\ \frac{1}{v(q+1)} \cdot {}_2F_1\left(1,q+1;q+2;1-\frac{u}{v}\right), & v>u \end{cases}. \end{aligned}$$

Theorem 2.2. *Let $f: I\subset(0,\infty)\to(0,\infty)$ be a three times differentiable mapping on I°, $n\in\mathbb{N}$ and $a,b\in I^\circ$ with $a<b$ such that $|f'''|^q\in L_1[a,b]$ is an arithmetic-harmonically convex function on the interval $[a,b]$ for some fixed $q>1$, then the following inequality holds:*

i) If $|f'''(A_{n,k+1})|^q-|f'''(A_{n,k})|^q\neq 0$, then

$$\begin{aligned} |J_n(f,a,b)| \leq{}& \sum_{k=1}^n \frac{(b-a)^3}{4n^4}\left(\frac{3}{2}\right)^{1+\frac{1}{p}} |f'''(A_{n,k})|\,|f'''(A_{n,k+1})| \\ &\times \beta_{\frac{2}{3}}^{\frac{1}{p}}(p+1,p+1)\, C_q^{\frac{1}{q}}\left(|f'''(A_{n,k+1})|^q, |f'''(A_{n,k})|^q\right), \end{aligned} \tag{2.1}$$

ii) If $|f'''(A_{n,k+1})|^q-|f'''(A_{n,k})|^q=0$, then

$$|J_n(f,a,b)| \leq \sum_{k=1}^n \frac{(b-a)^3}{4n^4}\left(\frac{3}{2}\right)^{1+\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \beta_{\frac{2}{3}}^{\frac{1}{p}}(p+1,p+1)\,|f'''(A_{n,k+1})|, \tag{2.2}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. i) Let $|f'''(A_{n,k+1})|^q-|f'''(A_{n,k})|^q\neq 0$. From the Lemma 1.8, the properties of modulus and the Hölder integral inequality, we write

$$\begin{aligned} &|J_n(f,a,b)| \\ &\leq \left|\sum_{k=1}^n \frac{(b-a)^3}{12n^4}\left[\int_0^1 t^2(3-2t)\,f'''(tA_{n,k}+(1-t)A_{n,k+1})\,dt\right]\right| \qquad (2.3) \\ &\leq \sum_{k=1}^n \frac{(b-a)^3}{12n^4}\left(\int_0^1 t^p(3-2t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t^q\,|f'''(tA_{n,k}+(1-t)A_{n,k+1})|^q\,dt\right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f'''|^q$ is an arithmetic-harmonically convex function on the interval $[a,b]$, the following inequality

$$|f'''(tA_{n,k}+(1-t)A_{n,k+1})|^q \leq \frac{|f'''(A_{n,k})|^q|f'''(A_{n,k+1})|^q}{t|f'''(A_{n,k+1})|^q+(1-t)|f'''(A_{n,k})|^q} \tag{2.4}$$

holds. If we use this inequality in (2.3), we get

$$\begin{aligned}
&|J_n(f,a,b)| \\
&\leq \sum_{k=1}^{n}\frac{(b-a)^3}{12n^4}\left(\int_0^1 t^p(3-2t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \frac{t^q|f'''(A_{n,k})|^q|f'''(A_{n,k+1})|^q\,dt}{t|f'''(A_{n,k+1})|^q+(1-t)|f'''(A_{n,k})|^q}\right)^{\frac{1}{q}} \qquad (2.5)\\
&=\sum_{k=1}^{n}\frac{(b-a)^3}{12n^4}|f'''(A_{n,k})||f'''(A_{n,k+1})|\left(\int_0^1 t^p(3-2t)^p\,dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\frac{t^q dt}{t|f'''(A_{n,k+1})|^q+(1-t)|f'''(A_{n,k})|^q}\right)^{\frac{1}{q}}\\
&=\sum_{k=1}^{n}\frac{(b-a)^3}{4n^4}\left(\frac{3}{2}\right)^{1+\frac{1}{p}}|f'''(A_{n,k})||f'''(A_{n,k+1})| \qquad (2.6)\\
&\times\beta_{\frac{2}{3}}^{\frac{1}{p}}(p+1,p+1)\,C_q^{\frac{1}{q}}\left(|f'''(A_{n,k+1})|^q,|f'''(A_{n,k})|^q\right),
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 t^p(3-2t)^p\,dt &= 3^p\left(\frac{3}{2}\right)^{p+1}\int_0^{\frac{2}{3}}x^p(1-x)^p\,dx\\
&= 3^p\left(\frac{3}{2}\right)^{p+1}\beta_{\frac{2}{3}}(p+1,p+1).
\end{aligned}$$

ii) Let $|f'''(A_{n,k+1})|^q-|f'''(A_{n,k})|^q=0$. Then, using (2.5), we have

$$\begin{aligned}
|J_n(f,a,b)| \leq\ & \sum_{k=1}^{n}\frac{(b-a)^3}{4n^4}\left(\frac{3}{2}\right)^{1+\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\\
&\times\beta_{\frac{2}{3}}^{\frac{1}{p}}(p+1,p+1)|f'''(A_{n,k+1})|.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 2.3. *By chosing $n=1$ in Theorem 2.2, we obtain the following inequalities:*
i) If $|f'''(A_{1,2})|^q-|f'''(A_{1,1})|^q\neq 0$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^3}{4}\left(\frac{3}{2}\right)^{1+\frac{1}{p}}|f'''(a)||f'''(b)|$$

$$\times\beta_{\frac{2}{3}}^{\frac{1}{p}}(p+1,p+1)\,C_q^{\frac{1}{q}}\left(|f'''(b)|^q,|f'''(a)|^q\right),$$

ii) If $|f'''(A_{1,2})|^q - |f'''(A_{1,1})|^q = 0$, *then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^3}{4}\left(\frac{3}{2}\right)^{1+\frac{1}{p}} \beta_{\frac{2}{3}}^{\frac{1}{p}}(p+1,p+1)\,|f'''(b)|\,.$$

Corollary 2.4. *By chosing* $n=2$ *in Theorem 2.2, we obtain the following Bullen type inequalities:*

i) If $|f'''(A_{2,k+1})|^q - |f'''(A_{2,k})|^q \neq 0$ *for* $k=1,2$, *then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^3}{64}\left(\frac{3}{2}\right)^{1+\frac{1}{p}} |f'''(a)|\left|f'''\left(\frac{a+b}{2}\right)\right| \\
&\times \beta_{\frac{2}{3}}^{\frac{1}{p}}(p+1,p+1)\,C_q^{\frac{1}{q}}\left(\left|f'''\left(\frac{a+b}{2}\right)\right|^q, |f'''(a)|^q\right) \\
&+\frac{(b-a)^3}{64}\left(\frac{3}{2}\right)^{1+\frac{1}{p}}\left|f'''\left(\frac{a+b}{2}\right)\right| |f'''(b)| \\
&\times \beta_{\frac{2}{3}}^{\frac{1}{p}}(p+1,p+1)\,C_q^{\frac{1}{q}}\left(|f'''(b)|^q, \left|f'''\left(\frac{a+b}{2}\right)\right|^q\right),
\end{aligned}$$

ii) If $|f'''(A_{2,k+1})|^q - |f'''(A_{2,k})|^q = 0$ *for* $k=1,2$, *then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^3}{64}\left(\frac{3}{2}\right)^{1+\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \beta_{\frac{2}{3}}^{\frac{1}{p}}(p+1,p+1)\left[\left|f'''\left(\frac{a+b}{2}\right)\right| + |f'''(b)|\right].
\end{aligned}$$

Theorem 2.5. *Let* $f: I \subset (0,\infty) \to (0,\infty)$ *be a three times differentiable mapping on* I°, $n \in \mathbb{N}$ *and* $a,b \in I^\circ$ *with* $a<b$ *such that* $|f'''|^q \in L_1[a,b]$ *is an arithmetic-harmonically convex function on the interval* $[a,b]$ *for some fixed* $q>1$, *then the following inequality holds:*

i) If $|f'''(A_{n,k+1})|^q - |f'''(A_{n,k})|^q \neq 0$, *then*

$$\begin{aligned}
|J_n(f,a,b)| \quad \leq\ & \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4}|f'''(A_{n,k})|\,|f'''(A_{n,k+1})| \\
& L_p(1,3)\,C_{2q}^{\frac{1}{q}}\left(|f'''(A_{n,k+1})|^q, |f'''(A_{n,k})|^q\right),
\end{aligned} \tag{2.7}$$

ii) If $|f'''(A_{n,k+1})|^q - |f'''(A_{n,k})|^q = 0$, *then*

$$|J_n(f,a,b)| \leq \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4} L_p(1,3)\left(\frac{1}{2q+1}\right)^{\frac{1}{q}} |f'''(A_{n,k+1})|\,. \tag{2.8}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. i) Let $|f'''(A_{n,k+1})|^q - |f'''(A_{n,k})|^q \neq 0$. From the Lemma 1.8 and the Hölder integral inequality, we write

$$\begin{aligned} & |J_n(f,a,b)| \\ & \leq \left| \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4} \left[\int_0^1 t^2 (3-2t) f'''(tA_{n,k} + (1-t)A_{n,k+1})\, dt \right] \right| \qquad (2.9) \\ & \leq \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4} \left(\int_0^1 (3-2t)^p\, dt \right)^{\frac{1}{p}} \left(\int_0^1 t^{2q} |f'''(tA_{n,k} + (1-t)A_{n,k+1})|^q\, dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f'''|^q$ is an arithmetic-harmonically convex function on the interval $[a,b]$, the following inequality

$$|f'''(tA_{n,k} + (1-t)A_{n,k+1})|^q \leq \frac{|f'''(A_{n,k})|^q |f'''(A_{n,k+1})|^q}{t|f'''(A_{n,k+1})|^q + (1-t)|f'''(A_{n,k})|^q}$$

holds. If we use this inequality in (2.9), we get

$$\begin{aligned} & |J_n(f,a,b)| \\ & \leq \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4} \left(\int_0^1 (3-2t)^p\, dt \right)^{\frac{1}{p}} \left(\int_0^1 \frac{t^{2q} |f'''(A_{n,k})|^q |f'''(A_{n,k+1})|^q}{t|f'''(A_{n,k+1})|^q + (1-t)|f'''(A_{n,k})|^q} dt \right)^{\frac{1}{q}} \qquad (2.10) \\ & = \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4} |f'''(A_{n,k})| |f'''(A_{n,k+1})| L_p(1,3) \times \\ & \qquad \left(\int_0^1 \frac{t^{2q} dt}{t|f'''(A_{n,k+1})|^q + (1-t)|f'''(A_{n,k})|^q} \right)^{\frac{1}{q}} \\ & = \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4} |f'''(A_{n,k})| |f'''(A_{n,k+1})| L_p(1,3)\, C_{2q}^{\frac{1}{q}} \left(|f'''(A_{n,k+1})|^q, |f'''(A_{n,k})|^q \right), \end{aligned}$$

where

$$\int_0^1 (3-2t)^p\, dt = \frac{3^{p+1}-1}{2(p+1)} = L_p^p(1,3).$$

ii) Let $|f'''(A_{n,k+1})|^q - |f'''(A_{n,k})|^q = 0$. Then, using (2.10), we have

$$|J_n(f,a,b)| \leq \sum_{k=1}^{n} \frac{(b-a)^3}{12n^4} L_p(1,3) \left(\frac{1}{2q+1} \right)^{\frac{1}{q}} |f'''(A_{n,k+1})|.$$

This completes the proof of theorem. □

Corollary 2.6. *By chosing $n=1$ in Theorem 2.5, we obtain the following inequalities:*
i) If $|f'''(A_{1,2})|^q - |f'''(A_{1,1})|^q \neq 0$, then

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ \leq \; & \frac{(b-a)^3}{12} L_p(1,3) |f'''(a)| |f'''(b)|\, C_q^{\frac{1}{q}} \left(|f'''(b)|^q, |f'''(a)|^q \right) \end{aligned}$$

ii) If $|f'''(A_{1,2})|^q - |f'''(A_{1,1})|^q = 0$, *then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12} L_p(1,3)\left(\frac{1}{2q+1}\right)^{\frac{1}{q}} |f'''(b)|.$$

Corollary 2.7. *By chosing* $n=2$ *in Theorem 2.5, we obtain the following Bullen type inequalities:*

i) If $|f'''(A_{2,k+1})|^q - |f'''(A_{2,k})|^q \neq 0$ *for* $k=1,2$, *then*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{(b-a)^3}{12} L_p(1,3)\,|f'''(a)|\left|f''\left(\frac{a+b}{2}\right)\right| C_{2q}^{\frac{1}{q}}\left(\left|f'''\left(\frac{a+b}{2}\right)\right|^q, |f'''(a)|^q\right) \\ &+ \frac{(b-a)^3}{12} L_p(1,3)\left|f''2\left(\frac{a+b}{2}\right)\right| |f'''(b)|\, C_{2q}^{\frac{1}{q}}\left(|f'''(b)|^q, \left|f'''\left(\frac{a+b}{2}\right)\right|^q\right),\end{aligned}$$

ii) If $|f'''(A_{2,k+1})|^q - |f'''(A_{2,k})|^q = 0$ *for* $k=1,2$, *then*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & \frac{(b-a)^3}{12} L_p(1,3)\left(\frac{1}{2q+1}\right)^{\frac{1}{q}}\left[\left|f'''\left(\frac{a+b}{2}\right)\right| + |f'''(b)|\right].\end{aligned}$$

3. Applications for special means

If $p \in (-1,0)$ then the function $f(x) = x^p, x > 0$ is an arithmetic harmonically-convex [5]. Using this function we obtain following propositions:

Proposition 3.1. Let $a, b \in (0,\infty)$ with $a < b$, $q > 1$ and $m \in (-1,0)$. Then, we have the following inequality:

$$\begin{aligned}&\prod_{s=1}^{3}\frac{q^3}{(m+sq)}\left|\sum_{k=1}^{n}\frac{1}{n}A\left((A_{n,k})^{\frac{m}{q}+3}, (A_{n,k+1})^{\frac{m}{q}+3}\right) - L_{\frac{m}{q}+3}^{\frac{m}{q}+3}(a,b)\right| \\ &\leq \sum_{k=1}^{n}\frac{(b-a)^3}{4n^4}\left(\frac{3}{2}\right)^{1+\frac{1}{p}}(A_{n,k})^{\frac{m}{q}}(A_{n,k+1})^{\frac{m}{q}}\beta_{\frac{2}{3}}^{\frac{1}{p}}(p+1,p+1)\, C_q^{\frac{1}{q}}\left((A_{n,k+1})^m, (A_{n,k})^m\right).\end{aligned}$$

Proof. The assertion follows from the inequality (2.1) in the Theorem 2.2. Let

$$f(x) = \frac{q^3 x^{\frac{m}{q}+3}}{(m+q)(m+2q)(m+3q)} = \prod_{s=1}^{3}\frac{q^3}{(m+sq)}x^{\frac{m}{q}+3}, \quad x \in (0,\infty).$$

Then $|f'''(x)|^q = x^m$ is an arithmetic harmonically-convex on $(0,\infty)$ and the result follows directly from Theorem 2.2. $\square$

Corollary 3.2. *If we take $n=1$ in Proposition 3.1, we get the following inequality:*

$$\prod_{s=1}^{3}\frac{q^3}{(m+sq)}\left|A\left((A_{1,1})^{\frac{m}{q}+3},(A_{1,2})^{\frac{m}{q}+3}\right)-L_{\frac{m}{q}+3}^{\frac{m}{q}+3}(a,b)\right|$$
$$\leq\frac{(b-a)^3}{4}\left(\frac{3}{2}\right)^{1+\frac{1}{p}}(A_{1,1})^{\frac{m}{q}}(A_{1,2})^{\frac{m}{q}}\beta_{\frac{2}{3}}^{\frac{1}{p}}(p+1,p+1)\,C_q^{\frac{1}{q}}\left((A_{1,2})^m,(A_{1,1})^m\right),$$

that is,

$$\prod_{s=1}^{3}\frac{q^3}{(m+sq)}\left|A\left(a^{\frac{m}{q}+3},b^{\frac{m}{q}+3}\right)-L_{\frac{m}{q}+3}^{\frac{m}{q}+3}(a,b)\right|\leq\frac{(b-a)^3}{4}\left(\frac{3}{2}\right)^{1+\frac{1}{p}}a^{\frac{m}{q}}b^{\frac{m}{q}}C_q^{\frac{1}{q}}(b^m,a^m).$$

Proposition 3.3. Let $a,b\in(0,\infty)$ with $a<b$, $q>1$ and $m\in(-1,0)$. Then, we have the following inequality:

$$\prod_{s=1}^{3}\frac{q^3}{(m+sq)}\left|\sum_{k=1}^{n}\frac{1}{n}A\left((A_{n,k})^{\frac{m}{q}+3},(A_{n,k+1})^{\frac{m}{q}+3}\right)-L_{\frac{m}{q}+3}^{\frac{m}{q}+3}(a,b)\right|$$
$$\leq\sum_{k=1}^{n}\frac{(b-a)^3}{12n^4}(A_{n,k})^{\frac{m}{q}}(A_{n,k+1})^{\frac{m}{q}}L_p(1,3)\,C_{2q}^{\frac{1}{q}}\left((A_{n,k+1})^m,(A_{n,k})^m\right).$$

Proof. The assertion follows from the inequality (2.7) in the Theorem 2.5. Let

$$f(x)=\frac{q^3x^{\frac{m}{q}+3}}{(m+q)(m+2q)(m+3q)}=\prod_{s=1}^{3}\frac{q^3}{(m+sq)}x^{\frac{m}{q}+3},\quad x\in(0,\infty).$$

Then $|f''(x)|^q=x^m$ is an arithmetic harmonically-convex on $(0,\infty)$ and the result follows directly from Theorem 2.5. □

Corollary 3.4. *If we take $n=1$ in Proposition 3.3, we get the following inequality:*

$$\prod_{s=1}^{3}\frac{q^3}{(m+sq)}\left|A\left((A_{1,1})^{\frac{m}{q}+3},(A_{1,2})^{\frac{m}{q}+3}\right)-L_{\frac{m}{q}+3}^{\frac{m}{q}+3}(a,b)\right|$$
$$\leq\frac{(b-a)^3}{12}L_p(1,3)(A_{1,1})^{\frac{m}{q}}(A_{1,2})^{\frac{m}{q}}C_{2q}^{\frac{1}{q}}\left((A_{1,2})^m,(A_{1,1})^m\right),$$

that is,

$$\prod_{s=1}^{3}\frac{q^3}{(m+sq)}\left|A\left((A_{1,1})^{\frac{m}{q}+3},(A_{1,2})^{\frac{m}{q}+3}\right)-L_{\frac{m}{q}+3}^{\frac{m}{q}+3}(a,b)\right|$$
$$\leq\frac{(b-a)^3}{12}L_p(1,3)\,a^{\frac{m}{q}}b^{\frac{m}{q}}C_{2q}^{\frac{1}{q}}(b^m,a^m).$$

References

[1] M. Alomari, M. Darus, and S. S. Dragomir, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are quasi-convex, Tamkang J. Math., 41(2010), 353-359.

[2] K. Bekar, Inequalities for three-times differentiable arithmetic-harmonically functions, Turkish Journal of Analysis and Number Theory, 7(2019), no. 3, 85-90.

[3] S. P. Bai, S. H. Wang, and F. Qi, Some Hermite-Hadamard type inequalities for n-time differentiable (α, m)-convex functions, J. Inequal. Appl., 2012(2012), 11 pages.

[4] P. Cerone, S. S. Dragomir, J. Roumeliotis, and J. Šunde, A new generalization of the trapezoid formula for n-time differentiable mappings and applications, Demonstratio Math., 33(2000), 719-736.

[5] S. S. Dragomir, Inequalities of Hermite-Hadamard type for AH-convex functions, Stud. Univ. Babeş-Bolyai Math., 61(2016), no. 4, 489-502.

[6] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.

[7] İ. İşcan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacet. J. Math. Stat., 43(2014), 935–942.

[8] İ. İşcan, S. Turhan, and S. Maden, Some Hermite-Hadamard-Fejer type inequalities for Harmonically convex functions via Fractional Integral, New Trends Math. Sci., 4(2016), 1-10.

[9] İ. İşcan, T. Toplu, and F. Yetgin, Some new inequalities on generalization of Hermite-Hadamard and Bullen type inequalities, Kragujevac Journal of Mathematics, (Accepted for publication), 2019.

[10] H. Kadakal, New Inequalities for Strongly r-Convex Functions, J. Funct. Spaces, 2019(2019), Article ID 1219237, 10 pages.

[11] M. Kadakal and İ. İşcan, Some new inequalities for differentiable arithmetic-harmonically convex functions, Comptes Rendus De L'Academie Bulgare Des Sciences, (Submitted to journal), 2019.

[12] M. Kadakal and İ. İşcan, Inequalities of Hermite-Hadamard and Bullen type for AH-convex functions, Universal Journal of Mathematics and Applications, 2(2019), no. 3, 152-158.

[13] M. Kadakal, H. Karaca, and İ. İşcan, Hermite-Hadamard type inequalities for multiplicatively geometrically P-functions, Poincare J. Anal. Appl., 2018(2018), no. 2(I), 77-85.

[14] U. S. Kirmaci, M. K. Bakula, M. E. Özdemir, J. Pečarić, Hadamard-type inequalities for s-convex functions, Appl. Math. Comp., 193(2007), 26-35.

[15] S. Maden, H. Kadakal, M. Kadakal, and İ. İşcan, Some new integral inequalities for n-times differentiable convex and concave functions, J. Nonlinear Sci. Appl., 10(2017), 6141-6148.

[16] J. E. Pečarić, F. Porschan, and Y. L. Tong, Convex Functions, Partial Orderings, and Statistical Applications, Academic Press, Boston, 1992.

[17] M. Z. Sarikaya and N. Aktan, On the generalization of some integral inequalities and their applications, Mathematical and Computer Modelling, 54(2011), 2175-2182.

[18] K. L. Tseng, S. R. Hwang, S. S. Dragomir, Fejér-type inequalities (I), J. Inequal. Appl., 2010, Article ID531976, 7 pp.

[19] F. Yetgin, M. Korkmaz, and İ. İşcan, Some new type general integral inequalities for s-convex functions and related results, 2017, doi: 10.13140/RG.2.2.22315.64805.

[20] T. Y. Zhang, and F. Qi, Integral Inequalities of Hermite–Hadamard Type for m-AH Convex Functions, Turkish Journal of Analysis and Number Theory, 2(2014), no. 3, 60-64.

(Huriye Kadakal) Ministry of Education, Giresun Hamdi Bozbağ Anatolian High School, Giresun-TÜRKİYE

E-mail address: huriyekadakal@hotmail.com

(Kerim Bekar) Department of Mathematics, Faculty of Sciences and Arts, Giresun University-Giresun-TÜRKİYE

E-mail address: kebekar@gmail.com