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EQUIVARIANT COMPLETE INVARIANCE PROPERTY ON A SUBSPACE OF TWO-INTERVAL WAVELETS

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Abstract. By providing a free action of the additive topological group of reals on a subspace ω of the space of all two-interval wavelets, we show that ω enjoys the complete invariance property. Further, considering ω as a topological transformation group with an action of the unit circle group, we obtain that ω possesses the equivariant complete invariance property.

1. Introduction

The notion of a flow has been axiomatized for providing precision to concepts available in dynamical systems. Its effect could be noticed in classifying topological spaces possessing properties such as the complete invariance property (CIP) and also the complete invariance property with respect to a homeomorphism (CIPH)([6], [7], [8]).

Although the notion of a wavelet is a recent development, its study rapidly grew two-fold. Based on the core topics in Mathematics such as algebra, analysis and topology, wavelets while growing theoretically supported by the distribution theory and harmonic analysis, did not leave users of Mathematics to remain out of its reach. Evolving wavelets through tools in analysis gave way to workers interested in pure mathematics to investigate wavelets and many subspaces of wavelets from the point of view of topology. Subspaces of the wavelet space which have been taken into account include those of MRA wavelets (wavelets that arise from a multiresolution analysis) and MSF (minimally supported frequency) wavelets. To each one-dimensional MSF wavelet there is associated a specific subset of $\mathbb{R}$ possessing Lebesgue measure one. Precisely, this is the support of Fourier transform of the MSF wavelet, called a *wavelet set.* An MSF wavelet for which the wavelet set possesses n-components is called an *n-interval wavelet.* Such wavelets have been completely characterized for $n = 2$ and symmetric $2n$ intervals ([5], [1]).

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In this paper, we consider a subspace ω of $\mathscr{W}_2$, where $\mathscr{W}_2$ is the space of all two-interval wavelets of $L^2(\mathbb{R})$, and provide a free action of $\mathbb{R}$ on it. As a consequence, we obtain that ω enjoys the CIP.

The equivariant version of the CIP and the CIPH have been investigated in our paper [4]. Considering ω as a topological transformation group with an action of the unit circle S^1, we obtain that ω possesses the equivariant complete invariance property.

2. Prerequisite

We denote by $\mathbb{R}$ the set of all real numbers and by $\mathbb{Z}$, the set of all integers. The symbol $L^2(\mathbb{R})$ is used to denote the set of all complex-valued square integrable functions on $\mathbb{R}$. An inner product on $L^2(\mathbb{R})$ is defined by

$$< f, g >= \int_{\mathbb{R}} f\bar{g}, \quad f,\ g \in L^2(\mathbb{R}).$$

According to the Plancherel theorem, $< f, g >= \frac{1}{2\pi} < \hat{f}, \hat{g} >$, where $\hat{f}, \hat{g}$ denote the Fourier transforms [9] of the respective functions.

We say that a member ψ in $L^2(\mathbb{R})$ is an *orthonormal wavelet*, or simply a *wavelet*, if the system $\{\psi_{j,k} \in L^2(\mathbb{R}) \mid j, k \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{R})$, where $\psi_{j,k}$ is defined by

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad x \in \mathbb{R}.$$

Definition 2.1 ([3])**.** A measurable subset W of $\mathbb{R}$ is called a *wavelet set* if $\frac{1}{\sqrt{2\pi}}\chi_W$ is the Fourier transform of a wavelet.

A wavelet which arises from a wavelet set is called an *MSF wavelet.* The following theorem due to Ha et al. characterizes a 2-interval wavelet.

Theorem 2.2 ([5])**.** *Let $\hat{\psi}(\xi) = \vartheta(\xi)\chi_W(\xi)$, where*

$$W = [2a - 4\pi, a - 2\pi) \cup [a, 2a)$$

for some $0 < a < 2\pi$ and $|\vartheta(\xi)| = 1$. Then ψ is a wavelet. Moreover, a 2-interval wavelet is of this form.

The above theorem shows that the set of all 2-interval wavelet sets is in bijection with $(0, 2\pi)$.

Definition 2.3 ([6])**.** A space is said to possess the *complete invariance property,* abbreviated CIP, if each of its nonempty closed subsets is the fixed point set of some continuous selfmap.

For a metric space (X, d), the distance of a point $x \in X$ from a closed set A of X is defined as $\varrho(x, A) = inf\{d(x, a) : a \in A\}$.

Let (X, d) be a metric space and φ be a free action of $\mathbb{R}$ on X. For a nonempty closed set A of X, define $h : X \to X$ by

$$h(x) = \varphi(x, \varrho(x, A)), \quad x \in X.$$

Then $h(x) = x$ if and only if $\varrho(x, A) = 0$, i.e. $x \in A$. Thus h has A as its fixed point set. This provides the following:

Proposition 2.4 ([2])**.** If there is a free action of $\mathbb{R}$ on a metric space X, then X possesses the CIP.

Definition 2.5 ([2])**.** A topological transformation group (X, G, ϑ) is said to possess the *equivariant complete invariance property*, abbreviated ECIP, if for each nonempty invariant closed set F of X, there is an equivariant continuous selfmap f on X such that $Fix\ f = F$.

Note that for metric spaces (X_1, d_1) and (X_2, d_2), the map $D : (X_1 \times X_2) \times (X_1 \times X_2) \to X_1 \times X_2$ defined by

$$D((x_1, y_1), (x_2, y_2)) = max\{d_1(x_1, x_2), d_2(y_1, y_2)\}, \quad x_1, x_2 \in X_1,\ y_1, y_2 \in X_2,$$

is a metric on the product $X_1 \times X_2$.

Throughout the paper, we use the following notation:

- Symbol 'd' represents the metric of $L^2(\mathbb{R})$.
- For $A \subset L^2(\mathbb{R})$, $\hat{A}$ denotes the image of A under the Fourier transform, while $\check{A}$ denotes the image of A under the inverse Fourier transform.
- For $a \in (0, 2\pi)$, W_a stands for the wavelet set $[2a - 4\pi, a - 2\pi) \cup [a, 2a)$.
- If $x \in \mathbb{R}$, then $\bar{x} = \pi \cdot \left(\frac{x}{1+|x|} + 1\right) \in (0, 2\pi)$.
- Define a map $\alpha : \mathbb{R} \to L^2(\mathbb{R})$ which sends $x \in \mathbb{R}$ to the inverse Fourier transform of the characteristic function on $W_{\bar{x}}$, denoted by $\tilde{x}$. The image of $\mathbb{R}$ under α is denoted by $\tilde{R}$ which is equipped with the subspace topology of $L^2(\mathbb{R})$.
- Symbol $\mathscr{S}$ is used to denote the set $\{(e^{i\alpha}\hat{\tilde{x}})\check{}\ |\ e^{i\alpha} \in S^1,\ \tilde{x} \in \tilde{R}\}$. It is a subspace of $L^2(\mathbb{R})$.
- The space of all 2-interval wavelets obtained by normalizing elements of $\mathscr{S}$ is denoted by ω.

3. A flow on ω.

In this section, we provide a free action of $\mathbb{R}$ on the space $\mathscr{S}$, which in turn, shows that $\mathscr{S}$ together with ω possesses the complete invariance property. Also, we show that $\mathscr{S}$ and ω enjoy ECIP with respect to the natural action of S^1.

We begin with the following lemmas:

Lemma 3.1. *Let $\tilde{x},\ \tilde{y} \in \tilde{R}$. Then*

$$[d(\tilde{x}, \tilde{y})]^2 = \frac{1}{2\pi}|(W_{\bar{x}} \setminus W_{\bar{y}}) \cup (W_{\bar{y}} \setminus W_{\bar{x}})|.$$

Proof. Since $\tilde{x}$ and $\tilde{y}$ are identified with $\check{\chi}_{W_{\bar{x}}}$ and $\check{\chi}_{W_{\bar{y}}}$, respectively, where $x,\ y \in \mathbb{R}$, we have

$$[d(\tilde{x}, \tilde{y})]^2 = \int_{\mathbb{R}} |\check{\chi}_{W_{\bar{x}}} - \check{\chi}_{W_{\bar{y}}}|^2$$

Using Plancherel's theorem, we get

$$\begin{aligned} [d(\tilde{x}, \tilde{y})]^2 &= \frac{1}{2\pi}\int_{\mathbb{R}} |\chi_{W_{\bar{x}}} - \chi_{W_{\bar{y}}}|^2 \\ &= \frac{1}{2\pi}|(W_{\bar{x}} \setminus W_{\bar{y}}) \cup (W_{\bar{y}} \setminus W_{\bar{x}})|. \end{aligned}$$

□

If W is a subset of $\mathbb{R}$, then denote by W^+ the set $W \cap (0,\infty)$ and by W^- the set $W \cap (-\infty, 0)$.

Lemma 3.2. *For $a,\ b \in (0, 2\pi)$, $|(W_a \setminus W_b) \cup (W_b \setminus W_a)| \leq 6|b-a|$.*

Proof. Without loss of generality assume that $b \geq a$. We have the following cases:
Case 1. When $W_a^+ \cap W_b^+ \neq \emptyset$ and $W_a^- \cap W_b^- \neq \emptyset$. Then $b < 2a$ and $2b < a + 2\pi$ and hence,

$$|(W_a \setminus W_b) \cup (W_b \setminus W_a)| = 6|b-a|.$$

Case 2. When $W_a^+ \cap W_b^+ = \emptyset$ and $W_a^- \cap W_b^- \neq \emptyset$. Then $b \geq 2a$ and $2b < a + 2\pi$ and hence,

$$|(W_a \setminus W_b) \cup (W_b \setminus W_a)| = 3|b-a| + (a+b).$$

Since $3(b-a) - (a+b) = 2b - 4a \geq 0$, we have $3(b-a) \geq (a+b)$. Thus, $|(W_a \setminus W_b) \cup (W_b \setminus W_a)| \leq 6|b-a|$.
Case 3. When $W_a^+ \cap W_b^+ \neq \emptyset$ and $W_a^- \cap W_b^- = \emptyset$. Then $b < 2a$ and $2b \geq a + 2\pi$ and hence, $(2\pi - b)$ and $(2\pi - a)$ satisfy Case 2. Thus

$$\begin{aligned}|(W_a \setminus W_b) \cup (W_b \setminus W_a)| &= |(W_{(2\pi-a)} \setminus W_{(2\pi-b)}) \cup (W_{(2\pi-b)} \setminus W_{(2\pi-a)})| \\ &\leq 6|b-a|.\end{aligned}$$

Case 4. When $W_a^+ \cap W_b^+ = \emptyset$ and $W_a^- \cap W_b^- = \emptyset$. Then $b \geq 2a$ and $2b \geq a + 2\pi$ and hence,

$$|(W_a \setminus W_b) \cup (W_b \setminus W_a)| = 4\pi.$$

The fact that $b \geq 2a$ and $2b \geq a + 2\pi$ imply $(b-a) \geq \frac{2\pi}{3}$. Thus, $6(b-a) \geq 4\pi$, and hence the result. □

Remark 3.3. For $a, b \in (0, 2\pi)$ satisfying any one of the cases $1, 2$ or 3 in the proof of above theorem, $|(W_a \setminus W_b) \cup (W_b \setminus W_a)| > 3|b-a|$.

Proposition 3.4. Let $\tilde{x},\ \tilde{y} \in \tilde{R}$. Then

$$[d(\tilde{x}, \tilde{y})]^2 \leq 3\left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right|.$$

Proof. Follows from lemmas 3.1 and 3.2. □

Define $\varphi : \tilde{R} \times \mathbb{R} \to \tilde{R}$ by

$$\varphi(\tilde{x},\ t) = \widetilde{x+t},$$

where $\tilde{x} \in \tilde{R},\ t \in \mathbb{R}$.

Theorem 3.5. *The map φ is an action of $\mathbb{R}$ on $\tilde{R}$.*

Proof. We simply show that φ is a continuous map. Note that the map from $(-1,1) \times \mathbb{R}$ to $\mathbb{R}$ which sends $(\frac{z}{1+|z|}, r)$ to $\frac{z+r}{1+|z+r|}$, where $z,\ r \in \mathbb{R}$, is a continuous map. Let $(\tilde{x}, t) \in \tilde{R} \times \mathbb{R}$ and $\varepsilon > 0$. Then there is a $\delta > 0$ such that

$$\left|\frac{x+t}{1+|x+t|} - \frac{z+r}{1+|z+r|}\right| < \frac{\varepsilon^2}{3}$$

whenever $\left|\frac{x}{1+|x|} - \frac{z}{1+|z|}\right| < \delta$ and $|t-r| < \delta$. While selecting δ we take care that δ is small enough such that the ball around $\frac{x}{1+|x|}$ of radius δ lies in $(-1, 1)$.

Now, consider $(\tilde{y}, s) \in \tilde{R} \times \mathbb{R}$ with $max\{d(\tilde{x}, \tilde{y}), |t-s|\} < \sqrt{\delta}$. Then,

$$d(\tilde{x}, \tilde{y}) < \sqrt{\delta}$$

or

$$[d(\tilde{x}, \tilde{y})]^2 < \delta.$$

By Lemma 3.1 and Remark 3.3, we have

$$\frac{3}{2\pi}|\bar{x} - \bar{y}| < \delta,$$

or

$$\frac{3}{2}\left|\frac{|x|}{1+|x|} - \frac{|y|}{1+|y|}\right| < \delta.$$

Thus,

$$\left|\frac{x+t}{1+|x+t|} - \frac{y+s}{1+|y+s|}\right| < \frac{\varepsilon^2}{3}.$$

Applying Proposition 3.4, we obtain that

$$\begin{aligned}[d(\widetilde{x+t}, \widetilde{y+s})]^2 &\leq 3\left|\frac{x+t}{1+|x+t|} - \frac{y+s}{1+|y+s|}\right| \\ &< \varepsilon^2,\end{aligned}$$

or

$$d(\widetilde{x+t}, \widetilde{y+s}) < \varepsilon.$$

This completes the proof. $\square$

Lemma 3.6. *The map* $p : \hat{\mathscr{S}} \to S^1 \times \hat{\tilde{R}}$ *defined by*

$$p(e^{i\vartheta}\hat{\tilde{x}}) = (e^{i\vartheta}, \hat{\tilde{x}}), \quad (e^{i\vartheta}\hat{\tilde{x}}) \in \hat{\mathscr{S}}$$

is a homeomorphism.

Proof. Let $e^{i\vartheta}\hat{\tilde{x}} \in \hat{\mathscr{S}}$ and $\varepsilon > 0$. Then for $\delta = min\{\varepsilon, \sqrt{\pi}\}$ and $e^{i\vartheta'}\hat{\tilde{y}} \in \hat{\mathscr{S}}$ with $d(e^{i\vartheta}\hat{\tilde{x}}, e^{i\vartheta'}\hat{\tilde{y}}) < \delta$, we have

$$\int_{\mathbb{R}} |\chi_{W_{\bar{x}}}|^2 + |\chi_{W_{\bar{y}}}|^2 - 2|\chi_{W_{\bar{x}}}||\chi_{W_{\bar{y}}}|cos(\vartheta - \vartheta') < \delta^2,$$

$$\text{or} \quad \int_{\mathbb{R}} |\chi_{W_{\bar{x}}}|^2 + |\chi_{W_{\bar{y}}}|^2 - 2|\chi_{W_{\bar{x}}}||\chi_{W_{\bar{y}}}| + 2|\chi_{W_{\bar{x}}}||\chi_{W_{\bar{y}}}| - 2|\chi_{W_{\bar{x}}}||\chi_{W_{\bar{y}}}|cos(\vartheta - \vartheta') < \delta^2,$$

$$\text{or} \quad [d(\hat{\tilde{x}}, \hat{\tilde{y}})]^2 + 2(1 - cos(\vartheta - \vartheta'))\int_{\mathbb{R}} |\chi_{W_{\bar{x}}}||\chi_{W_{\bar{y}}}| < \delta^2,$$

$$\text{or} \quad [d(\hat{\tilde{x}}, \hat{\tilde{y}})]^2 + |e^{i\vartheta} - e^{i\vartheta'}|^2|W_{\bar{x}} \cap W_{\bar{y}}| < \delta^2,$$

Thus,

$$d(\hat{\tilde{x}}, \hat{\tilde{y}}) < \delta$$

and

$$|e^{i\vartheta} - e^{i\vartheta'}|^2|W_{\bar{x}} \cap W_{\bar{y}}| < \delta^2.$$

Using the fact that measure of $(W_{\bar{x}} \cup W_{\bar{y}})$ can not exceed 2π and Lemma 3.1, we have

$$\begin{aligned} |W_{\bar{x}} \cap W_{\bar{y}}| &= |W_{\bar{x}} \cup W_{\bar{y}}| - |(W_{\bar{x}} \setminus W_{\bar{y}}) \cup (W_{\bar{y}} \setminus W_{\bar{x}})| \\ &> 2\pi - \delta^2 > \pi, \end{aligned}$$

which implies that $|e^{i\vartheta} - e^{i\vartheta'}| < \delta$, or $|e^{i\vartheta} - e^{i\vartheta'}| < \varepsilon$. Thus $D((e^{i\vartheta}, \hat{\tilde{x}}), (e^{i\vartheta'}, \hat{\tilde{y}})) < \varepsilon$. This shows that the map p is continuous.

Define $q : S^1 \times \hat{\tilde{R}} \to \hat{\mathscr{S}}$ by

$$q((e^{i\vartheta}, \hat{\tilde{x}})) = e^{i\vartheta}\hat{\tilde{x}}, \qquad e^{i\vartheta} \in S^1,\ \hat{\tilde{x}} \in \hat{\tilde{R}}.$$

For the continuity of q, consider a point $(e^{i\vartheta}, \hat{\tilde{x}})$ in $S^1 \times \hat{\tilde{R}}$ and $\varepsilon > 0$. Put $\delta = \sqrt{\frac{\varepsilon}{1+2\pi}}$. Then for $(e^{i\psi}, \hat{\tilde{y}}) \in S^1 \times \hat{\tilde{R}}$ with $D((e^{i\vartheta}, \hat{\tilde{x}}), (e^{i\psi}, \hat{\tilde{y}})) < \delta$, we have

$$\begin{aligned} [d(e^{i\vartheta}\hat{\tilde{x}}, e^{i\psi}\hat{\tilde{y}})]^2 &= [d(\hat{\tilde{x}}, \hat{\tilde{y}})]^2 + |e^{i\vartheta} - e^{i\psi}|^2 |W_{\bar{x}} \cap W_{\bar{y}}| \\ &< \delta^2 + 2\pi\delta^2 < \varepsilon. \end{aligned}$$

Since $p^{-1} = q$, the result follows. □

Theorem 3.7. *The map $\Phi : \mathscr{S} \times \mathbb{R} \to \mathscr{S}$ defined by*

$$\Phi((e^{i\vartheta}\hat{\tilde{x}})\check{}, t) = (e^{i\vartheta}\widehat{\widetilde{x + t}})\check{}, \quad (e^{i\vartheta}\hat{\tilde{x}})\check{} \in \mathscr{S},\ t \in \mathbb{R},$$

describes an action of $\mathbb{R}$ on $\mathscr{S}$.

Proof. Since Φ is the composition of the following continuous maps

(i) $\wedge \times I_{\mathbb{R}} : \mathscr{S} \times \mathbb{R} \to \hat{\mathscr{S}} \times \mathbb{R}$,
(ii) $p \times I_{\mathbb{R}} : \hat{\mathscr{S}} \times \mathbb{R} \to S^1 \times \hat{\tilde{R}} \times \mathbb{R}$,
(iii) $I_{S^1} \times \vee \times I_{\mathbb{R}} : S^1 \times \hat{\tilde{R}} \times \mathbb{R} \to S^1 \times \tilde{R} \times \mathbb{R}$,
(iv) $I_{S^1} \times \varphi : S^1 \times \tilde{R} \times \mathbb{R} \to S^1 \times \tilde{R}$,
(v) $I_{S^1} \times \wedge : S^1 \times \tilde{R} \to S^1 \times \hat{\tilde{R}}$,
(vi) $q : S^1 \times \hat{\tilde{R}} \to \hat{\mathscr{S}}$, and
(vii) $\vee : \hat{\mathscr{S}} \to \mathscr{S}$,

we have the result. □

It may be noted that the action of $\mathbb{R}$ on $\mathscr{S}$ is free. Taking Proposition 2.4 into account, we have the following:

Theorem 3.8. *The space $\mathscr{S}$ enjoys the* CIP.

Corollary 3.9. *The space ω being homeomorphic to $\mathscr{S}$ possesses the* CIP.

Theorem 3.10. *The topological transformation group $(S^1, \mathscr{S}, \vartheta)$, where $\vartheta : S^1 \times \mathscr{S} \to \mathscr{S}$ is defined by*

$$\vartheta(e^{i\psi}, (e^{i\alpha}\hat{\tilde{x}})\check{}) = (e^{i(\alpha+\psi)}\hat{\tilde{x}})\check{}, \quad (e^{i\alpha}\hat{\tilde{x}})\check{} \in \mathscr{S},\ e^{i\psi} \in S^1,$$

enjoys the ECIP.

Proof. Let F be a nonempty invariant closed set of $\mathscr{S}$. Then define $h : \mathscr{S} \to \mathscr{S}$ by

$$h(z) = \Phi\left(z, \frac{1}{2}\varrho(z, F)\right), \qquad z \in \mathscr{S}.$$

Since Φ is a free action, the fixed point set of h is F. For $g \in S^1$ and $z \in \mathscr{S}$, note the following:

(i) $\Phi(\vartheta(g,z),t) = \vartheta(g, \Phi(z,t))$, and

(ii) since d is invariant with respect to the action ϑ and $\vartheta(S^1 \times F) = F$, $\varrho(\vartheta(g,z),\ F) = \varrho(z,F)$.

In view of above, we have

$$\vartheta(g, h(z)) = h(\vartheta(g,z)).$$

This proves the equivariance of h, and hence the result follows. □

Corollary 3.11. *The topological transformation group $(\omega, S^1, \bar{\vartheta})$, where $\bar{\vartheta}$ is the action of S^1 on ω induced from ϑ, possesses the* ECIP.

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