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CONVERGENCE ANALYSIS OF ALGORITHMS FOR VARIATIONAL INEQUALITIES INVOLVING STRICTLY PSEUDO-CONTRACTIVE OPERATORS

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Abstract. The purpose of the present work is to analyze the convergence of two new viscosity iterative algorithms for finding common solutions to variational inequality problem and fixed point problem of a finite system of strictly pseudo-contractive operators in a q-uniformly smooth Banach space. Under suitable conditions, some strong convergence theorems are obtained. Using Matlab software, we numerically solve a convex feasibility problem. Our results improve some existing known results.

1. Introduction

Consider the well known convex feasibility problem (CFP):

$$\text{finding an } x \in \bigcap_{i=1}^{k} C_i \tag{1.1}$$

where $C_1, C_2, \dots, C_k$ are intersecting closed convex subsets of a Banach space X. The CFP lies in center of many problems of mathematics and the physical sciences such as computerized tomography [14], radiation therapy treatment planning [7, 11, 15], sensor networking [3], image restoration [13]. A comprehensive study on algorithms for solving the CFP can be found in [2, 13].

Let X^* be dual of a real Banach space X. For $q > 1$, the generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(u) = \left\{u^* \in X^* : \langle u, u^* \rangle = \|u\|^q, \|u^*\| = \|u\|^{q-1}\right\}, \quad \forall\ u \in X,$$

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where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. In particular, $J_q = J_2$ is called the normalized duality mapping and further, J_q is single valued (denoted by j_q), if X is smooth [18].

A Banach space X is called strictly convex if $\|u + v\| < 2$ for all $u, v \in X$ with $\|u\| = \|v\| = 1$ and $u \neq v$. Furthermore, Banach space X is said to be uniformly convex if for each $\varepsilon > 0$, there is a $\delta > 0$ such that if $\|u\| = \|v\| = 1$ and $\|u - v\| \geq \epsilon$ then $\|\frac{u+v}{2}\| \leq 1 - \delta$. Clearly, uniform convexity implies strict convexity.

A modulus of smoothness of X is a function $\rho_X : [0, \infty) \to [0, \infty)$ described as

$$\rho_X(r) = \sup \left\{ \frac{1}{2} (\|u + v\| + \|u - v\|) - 1 : \|u\| = 1, \|v\| \leq r \right\}.$$

A Banach space X is called uniformly smooth if $\frac{\rho_X(r)}{r} \to 0$ as $r \to 0$ and it is called q-uniformly smooth, if there exists $d > 0$ (fixed constant) such that $\rho_X(r) \leq dr^q$. It is well known that if X is q-uniformly smooth then $q \leq 2$ and X is uniformly smooth.

Let $D \subseteq X$ be a closed convex set and $D_1 \subseteq D$ then a mapping $Q : D \to D_1$ is called sunny if

$$Q(Qu + r(u - Qu)) = Qu,$$

whenever $Qu + r(u - Qu) \in D$ for $u \in D$ and $r \geq 0$. Furthermore, $Q\colon D \to D_1$ is sunny nonexpansive retraction if Q is a retraction (i.e., $Qu = u \ \forall \ u \in D_1$) which is also sunny and nonexpansive. In a real Hilbert space, a sunny nonexpansive retraction Q coincides with the metric projection mapping.

A mapping $S : D \to D$ is said to be γ-Lipschitzian if there exists a constant $\gamma > 0$ such that

$$\|Su - Sv\| \leq \gamma \|u - v\| \ \forall \ u, v \in D.$$

S is a contraction if $\gamma \in (0, 1)$ and it is called nonexpansive if $\gamma = 1$. Here, Ω_D will be used to denote the collection of all contractions on D.

A mapping $S : D \to D$ is called β-strictly pseudo-contractive if there exists $\beta > 0$ and $j_q(u - v) \in J_q(u - v)$ such that

$$\langle Su - Sv, j_q(u - v) \rangle \leq \|u - v\|^q - \beta \|(I - S)u - (I - S)v\|^q \ \forall \ u, v \in D.$$

The class of β-strictly pseudo-contractive introduced by Browder and Petryshyn [4] in 1967 is a generalization of the class of nonexpansive mappings.

Here, $F(S)$ will be used to denote the set of all fixed points of mapping S.

A mapping $A : D \to X$ is called μ-inverse strongly accretive if there exists a constant $\mu > 0$ and $j_q(u - v) \in J_q(u - v)$ such that

$$\langle Au - Av, j_q(u - v) \rangle \geq \mu \|Au - Av\|^q \ \forall \ u, v \in D.$$

Recall that the variational inequality problem is to find $\bar{u} \in D$ such that

$$\langle A\bar{u}, j_q(u - \bar{u}) \rangle \geq 0, \quad \forall \ u \in D, \tag{1.2}$$

where $j_q(u - \bar{u}) \in J_q(u - \bar{u})$. Usually, its solution set is denoted by $VI(D, A)$. In a Hilbert space, this problem reduces to find $\bar{u} \in D$ such that

$$\langle A\bar{u}, u - \bar{u} \rangle \geq 0, \quad \forall \ u \in D. \tag{1.3}$$

In this paper, first we shall consider the case when C_i in CFP (1.1) is the set of solutions of the variational inequality (1.2) and the fixed point set of λ-strictly pseudo-contractive mapping. Further, we shall consider the case when C_i is arbitrary closed and convex subset in a Banach space setting.

In [16], Yu Li gave the following iterative algorithm to solve CFP

$$u_1 \in D, \quad u_{n+1} = \alpha_n x + \beta_n u_n + \gamma_n S \sum_{i=1}^{k} \theta_i P_D(u_n - \lambda_i A_i u_n), \quad n \geq 1, \tag{1.4}$$

where $x \in D$ is arbitrary (but fixed), P_D is the metric projection from Hilbert space X onto D, $A_i : D \to X$, $i = 1, 2, \ldots, k$ is a family of η_i-inverse strongly monotone mappings, $S : D \to D$ is a nonexpansive mapping and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\theta_i\}$, $\{\lambda_i\}$ fulfill certain conditions. He proved the strong convergence of $\{u_n\}$ to $P_F x$, where $F = \bigcap_{i=1}^{k} VI(D, A_i) \bigcap F(S) \neq \phi$.

Furthermore, Cho and Kang [12] introduced the following iterative algorithm in a Hilbert space:

$$\begin{cases} u_1 \in D, \\ v_n = \alpha_n x + (1 - \alpha_n) \sum\limits_{i=1}^{k} \left(\theta_n^i P_D(u_n - \lambda_i A_i u_n)\right), \\ u_{n+1} = \beta_n u_n + (1 - \beta_n)(\gamma_n v_n + (1 - \gamma_n) S v_n), \quad n \geq 1, \end{cases} \tag{1.5}$$

where x is a fixed element in D, $A_i : D \to X$, $i = 1, 2, \ldots, k$ is a family of η_i-inverse strongly monotone mappings, $S : D \to D$ is a λ-strictly pseudo-contractive with a fixed point. They proved that the sequence $\{u_n\}$ in (1.5) converges strongly to a point in $F = \bigcap_{i=1}^{k} VI(D, A_i) \bigcap F(S)$ provided $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\theta_n^i\}$, $\{\lambda_i\}$, $i = 1, 2, \ldots, k$ satisfy certain conditions and $F \neq \phi$.

Recently, Zegeye and Shahzad [21] constructed the following iterative process in a uniformly convex and q-uniformly smooth Banach space:

$$u_0,\ x \in D, \quad u_{n+1} = \alpha_n x + (1 - \alpha_n) Q_D [I - \gamma A]((1 - \beta) u_n + \beta S u_n), \quad \forall\, n \geq 0, \tag{1.6}$$

where $S = (1 - \mu) I + \mu T$ for $T = \beta_1 T_1 + \beta_2 T_2 + \ldots + \beta_N T_N$, $T_l : D \to D$, $l = 1, 2, \ldots, N$ are λ_l-strictly pseudo-contractive mappings, $A : D \to X$ is η-inverse strongly accretive mapping, Q_D is sunny nonexpansive retraction from X onto D and $\{\alpha_n\}$, β, γ, μ, $\{\beta_l\}$, $l = 1, 2, \ldots, N$ satisfy certain assumptions. They proved strong convergence of (1.6) to $Q_F(x)$ where $F = VI(D, A) \bigcap \left(\bigcap_{l=1}^{N} F(T_l) \right)$, which also solves following variational inequality:

$$\text{find } \bar{u} \in \bigcap_{l=1}^{N} F(T_l) \text{ such that } \langle A\bar{u}, j(u - \bar{u}) \rangle \geq 0, \quad \forall\, u \in D, \tag{1.7}$$

and in particular

$$\text{find } \bar{u} \in \bigcap_{l=1}^{N} F(T_l) \text{ such that } \langle A\bar{u}, j(u - \bar{u}) \rangle \geq 0, \quad \forall\, u \in \bigcap_{l=1}^{N} F(T_l). \tag{1.8}$$

Recently, Censor et al. introduced the Common Solutions to Variational Inequality Problem (CSVIP) (see [8], [9], [10]). The general form of the CSVIP is the following (for single valued operator):

Let $\{C_i\}_{i=1}^k$ be nonempty intersecting closed convex subsets of a Hilbert space H and $\{A_i\}_{i=1}^k$ be operators on H. The CSVIP is to find a point $\bar{u} \in \bigcap_{i=1}^{k} C_i$ such that

$$\langle A_i\bar{u}, u - \bar{u}\rangle \geq 0, \quad \forall\, u \in C_i \text{ and } i = 1, 2, \ldots, k.$$

In this research article, inspired by the work of Yu Li [16], Cho et al. [12], Zegeye and Shahzad [21] and Censor et al. [8, 9, 10], we construct two viscosity iterative algorithms which solve both variational inequality problem and fixed point problem at the same time.

2. Preliminaries

We collect several lemmas to establish our main results.

Lemma 2.1 ([17]). *For arbitrary positive real numbers r, s*

$$rs \leq \frac{1}{p} r^p + \frac{p-1}{p} s^{\frac{p}{p-1}},$$

where $p > 1$.

Lemma 2.2 ([19]). *In a real q-uniformly smooth Banach space X, there exists a constant $D_q > 0$ such that*

$$\|u + v\|^q \leq \|u\|^q + q\langle v, j_q(u)\rangle \ + D_q\|v\|^q \ \forall u, v \in X.$$

Lemma 2.3 ([20]). *Assume that $\{a_n\} \subset [0, \infty)$, $\{c_n\} \subset [0, \infty)$, $\{\psi_n\} \subset [0, 1]$ and $\{b_n\}$ are real number sequences satisfying*

$$a_{n+1} \leq (1 - \psi_n)a_n + \psi_n b_n + c_n, \quad \forall\, n \geq 0.$$

Assume that $\sum\limits_{n=1}^{\infty} \psi_n = \infty$, $\limsup\limits_{n\to\infty} b_n \leq 0$ and $\sum\limits_{n=1}^{\infty} c_n < \infty$. Then, $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.4 ([22]). *Let $S : D \to D$ be a λ-strictly pseudo-contractive mapping, where D is a nonempty convex subset of a real $q-$ uniformly smooth Banach space X. Define $S_\alpha u = (1 - \alpha)u + \alpha S u$ where $\alpha \in (0, \mu], \mu = \min\left\{1, \left\{\frac{q\lambda}{D_q}\right\}^{\frac{1}{q-1}}\right\}$ then $S_\alpha : D \to D$ is nonexpansive and $F(S_\alpha) = F(S)$.*

Lemma 2.5 ([6]). *Let $S : D \to D$ be a nonexpansive mapping with $F(S) \neq \phi$ and $g \in \Omega_D$, where D is a closed convex subset of a uniformly smooth Banach space X. If we define $\{u_r\}$ by $u_r = rg(u_r) + (1 - r)Su_r$ for $r \in (0, 1)$ then $\{u_r\}$ converges strongly to an element in $F(S)$. Further, the map $Q : \Omega_D \to F(S)$ defined by*

$$Q(g) := \lim_{r\to 0} u_r, \quad g \in \Omega_D,$$

solves the following variational inequality:

$$\langle (I - g)Q(g), j_q(Q(g) - p)\rangle \leq 0, \quad g \in \Omega_D,\ p \in F(S).$$

Lemma 2.6 ([6]). *Let $S : D \to D$ be a nonexpansive mapping with $F(S) \neq \phi$ and $g \in \Omega_D$, where D is a closed convex subset of a uniformly smooth Banach space X. Let $\{u_n\}$ be a bounded sequence such that $u_n - Su_n \to 0$ as $n \to \infty$ and $u_r = rg(u_r) + (1 - r)Su_r$ for $r \in (0, 1)$. Suppose that $Q(g) := \lim\limits_{r\to 0} u_r$ exists. Then*

$$\limsup_{n\to\infty} \langle (g - I)Q(g), j_q(u_n - Q(g))\rangle \leq 0.$$

Lemma 2.7 ([6]). *Let $A : D \to X$ be η-inverse strongly accretive mapping where D is a closed convex subset of a real $q-$ uniformly smooth Banach space X. Then, for $0 < \gamma < \left(\dfrac{q\eta}{D_q}\right)^{\frac{1}{q-1}}$, the mapping $A_\gamma u = (u - \gamma A u)$ is nonexpansive.*

Lemma 2.8 ([1]). *Let Q_D be a sunny nonexpansive retraction from X onto D and $B\colon D \to X$ be an accretive operator where D is a nonempty closed convex subset of a smooth Banach space X. Then, we have*

$$VI(D,B) = F(Q_D(I - \lambda B)) \ \forall\ \lambda > 0,$$

where $VI(D,B) = \{u^* \in D : \langle Bu^*, j(u - u^*)\rangle \geq 0,\ \forall\ u \in D\}$.

Lemma 2.9 ([5]). *Let $\{T_i\}_{i=1}^r$ be nonexpansive mappings on D with $\bigcap\limits_{i=1}^{r} F(T_i) \neq 0$, where D is a nonempty closed convex subset of a strictly convex Banach space X. Consider positive number sequence $\{\mu_i\}$ such that $\sum\limits_{i=1}^{r} \mu_i = 1$. Then $\sum\limits_{i=1}^{r} \mu_i T_i\colon D \to D$ is nonexpansive mapping such that $F(\sum\limits_{i=1}^{r} \mu_i T_i) = \bigcap\limits_{i=1}^{r} F(T_i)$.*

Lemma 2.10 ([23]). *Let D be a nonempty convex subset of a smooth Banach space X. Given a natural number N, suppose for each $i \in \Lambda$, $T_i : D \to D$ is a λ_i-strict pseudo-contraction for some $\lambda_i \in [0,1)$. Consider positive number sequence $\{\eta_i\}_{i=1}^N$ such that $\sum\limits_{i=1}^{N} \eta_i = 1$, then $\sum\limits_{i=1}^{N} \eta_i T_i : D \to D$ is a λ-strict pseudo-contraction with $\lambda = \min\{\lambda_i : 1 \leq i \leq N\}$. Furthermore, if $\bigcap\limits_{i=1}^{N} F(T_i) \neq \phi$ then $F\left(\sum\limits_{i=1}^{N} \eta_i T_i\right) = \bigcap\limits_{i=1}^{N} F(T_i)$.*

Lemma 2.11 ([21]). *Let $\{T_l\}_{l=1}^N$ be λ_l- strictly pseudo-contractive mappings on D and $\bar{A} : D \to X$ be η-inverse strongly accretive mapping, where D is a nonempty closed convex subset of a real uniformly convex and $q-$ uniformly smooth Banach space X. Assume that $F = \left(\bigcap\limits_{l=1}^{N} F(T_l)\right) \cap VI(D, \bar{A}) \neq \phi$. Let $A : D \to D$ be defined by $A = Q_D[I - \gamma\bar{A}]((1-\delta)I + \delta B)$, where $B := [(1-\rho)I + \rho T]$ for $T := \beta_1 T_1 + \beta_2 T_2 + \ldots + \beta_N T_N$ with $\sum\limits_{l=1}^{N} \beta_l = 1$, $\delta, \rho \in (0, \theta)$, $\theta = \min\left\{1, \left\{\dfrac{q\lambda}{D_q}\right\}^{\frac{1}{q-1}}\right\}$ for $\lambda = \min\{\lambda_l : l = 1, 2, \ldots, N\}$ and $0 < \gamma < \left(\dfrac{q\eta}{D_q}\right)^{\frac{1}{q-1}}$ for D_q a real number in Lemma 2.2. Then A is nonexpansive and $F(A) = F(B) \cap VI(D, \bar{A})$.*

3. Main results

Lemma 3.1. *Let D be a nonempty closed convex subset of a real strictly convex and q-uniformly smooth Banach space X $(1 < q < \infty)$. Let $\{T_l\}_{l=1}^N$ be λ_l-strictly pseudo-contractive mappings on D and $\{A_i\}_{i=1}^k$ be η_i-inverse strongly accretive mappings from D to X such that*

$$F = \left(\bigcap_{l=1}^{N} F(T_l)\right) \bigcap \left(\bigcap_{i=1}^{k} VI(D, A_i)\right) \neq \phi.$$

Let $A : D \to D$ *be defined by* $A = \sum_{i=1}^{k} \theta_i Q_D[I - \gamma_i A_i]((1-\delta)I + \delta B)$*, where* $B = [(1-\rho)I + \rho T]$ *for* $T = \beta_1 T_1 + \beta_2 T_2 + \ldots + \beta_N T_N$ *with* $\sum_{l=1}^{N} \beta_l = 1$*,* $\sum_{i=1}^{k} \theta_i = 1$*,* $0 < \delta$*,* $\rho < \min\left\{1, \left\{\frac{q\lambda}{D_q}\right\}^{\frac{1}{q-1}}\right\}$ *for* $\lambda = \min\{\lambda_l : l = 1, 2, \ldots, N\}$ *and* $0 < \gamma_i < \left(\frac{q\eta_i}{D_q}\right)^{\frac{1}{q-1}}$ *for* $i = 1, 2, \ldots, k$ *and* D_q *a real number in Lemma 2.2. Then* A *is nonexpansive and* $F(A) = F$*.*

Proof. In view of Lemma 2.10, T is λ-strictly pseudo-contractive. Further, by Lemmas 2.4 and 2.7, we have B and $I - \gamma_i A_i$ $(1 \le i \le k)$ are nonexpansive respectively. In consequence, $Q_D[I - \gamma_i A_i]((1-\delta)I + \delta B)$ is nonexpansive. Moreover, by using Lemma 2.9, we get that A is nonexpansive.

Now, we prove that $F(A) = F$.

Indeed by using Lemmas 2.4, 2.9, 2.10 and 2.11, we obtain

$$\begin{aligned}
F(A) &= F\left(\sum_{i=1}^{k} \theta_i Q_D[I - \gamma_i A_i]((1-\delta)I + \delta B)\right) \\
&= \bigcap_{i=1}^{k} F\left(Q_D[I - \gamma_i A_i]((1-\delta)I + \delta B)\right) \\
&= \bigcap_{i=1}^{k} \left(F(B) \bigcap VI(D, A_i)\right) \\
&= F(B) \bigcap \left(\bigcap_{i=1}^{k} VI(D, A_i)\right) \\
&= F(T) \bigcap \left(\bigcap_{i=1}^{k} VI(D, A_i)\right) \\
&= \left(\bigcap_{l=1}^{N} F(T_l)\right) \bigcap \left(\bigcap_{i=1}^{k} VI(D, A_i)\right) \\
&= F.
\end{aligned}$$

□

Theorem 3.2. *Let* $\{T_l\}_{l=1}^{N}$ *be* λ_l*-strictly pseudo-contractive mappings on* D *and* $\{A_i\}_{i=1}^{k}$ *be* η_i*-inverse strongly accretive mappings from* D *to* X *such that* $F = \left(\bigcap_{l=1}^{N} F(T_l)\right) \bigcap \left(\bigcap_{i=1}^{k} VI(D, A_i)\right) \neq \phi$*. Let* $\{u_n\}$ *be a sequence defined by* $u_1 \in D$ *and*

$$u_{n+1} = \alpha_n g(u_n) + (1-\alpha_n) \sum_{i=1}^{k} \theta_i Q_D[I - \gamma_i A_i]((1-\delta)u_n + \delta B u_n), \tag{3.1}$$

where B*,* δ*,* $\{\gamma_i\}$*,* $\{\theta_i\}$ *are as in Lemma 3.1,* $g \in \Omega_D$ *with contractive constant* $\gamma \in (0,1)$*. Suppose following conditions are satisfied by* $\{\alpha_n\} \subset (0,1)$*:*

(i) $\lim_{n\to\infty} \alpha_n = 0$*;*

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim\limits_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$.

Then $\{u_n\}$ *converges strongly to* $Q(g) \in F$*, which also solves following variational inequality:*

$$\langle (I-g)Q(g), j_q(Q(g)-p)\rangle \leq 0, \quad g \in \Omega_D,\ p \in F. \tag{3.2}$$

Proof. Rewrite (3.1) as

$$u_{n+1} = \alpha_n g(u_n) + (1-\alpha_n)Au_n, \tag{3.3}$$

where A is as in Lemma 3.1.

Then, by Lemma 3.1,

$$F(A) = F.$$

Further, by Lemma 2.5, there exists a solution $Q(g)$ of the variational inequality in (3.2), where $Q(g) := \lim\limits_{r\to 0} u_r \in F$ and

$$u_r = rg(u_r) + (1-r)Au_r, \quad \text{for } r \in (0,1).$$

Next, we claim that $\{u_n\}$ is bounded.

Take $p \in F$, noting that

$$\begin{aligned}
\|u_{n+1} - p\| &= \|\alpha_n(g(u_n) - g(p)) + (1-\alpha_n)(Au_n - p) + \alpha_n(g(p)-p)\| \\
&\leq \alpha_n\gamma\|u_n - p\| + (1-\alpha_n)\|u_n - p\| + \alpha_n\|g(p)-p\| \\
&\leq [1-\alpha_n(1-\gamma)]\|u_n - p\| + \alpha_n(1-\gamma)\frac{\|g(p)-p\|}{1-\gamma} \\
&\leq \max\left\{\|u_n - p\|, \frac{\|g(p)-p\|}{1-\gamma}\right\}
\end{aligned}$$

By induction,

$$\|u_{n+1} - p\| \leq \max\left\{\|u_1 - p\|, \frac{\|g(p)-p\|}{1-\gamma}\right\}$$

This proves the boundedness of $\{u_n\}$ and therefore, $\{g(u_n)\}$ and $\{Au_n\}$ are bounded.

Next, we show that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.4}$$

Using (3.4), we have

$$\begin{aligned}\|u_{n+1}-u_n\| &= \|\alpha_n g(u_n)+(1-\alpha_n)Au_n-\alpha_{n-1}g(u_{n-1})\\ &\quad -(1-\alpha_{n-1})Au_{n-1}\|\\ &= \|(1-\alpha_n)(Au_n-Au_{n-1})+\alpha_n(g(u_n)-g(u_{n-1}))\\ &\quad +(\alpha_n-\alpha_{n-1})(g(u_{n-1})-Au_{n-1})\|\\ &\le (1-\alpha_n)\|u_n-u_{n-1}\|+\alpha_n\gamma\|u_n-u_{n-1}\|\\ &\quad +|\alpha_n-\alpha_{n-1}|\,\|g(u_{n-1})-Au_{n-1}\|\\ &\le [1-\alpha_n(1-\gamma)]\|u_n-u_{n-1}\|+M\,|\alpha_n-\alpha_{n-1}|\end{aligned}$$

where $M>0$ is a constant such that

$$M \ge \sup\left\{\|g(u_{n-1})\|+\|A(u_{n-1})\|\right\},\quad \text{for all } n.$$

(I) If $\{\alpha_n\}$ satisfy the condition $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ then take $a_n=\|u_n-u_{n-1}\|$, $b_n=0, c_n=M|\alpha_n-\alpha_{n-1}|$ and $\psi_n=(1-\gamma)\alpha_n$ in Lemma 2.3. Clearly, all conditions in Lemma 2.3 are satisfied. Therefore, (3.4) is satisfied.

(II) If $\{\alpha_n\}$ satisfy the condition $\lim_{n\to\infty}\dfrac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}=0$ then take $a_n=\|u_n-u_{n-1}\|$, $\psi_n=(1-\gamma)\alpha_n$, $b_n=\dfrac{M|\alpha_n-\alpha_{n-1}|}{(1-\gamma)\alpha_n}$ and $c_n=0$. Clearly, using Lemma 2.3, we obtain (3.4).

Next, using boundedness of $\{Au_n\},\{g(u_n)\}$ and given condition $\alpha_n\to 0$, we get

$$\|u_{n+1}-Au_n\|=\alpha_n\|g(u_n)-Au_n\|\to 0 \ as\ n\to\infty. \tag{3.5}$$

Thus, (3.4) and (3.5) implies that

$$\|u_n-Au_n\|\le\|u_n-u_{n+1}\|+\|u_{n+1}-Au_n\|\to 0\ as\ n\to\infty. \tag{3.6}$$

Now, by (3.6) and Lemma 2.6, we get

$$\limsup_{n\to\infty}\langle (g-I)Q(g), j_q(u_n-Q(g))\rangle\le 0.$$

Finally, we prove that $u_n\to z\in F$, where $z=Q(g)$.
In fact, by Lemma 2.1, we have

$$\begin{aligned}\|u_{n+1}-z\|^q &= \|\alpha_n(g(u_n)-z)+(1-\alpha_n)(Au_n-z)\|^q\\ &= \alpha_n\langle g(u_n)-z, j_q(u_{n+1}-z)\rangle+(1-\alpha_n)\langle Au_n-z, j_q(u_{n+1}-z)\rangle\\ &= \alpha_n\langle g(u_n)-g(z), j_q(u_{n+1}-z)\rangle+\alpha_n\langle g(z)-z, j_q(u_{n+1}-z)\rangle\\ &\quad +(1-\alpha_n)\langle Au_n-z, j_q(u_{n+1}-z)\rangle\\ &\le [1-\alpha_n(1-\gamma)]\|u_n-z\|\|u_{n+1}-z\|^{q-1}\\ &\quad +\alpha_n\langle g(z)-z, j_q(u_{n+1}-z)\rangle\\ &\le [1-\alpha_n(1-\gamma)]\left(\frac{1}{q}\|u_n-z\|^q+\frac{q-1}{q}\|u_{n+1}-z\|^q\right)\\ &\quad +\alpha_n\langle g(z)-z, j_q(u_{n+1}-z)\rangle,\end{aligned}$$

which implies

$$\|u_{n+1}-z\|^q \le [1-\alpha_n(1-\gamma)]\|u_n-z\|^q+\alpha_n(1-\gamma)\frac{q\langle g(z)-z, j_q(u_{n+1}-z)\rangle}{1-\gamma},$$

and hence, in view of Lemma 2.3, $u_n \to z$ as $n \to \infty$. □

Corollary 3.3 (Zegeye and Shahzad [21, Theorem 3.2]). *Let $\{T_l\}_{l=1}^N$ be λ_l - strictly pseudo - contractive mappings on D and A_1 be η-inverse strongly accretive mapping from D to X, where D is nonempty closed convex subset of a uniformly convex and $q-$ uniformly smooth Banach space X such that $F=\left(\bigcap_{l=1}^{N} F(T_l)\right)\bigcap VI(D,A_1)\neq\phi$. Let $\{u_n\}$ be a sequence defined by $u_1\in D$ and*

$$u_{n+1}=\alpha_n x+(1-\alpha_n)Q_D[I-\gamma_1 A_1]((1-\delta)u_n+\delta Bu_n),$$

where $x\in D$ is arbitrary (but fixed), B, δ, γ_1 are as in Lemma 2.11. Suppose conditions (i)-(iii) in Theorem 3.2 are satisfied by $\{\alpha_n\}\subset(0,1)$. Then $\{u_n\}$ converges strongly to $Q_F(x)$, which solves the variational inequalities (1.7) *and* (1.8)*.*

Proof. Taking X a uniformly convex and $q-$ uniformly smooth Banach space, $g\equiv x$(constant map) and $k=1$ in Theorem 3.2, which yields the desired result. □

Now, we solve the CSVIP by introducing a new iterative algorithm.

Lemma 3.4. *Let $C_1, C_2, \ldots, C_k$ be nonempty closed convex subsets of a real strictly convex and q-uniformly smooth Banach space X $(1<q<\infty)$. Let $T_l : X\to X$ for $l=1,2,\ldots,N$ be λ_l-strictly pseudo-contractive mappings and $A_i: C_i\to X$ for $i=1,2,\ldots,k$ be η_i-inverse strongly accretive mappings such that $F=\left(\bigcap_{l=1}^{N}F(T_l)\right)\bigcap\left(\bigcap_{i=1}^{k}VI(C_i,A_i)\right)\neq\phi$. Let $A:X\to X$ be defined by $A=\sum_{i=1}^{k}\theta_i Q_{C_i}[I-\gamma_i A_i]Q_{C_i}((1-\delta)I+\delta B)$, where $B=[(1-\rho)I+\rho T]$ for $T=\beta_1T_1+\beta_2T_2+\ldots+\beta_NT_N$ with $\sum_{l=1}^{N}\beta_l=1$, $\sum_{i=1}^{k}\theta_i=1$, $0<\delta,\ \rho<\min\left\{1,\left\{\frac{q\lambda}{D_q}\right\}^{\frac{1}{q-1}}\right\}$ for $\lambda=\min\{\lambda_l : l=1,2,\ldots,N\}$ and $0<\gamma_i<\left(\frac{q\eta_i}{D_q}\right)^{\frac{1}{q-1}}$ for $i=1,2,\ldots,k$ and D_q a real number in Lemma 2.2. Then A is nonexpansive and $F(A)=F$.*

Proof. Following the proof lines of Lemma 3.1, we can reach to the conclusion that A is nonexpansive.

Now we prove that $F(A)=F$.

We first show that $F\left(Q_{C_i}[I-\gamma_iA_i]Q_{C_i}((1-\delta)I+\delta B)\right)=F(B)\cap VI(C_i,A_i)$ for each $i=1,2,\ldots,k$.

Also, note that following the proof lines of Lemma 3.1 in Zegeye and Shahzad [21], we can easily show that $F\left(Q_{C_i}[I-\gamma_iA_i]Q_{C_i}((1-\delta)I+\delta B)\right)=F(B)\cap VI(C_i,A_i)$. Now, by using Lemmas 2.4, 2.9 and 2.10, we obtain

$$\begin{aligned}
F(A) &= F\left(\sum_{i=1}^{k}\theta_i Q_{C_i}[I-\gamma_i A_i]Q_{C_i}((1-\delta)I+\delta B)\right)\\
&= \bigcap_{i=1}^{k} F\left(Q_{C_i}[I-\gamma_i A_i]Q_{C_i}((1-\delta)I+\delta B)\right)\\
&= \bigcap_{i=1}^{k}\Big(F(B)\bigcap VI(C_i,A_i)\Big)\\
&= F(T)\bigcap\left(\bigcap_{i=1}^{k} VI(C_i,A_i)\right)\\
&= \left(\bigcap_{l=1}^{N} F(T_l)\right)\bigcap\left(\bigcap_{i=1}^{k} VI(C_i,A_i)\right)\\
&= F\,.
\end{aligned}$$

□

Theorem 3.5. *Let $C_1, C_2, \dots, C_k$ be nonempty closed convex subsets of a real strictly convex and q-uniformly smooth Banach space X $(1 < q < \infty)$. Let $T_l : X \to X$ for $l = 1, 2, \dots, N$ be λ_l-strictly pseudo-contractive mappings and $A_i : C_i \to X$ for $i = 1, 2, \dots, k$ be η_i-inverse strongly accretive mappings such that $F = \left(\bigcap_{l=1}^{N} F(T_l)\right)\bigcap\left(\bigcap_{i=1}^{k} VI(C_i,A_i)\right) \neq \phi$. Let $\{u_n\}$ be a sequence defined by $u_1 \in X$ and*

$$u_{n+1} = \alpha_n g(u_n) + (1-\alpha_n)\sum_{i=1}^{k}\theta_i Q_{C_i}[I-\gamma_i A_i]Q_{C_i}((1-\delta)u_n+\delta B u_n), \quad (3.7)$$

where B, δ, $\{\gamma_i\}$, $\{\theta_i\}$ are as in Lemma 3.4, $g \in \Omega_X$ with contractive constant $\gamma \in (0,1)$. Suppose that $\{\alpha_n\} \subset (0,1)$ satisfying the conditions (i)-(iii) in Theorem 3.2. Then $\{u_n\}$ converges strongly to $Q(g) \in F$, which also solves the variational inequality (3.2).

Proof. First we rewrite (3.7) as

$$u_{n+1} = \alpha_n g(u_n) + (1-\alpha_n)Au_n,$$

where A is as in Lemma 3.4.

Now, the proof is analogous to that of Theorem 3.2 and therefore omitted. □

Now, we solve CFP by considering the case when C_i in (1.1) is arbitrary closed and convex subset.

Corollary 3.6. *Let $C_1, C_2, \dots, C_k$ be nonempty closed convex subsets of a real strictly convex and q-uniformly smooth Banach space X $(1 < q < \infty)$ with nonempty intersection. Let $\{u_n\}$ be a sequence defined by $u_1 \in X$ and*

$$u_{n+1} = \alpha_n g(u_n) + (1-\alpha_n)\sum_{i=1}^{k}\theta_i Q_{C_i}u_n, \quad (3.8)$$

where $g \in \Omega_X$ *and* $\sum_{i=1}^{k} \theta_i = 1$. *Suppose that* $\{\alpha_n\} \subset (0,1)$ *satisfying the conditions (i)-(iii) in Theorem 3.2. Then* $\{u_n\}$ *converges strongly to an element in* $\bigcap_{i=1}^{k} C_i$.

Proof. In Theorem 3.5, put $A_1 = A_2 = ... = A_k = 0$ and $T_1 = T_2 = ... = T_N = I$. Then $F = \bigcap_{i=1}^{k} C_i \neq \phi$. So by Theorem 3.5, we obtain the desired result. □

4. Numerical Example

In this section, using Corollary 3.6, we numerically solve a CFP.

We consider six closed and convex sets C_i, $i = 1,2,3,4,5,6$ in $\mathbb{R}^2$, these are given by the following expressions:

$$C_1 = \{(x,y) : x \geq 0\}; \qquad C_2 = \{(x,y) : x \leq 1\};$$
$$C_3 = \{(x,y) : y \geq 0\}; \qquad C_4 = \{(x,y) : y \leq 1\};$$
$$C_5 = \{(x,y) : x = 0.5\}; \qquad C_6 = \{(x,y) : y = 0.5\}.$$

Clearly, their intersection is nonempty; in fact $z = (0.5, 0.5)$ is the only point in their intersection.

Let $\{u_n\}$ be a sequence defined in (3.8), where $g(u) = \frac{u}{3}$, $\alpha_n = \frac{1}{2(n+1)}$, $n \geq 1$, $\theta_i = \frac{1}{6}$, $i = 1,2,3,4,5,6$. We examine three possible cases:

Case 1. Using the initial point $u_1 = (5, 0.7)$, we make the observations as in Table 1.

Table 1. Numerical results for Case 1

n	u_{n+1}	$\|u_{n+1} - z\|$
0	(5,0.7)	4.504442
1	(3.10417,0.558333)	2.604823
4	(1.1969,0.426899)	0.700724
5	(0.993852,0.414351)	0.501224
99	(0.48961,0.48961)	0.014694
100	(0.489718,0.489718)	0.014541
499	(0.497986,0.497986)	0.002848
500	(0.49799,0.49799)	0.002842
999	(0.498996,0.498996)	0.001419
1000	(0.498997,0.498997)	0.001418

From Table 1, it can be observed that the sequence $\{u_n\}$ is converging towards $z = (0.5, 0.5)$ as $n \to \infty$.

Case 2. We use the initial point $u_1 = (3, -0.8)$ and make the observations as in Table 2.

TABLE 2. Numerical results for Case 2

n	u_{n+1}	$\|u_{n+1} - z\|$
0	(3,-0.8)	2.817800
1	(1.9375,-0.404167)	1.698212
4	(0.914685,0.0509018)	0.611271
5	(0.800515,0.116686)	0.487072
99	(0.48961,0.48961)	0.014694
100	(0.489718,0.489718)	0.014541
499	(0.497986,0.497986)	0.002848
500	(0.49799,0.49799)	0.002842
999	(0.498996,0.498996)	0.001419
1000	(0.498997,0.498997)	0.001418

One can easily see that $\|u_n - z\|$ is converging to zero as $n \to \infty$, i.e., $\{u_n\}$ is converging to $z = (0.5, 0.5)$ as $n \to \infty$.

Case 3. Taking initial point $u_1 = (-4, 2)$, we make the observations as in Table 3.

TABLE 3. Numerical results for Case 3

n	u_{n+1}	$\|u_{n+1} - u\|$
0	(-4,2)	4.743416
1	(-2.27083, 1.35417)	2.899501
4	(-0.400641,0.753504)	0.935638
5	(-0.179576,0.672913)	0.701229
99	(0.48961,0.48961)	0.014694
100	(0.489718,0.489718)	0.014541
499	(0.497986,0.497986)	0.002848
500	(0.49799,0.49799)	0.002842
999	(0.498996,0.498996)	0.001419
1000	(0.498997,0.498997)	0.001418

In this case also $\{u_n\}$ is converging to $z = (0.5, 0.5)$ as $n \to \infty$.

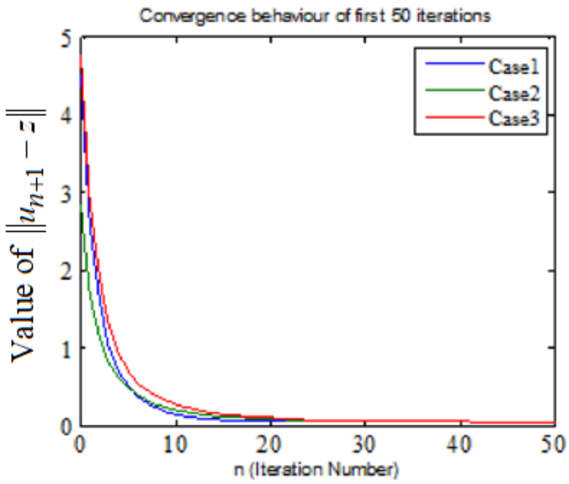

FIGURE 1. Convergence behaviour of first 50 iterations

From the tables and the graph, it can be observed that the sequence $\{u_n\}$ defined by (3.8) converging towards $z = (0.5, 0.5) \in \bigcap_{i=1}^{6} C_i$, for any initial point u_1.

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